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Wave Reflection and Refraction at the Interface of Triclinic and Liquid Medium

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ABSTRACT

A Mathematical model has been considered to study the reflection and refraction phenomenon of plane wave at the interface of an isotropic liquid medium and a triclinic (anisotropic) half-space. The incident plane *qP* wave generates three types of reflected waves namely quasi-*P* (*qP*), quasi-*SV* (*qSV*) and quasi-*SH* (*qSH*) waves in the triclinic medium and one refracted *P* wave in the isotropic liquid medium. Expression of phase velocities of all the three quasi waves have been calculated. It has been considered that the direction of particle motion is neither parallel nor perpendicular to the direction of propagation in anisotropic medium. Some specific relations have been established between directions of motion and propagation. The expressions for reflection coefficients of *qP, qSV, qSH* and refracted *P* waves with respect to incident qP wave are obtained. Numerical computation and graphical representations have been performed for the reflection coefficient of reflected *qP*, reflected *qSV*, reflected *qSH* and refraction coefficient of refracted *P* wave with incident *qP* wave.

Keywords: Reflection; Refraction; Plane wave; Triclinic.

1 INTRODUCTION

THE fundamentals of seismic wave propagation are developed using a physical approach and then teleseismic techniques are used to study reflection and refraction phenomena in the composite structure related to the problem. Study of Seismic waves investigate the plate tectonic processes that cause earthquakes. From recordings of the earthquake-generated waves, information about the structure of the Earth may be derived. It is well accepted that the Earth is anisotropic, but from the seismological and mineralogical point of view, this anisotropy is sometimes difficult to measure and interpret. In an anisotropic medium, physical properties depends on the direction of propagation. When Seismic waves travel through such a medium the velocities vary as functions of polarization and direction of propagation. The study of wave propagation in anisotropy medium have attracted the researchers over past few decades. Keith and Crampin [2] studied the reflection and transmission of plane waves in anisotropic medium, following the same media some milestone works have been carried out by Chattopadhyay [3-4] and [5]. In

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order to understand the phenomenon better Chattopadhyay and Michel [6] studied the reflection and refraction of wave in anisotropic media. Carcione [7] investigated that materials displaying anisotropy must have its effective elasticity constants arranged in some form of crystalline symmetry. Crampin [8] showed that the surface waves propagating through a layer of anisotropy material with certain symmetric relations have distinct particle motion. Some possible application of reflection and refraction of elastic waves in seismology has been discussed by Knott [9]. The phenomenon of reflection and transmission of seismic waves may help to understand the internal structure of the Earth. Particularly, refraction tends to cause *P* and *S* waves to become vertically orientated as they approach the surface. Reflection of plane wave at the free surface of an anisotropic half-space has been studied earlier by many authors. Particularly, the work done by Ditri and Rose [10], Zilmer et. al. [12] and Singh and Khurana [11] has remarkable impact on the field of seismology. Chatterjee et. al. [3] obtained the solution for a three dimensional wave scattering phenomenon between two dissimilar anisotropic half spaces under initial stress. Paswan et. al. [14] derived the amplitude ratios and showed the energy conservation in a two dimensional model of fluid layer sandwiched between two monoclinic half-spaces. Singh et. al. [15] studied similar type of problem considering an intermediate layer lying between two semi-infinite media.

In this paper, we have studied the reflection and refraction of quasi-*P* wave at the interface between an isotropic homogeneous liquid half-space and a triclinic half-space.(Relations between directions of motion and propagation have been expressed.) The incident *qP* wave gives rise to three reflected waves, namely *qP, qSV* and *qSH* waves in triclinic (anisotropic) medium and one refracted *qP* wave in homogeneous isotropic liquid medium. The expression for the phase velocities of reflected *qP, qSV* and *qSH* waves in triclinic medium have been derived. The amplitude ratios of reflected *qP, qSV, qSH* waves and transmitted *P* wave with respect to incident *qP* wave have been obtained. Numerical example has been given and variations of amplitude ratios with the incident angle have been illustrated graphically. It has been observed that triclinic medium plays a significant role in case of reflection and refraction.

2 FORMULATION OF THE PROBLEM

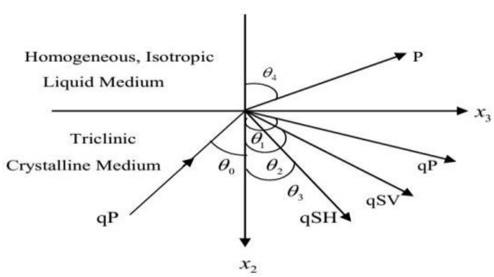

Fig.1
Geometry of the problem.

The stress-strain relations for a homogeneous triclinic medium with twenty-one elastic constants are

$$\left.\begin{aligned}
\tau_{11} &= C_{11}e_{11} + C_{12}e_{22} + C_{13}e_{33} + C_{14}e_{23} + C_{15}e_{13} + C_{16}e_{12}\,, \\
\tau_{22} &= C_{12}e_{11} + C_{22}e_{22} + C_{23}e_{33} + C_{24}e_{23} + C_{25}e_{13} + C_{26}e_{12}\,, \\
\tau_{33} &= C_{13}e_{11} + C_{23}e_{22} + C_{33}e_{33} + C_{34}e_{23} + C_{35}e_{13} + C_{36}e_{12}\,, \\
\tau_{23} &= C_{14}e_{11} + C_{24}e_{22} + C_{34}e_{33} + C_{44}e_{23} + C_{45}e_{13} + C_{46}e_{12}\,, \\
\tau_{13} &= C_{15}e_{11} + C_{25}e_{22} + C_{35}e_{33} + C_{45}e_{23} + C_{55}e_{13} + C_{56}e_{12}\,, \\
\tau_{12} &= C_{16}e_{11} + C_{26}e_{22} + C_{36}e_{33} + C_{46}e_{23} + C_{56}e_{13} + C_{66}e_{12}
\end{aligned}\right\} \tag{1}$$

where

$$\begin{aligned}
C_{ij} = C_{ji}\,, e_{ij} &= \left(u_{i,j} + u_{j,i}\right), \quad \text{for } i \neq j \\
&= u_{i,i}\,, \quad \text{for } i = j
\end{aligned} \tag{2}$$

and u_i $(i = 1, 2, 3)$ are the displacement components.

We have considered the plane of symmetry as the (x_2, x_3) plane and x_2 axis vertically downwards. For plane wave propagating in the (x_2, x_3) plane, we have

$$\frac{\partial}{\partial x_1} = 0 \tag{3a}$$

The equations of motion without body forces are

$$\tau_{ij,j} = \rho \ddot{u}_i \,, i = 1, 2, 3. \tag{3b}$$

Using Eqs. (1), (3a) and (3b), the equations of motion reduces to

$$\begin{aligned}&\left(C_{55}\frac{\partial^2 u_1}{\partial x_3^2} + 2C_{56}\frac{\partial^2 u_1}{\partial x_2 \partial x_3} + C_{66}\frac{\partial^2 u_1}{\partial x_2^2}\right) + \left\{C_{45}\frac{\partial^2 u_2}{\partial x_3^2} + \left(C_{46} + C_{25}\right)\frac{\partial^2 u_2}{\partial x_2 \partial x_3} + C_{26}\frac{\partial^2 u_2}{\partial x_2^2}\right\}\\ &+\left\{C_{35}\frac{\partial^2 u_3}{\partial x_3^2} + \left(C_{36} + C_{45}\right)\frac{\partial^2 u_3}{\partial x_2 \partial x_3} + C_{46}\frac{\partial^2 u_3}{\partial x_2^2}\right\} = \rho\frac{\partial^2 u_1}{\partial t^2},\end{aligned} \tag{4a}$$

$$\begin{aligned}&\left(C_{45}\frac{\partial^2 u_1}{\partial x_3^2} + \left(C_{46} + C_{25}\right)\frac{\partial^2 u_1}{\partial x_2 \partial x_3} + C_{26}\frac{\partial^2 u_1}{\partial x_2^2}\right) + \left\{C_{44}\frac{\partial^2 u_2}{\partial x_3^2} + 2C_{24}\frac{\partial^2 u_2}{\partial x_2 \partial x_3} + C_{22}\frac{\partial^2 u_2}{\partial x_2^2}\right\}\\ &+\left\{C_{34}\frac{\partial^2 u_3}{\partial x_3^2} + \left(C_{23} + C_{44}\right)\frac{\partial^2 u_3}{\partial x_2 \partial x_3} + C_{24}\frac{\partial^2 u_3}{\partial x_2^2}\right\} = \rho\frac{\partial^2 u_2}{\partial t^2},\end{aligned} \tag{4b}$$

$$\begin{aligned}&\left(C_{35}\frac{\partial^2 u_1}{\partial x_3^2} + \left(C_{45} + C_{36}\right)\frac{\partial^2 u_1}{\partial x_2 \partial x_3} + C_{46}\frac{\partial^2 u_1}{\partial x_2^2}\right) + \left\{C_{34}\frac{\partial^2 u_2}{\partial x_3^2} + \left(C_{23} + C_{44}\right)\frac{\partial^2 u_2}{\partial x_2 \partial x_3} + C_{24}\frac{\partial^2 u_2}{\partial x_2^2}\right\}\\ &+\left\{C_{33}\frac{\partial^2 u_3}{\partial x_3^2} + 2C_{34}\frac{\partial^2 u_3}{\partial x_2 \partial x_3} + C_{44}\frac{\partial^2 u_3}{\partial x_2^2}\right\} = \rho\frac{\partial^2 u_3}{\partial t^2}.\end{aligned} \tag{4c}$$

Let us assume

$$u_i = u_i\left(x_2, x_3, t\right), i = 1, 2, 3. \tag{5a}$$

Let $\vec{p}\left(0, p_2^{(n)}, p_3^{(n)}\right)$ denote the unit propagation vector, c_n is the phase velocity and k_n is the wave number of plane waves propagating in the (x_2, x_3) plane.

Consider plane wave solution of Eqs. (4a), (4b) and (4c) as:

$$\begin{pmatrix} u_1^{(n)} \\ u_2^{(n)} \\ u_3^{(n)} \end{pmatrix} = A_n \begin{pmatrix} d_1^{(n)} \\ d_2^{(n)} \\ d_3^{(n)} \end{pmatrix} \exp\left(i\,\overline{\eta}_n\right) \tag{5b}$$

where $\bar{d} = \left(d_1^{(n)}, d_2^{(n)}, d_3^{(n)}\right)$ and $\overline{\eta}_n = k_n\left(x_2 p_2^{(n)} + x_3 p_3^{(n)} - c_n t\right)$.

Substituting Eqs. (5a) and (5b) in Eqs. (4a), 4(b) and 4(c), we have

$$\left(S-\bar{c}^2\right)d_1^{(n)}+Td_2^{(n)}+Pd_3^{(n)}=0 \tag{6}$$

$$Td_1^{(n)}+\left(Q-\bar{c}^2\right)d_2^{(n)}+Rd_3^{(n)}=0 \tag{7}$$

$$Pd_1^{(n)}+Rd_2^{(n)}+\left(W-\bar{c}^2\right)d_3^{(n)}=0 \tag{8}$$

where $\bar{c}^2, P, Q, R, S, T$ and W are given in Appendix A.

From Eqs. (6), (7) and (8), we have

$$\frac{d_1^{(n)}}{D_1}=\frac{d_2^{(n)}}{D_2}=\frac{d_3^{(n)}}{D_3} \tag{9}$$

where

$$\begin{aligned} D_1 &= \left(Q-\bar{c}^2\right)\left(W-\bar{c}^2\right)-R^2, \\ D_2 &= PR-T\left(W-\bar{c}^2\right), \\ D_3 &= TR-P\left(Q-\bar{c}^2\right). \end{aligned} \tag{10}$$

Now $\bar{d}$ can be calculated in terms of $\bar{p}$ using the Eq. (9).

Eliminating $d_1^{(n)}, d_2^{(n)}$ and $d_3^{(n)}$ form Eqs. (6), (7) and (8), we have

$$\begin{vmatrix} \left(S-\bar{c}^2\right) & T & P \\ T & \left(Q-\bar{c}^2\right) & R \\ P & R & \left(W-\bar{c}^2\right) \end{vmatrix}=0$$

Solving the above determinant, we get

$$\begin{aligned} &\bar{c}^6-\left(S+W+Q\right)\bar{c}^4+\left(WQ+SQ+SW-R^2-T^2-P^2\right)\bar{c}^2+ \\ &\left(SR^2+WT^2+QP^2-2TPR-SWQ\right)=0 \\ &\text{or } \bar{c}^6+a_1\bar{c}^4+a_2\bar{c}^2+a_3=0 \end{aligned} \tag{11}$$

where

$$\begin{aligned} a_1 &= -\left(S+W+Q\right), \\ a_2 &= \left(WQ+SQ+SW-R^2-T^2-P^2\right), \\ a_3 &= \left(SR^2+WT^2+QP^2-2TPR-SWQ\right). \end{aligned} \tag{12}$$

Three real roots of $\bar{c}^2$ for *qP, qSV* and *qSH* may be obtained from Eq. (11). The largest root is for the phase velocity of quasi-*P* (*qP*) waves. The second largest root is for the phase velocity of quasi-*SV* (*qSV*) waves and the smallest root for the phase velocity of quasi-*SH* (*qSH*) waves.

The phase velocities are as follows [Chattopadhyay [3]]

$$\bar{c}_L^2 = \frac{\rho c_L^2}{\mu} = -2r\cos\left(\frac{\phi}{3}\right) - \frac{a_1}{3} \tag{13}$$

$$\bar{c}_{SV}^2 = \frac{\rho c_{SV}^2}{\mu} = 2r\cos\left(60 + \frac{\phi}{3}\right) - \frac{a_1}{3} \tag{14}$$

$$\bar{c}_{SH}^2 = \frac{\rho c_{SH}^2}{\mu} = 2r\cos\left(60 - \frac{\phi}{3}\right) - \frac{a_1}{3} \tag{15}$$

where

$$\begin{aligned} 2q &= \frac{2a_1^3}{27} - \frac{a_1 a_2}{3} + a_3, \\ 3p &= \frac{3a_2 - a_1^2}{3}, \\ r &= -\sqrt{|p|}, \\ \phi &= \cos^{-1}\left(\frac{q}{r^3}\right). \end{aligned} \tag{16}$$

Form Eqs. (14) and (15), it is clear that the phase velocities of quasi-transverse waves (*qSV* and *qSH*) are not identical in the case of triclinic medium.

For isotropic case

$$\begin{aligned} C_{11} &= C_{22} = C_{33} = \lambda' + 2\mu', \\ C_{12} &= C_{13} = C_{23} = \lambda', \\ C_{44} &= C_{55} = C_{66} = \mu' \end{aligned} \tag{17}$$

where λ' and μ' are Lame's constants for the isotropic medium, all other elastic constants are zero.

Substituting Eq. (17) in Eqs. (13), (14) and (15), we have

$$\begin{aligned} \bar{c}^2 &= \frac{\rho c_n^2}{C_{44}}, \bar{C}_{ij} = \frac{C_{ij}}{C_{44}}, \\ S &= \left\{p_3^{(n)}\right\}^2 + \left\{p_2^{(n)}\right\}^2, \quad T = 0, \quad P = 0, \\ Q &= \left\{p_3^{(n)}\right\}^2 + \left(\frac{\lambda' + 2\mu'}{\mu'}\right)\left\{p_2^{(n)}\right\}^2, \\ R &= \left(\frac{\lambda' + \mu'}{\mu'}\right) p_2^{(n)} p_3^{(n)}, \\ W &= \left(\frac{\lambda' + 2\mu'}{\mu'}\right)\left\{p_3^{(n)}\right\}^2 + \left\{p_2^{(n)}\right\}^2, \end{aligned}$$

The wave propagation is along x_3 axis, so there will be no components along x_2 axis.

Unit propagation vector becomes $\vec{p}(0,0,1)$, i.e. $p_2^{(n)} = 0, p_3^{(n)} = 1$

Using above values of *S, Q, R, W* in Eq. (12) and $\left\{p_3^{(n)}\right\}^2 + \left\{p_2^{(n)}\right\}^2 = 1$

We get

$$a_1 = -\left(\frac{\lambda' + 4\mu'}{\mu'}\right), a_2 = \left\{1 + 2\left(\frac{\lambda' + 2\mu'}{\mu'}\right)\right\}, a_3 = -\left(\frac{\lambda' + 2\mu'}{\mu'}\right).$$

and

$$q = -\left(\frac{\lambda' + \mu'}{3\mu'}\right)^3,$$
$$p = -\left(\frac{\lambda' + \mu'}{3\mu'}\right)^3,$$
$$r = -\left(\frac{\lambda' + \mu'}{\mu'}\right),$$
$$\phi = 0.$$

Substituting the above parameters in Eqs. (13), (14) and (15), we obtained the expression for the compressible wave velocity c_L and same value for the repeated roots for shear wave velocities (i.e. c_{SV} and c_{SH}) as:

$$\bar{c}_L^2 = \left(\frac{\lambda' + 2\mu'}{\rho'}\right),$$
$$\bar{c}_{SV}^2 = \left(\frac{\mu'}{\rho'}\right), \tag{18}$$
$$\bar{c}_{SH}^2 = \left(\frac{\mu'}{\rho'}\right).$$

3 SOLUTION OF THE PROBLEM

Consider X_3- axis along the interface of two half-spaces. The lower half-space is of triclinic nature and the upper half-space is isotropic homogeneous liquid medium. The elastic constants of the lower medium are given in Eq. (1) and the density is ρ occupying the region $X_2 \geq 0$ (lower medium). The homogenous liquid half-space with Lame's constants λ' and μ' (where $\mu' = 0$), density ρ' is occupying the region $X_2 \leq 0$ (upper medium). The X_2-axis is directed vertically downwards (Fig.1.) A quasi-*P* wave is incident on the interface at $X_2 = 0$ will generate reflected *qP*, reflected *qSV*, reflected *qSH* waves and also refracted *P* wave, as shown in Fig.1. Form Eqs. (4a), (4b) and (4c), it is inferred that all the displacement components are coupled for triclinic medium. Assuming *n*=0, 1, 2, 3, 4 for the incident *qP* wave, reflected *qP*, reflected *qSV*, reflected *qSH*, refracted *P* waves respectively. The angle made by incident *qP* wave, reflected *qP*, reflected *qSV*, reflected *qSH*, refracted *P* waves with the normal to the interface are θ_0, θ_1, θ_2, θ_3 and θ_4 respectively.

For incident *qP* wave (*n*=0): $p_2^{(0)} = -\cos\theta_0, p_3^{(0)} = \sin\theta_0, c_0 = c_L$.

For reflected *qP* wave (*n*=1): $p_2^{(1)} = \cos\theta_1, p_3^{(1)} = \sin\theta_1, c_1 = c_{L1}$.

For reflected *qSV* wave (*n*=2): $p_2^{(2)} = \cos\theta_2, p_3^{(2)} = \sin\theta_2, c_2 = c_T$.

For reflected *qSH* wave (*n*=3): $p_2^{(3)} = \cos\theta_3, p_3^{(3)} = \sin\theta_3, c_3 = c_{T1}$.

For refracted *P* wave (*n*=4): $d_2^{(4)} = -\cos\theta_4, d_3^{(4)} = \sin\theta_4,\ p_2^{(4)} = -\cos\theta_4,\ p_3^{(4)} = \sin\theta_4,\ c_4 = c_L{}' = \sqrt{\dfrac{\lambda'}{\rho'}}$.

where c_L, c_{L1}, c_T, c_{T1} and $c_L{}'$ are the phase velocities of the incident *qP*, reflected *qP*, reflected *qSV*, reflected *qSH*, refracted *P* waves, respectively.

In the plane $X_2 = 0$, the displacements and stresses of incident, reflected waves may be expressed as follows:

$$
\begin{aligned}
u_j^{(n)} &= A_n d_j^{(n)} \exp(i\,\overline{\eta}_n), \quad \text{for } j=1,2,3. \\
\tau_{12}^{(n)} &= P_n i k_n A_n \exp(i\,\overline{\eta}_n), \\
\tau_{22}^{(n)} &= Q_n i k_n A_n \exp(i\,\overline{\eta}_n), \\
\tau_{23}^{(n)} &= R_n i k_n A_n \exp(i\,\overline{\eta}_n).
\end{aligned}
\tag{19}
$$

where

$$
\overline{\eta}_n = k_n \left(x_2 \mathrm{p}_2^{(\mathrm{n})} + x_3 \mathrm{p}_3^{(\mathrm{n})} - c_n t \right), \ \text{for } n=0,1,2,3. \tag{20}
$$

Thus for $\mathrm{n} = 0, 1, 2, 3$.

$$
P_n = C_{26} p_2^{(n)} d_2^{(n)} + C_{36} p_3^{(n)} d_3^{(n)} + C_{46} \left\{ p_3^{(n)} d_2^{(n)} + p_2^{(n)} d_3^{(n)} \right\} + C_{56} p_3^{(n)} d_1^{(n)} + C_{66} p_2^{(n)} d_1^{(n)}, \tag{21}
$$

$$
Q_n = C_{22} p_2^{(n)} d_2^{(n)} + C_{23} p_3^{(n)} d_3^{(n)} + C_{24} \left\{ p_3^{(n)} d_2^{(n)} + p_2^{(n)} d_3^{(n)} \right\} + C_{25} p_3^{(n)} d_1^{(n)} + C_{26} p_2^{(n)} d_1^{(n)}, \tag{22}
$$

$$
R_n = C_{24} p_2^{(n)} d_2^{(n)} + C_{34} p_3^{(n)} d_3^{(n)} + C_{44} \left\{ p_3^{(n)} d_2^{(n)} + p_2^{(n)} d_3^{(n)} \right\} + C_{45} p_3^{(n)} d_1^{(n)} + C_{46} p_2^{(n)} d_1^{(n)}, \tag{23}
$$

For the refracted waves in liquid medium we have $\mu' = 0$. In the plane $X_2 = 0$, the displacements and stresses of the refracted waves may be expressed as follows:

$$
\begin{aligned}
u_1^{(n)} &= 0, \\
u_j^{(n)} &= A_n d_j^{(n)} \exp(i\,\overline{\eta}_n), \ \text{for j} = 2,3.
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\tau_{12}^{(n)} &= 0, \\
\tau_{22}^{(n)} &= Q_n i k_n A_n \exp(i\,\overline{\eta}_n), \\
\tau_{23}^{(n)} &= 0.
\end{aligned}
\tag{25}
$$

and

$$
\begin{aligned}
P_n &= 0, \\
Q_n &= \lambda' \left(p_2^{(n)} d_2^{(n)} + p_3^{(n)} d_3^{(n)} \right), \\
R_n &= 0.
\end{aligned}
\tag{26}
$$

where,

$$
\overline{\eta}_n = k_n \left(x_2 \mathrm{p}_2^{(\mathrm{n})} + x_3 \mathrm{p}_3^{(\mathrm{n})} - c_n t \right), \ \text{for } n=4.
$$

4 BOUNDARY CONDITIONS

The boundary conditions at the interface $\left(x_2 = 0 \right)$ are

$$
u_2^{(0)} + u_2^{(1)} + u_2^{(2)} + u_2^{(3)} = u_2^{(4)}, \tag{27}
$$

$$u_3^{(0)}+u_3^{(1)}+u_3^{(2)}+u_3^{(3)}=u_3^{(4)}, \tag{28}$$

$$\tau_{22}^{(0)}+\tau_{22}^{(1)}+\tau_{22}^{(2)}+\tau_{22}^{(3)}=\tau_{22}^{(4)}, \tag{29}$$

$$\tau_{23}^{(0)}+\tau_{23}^{(1)}+\tau_{23}^{(2)}+\tau_{23}^{(3)}=0. \tag{30}$$

From the boundary conditions (27) to (30) and Eqs. (19) to (26), we get

$$A_0 d_2^{(0)}\exp(i\bar{\eta}_0)+A_1 d_2^{(1)}\exp(i\bar{\eta}_1)+A_2 d_2^{(2)}\exp(i\bar{\eta}_2)+A_3 d_2^{(3)}\exp(i\bar{\eta}_3)=A_4 d_2^{(4)}\exp(i\bar{\eta}_4), \tag{31}$$

$$A_0 d_3^{(0)}\exp(i\bar{\eta}_0)+A_1 d_3^{(1)}\exp(i\bar{\eta}_1)+A_2 d_3^{(2)}\exp(i\bar{\eta}_2)+A_3 d_3^{(3)}\exp(i\bar{\eta}_3)=A_4 d_3^{(4)}\exp(i\bar{\eta}_4), \tag{32}$$

$$Q_0 k_0 A_0\exp(i\bar{\eta}_0)+Q_1 k_1 A_1\exp(i\bar{\eta}_1)+Q_2 k_2 A_2\exp(i\bar{\eta}_2)+Q_3 k_3 A_3\exp(i\bar{\eta}_3)=Q_4 k_4 A_4\exp(i\bar{\eta}_4), \tag{33}$$

and

$$R_0 k_0 A_0\exp(i\bar{\eta}_0)+R_1 k_1 A_1\exp(i\bar{\eta}_1)+R_2 k_2 A_2\exp(i\bar{\eta}_2)+R_3 k_3 A_3\exp(i\bar{\eta}_3)=0. \tag{34}$$

The above equations are valid for all values of X_3 and t, therefore we have

$$\bar{\eta}_0=\bar{\eta}_1=\bar{\eta}_2=\bar{\eta}_3=\bar{\eta}_4.$$

or we can write

$$\begin{aligned}&k_0\left(-x_2\cos\theta_0+x_3\sin\theta_0-c_L t\right)=k_1\left(x_2\cos\theta_1+x_3\sin\theta_1-c_{L1}t\right)=\\&k_2\left(x_2\cos\theta_2+x_3\sin\theta_2-c_T t\right)=k_3\left(x_2\cos\theta_3+x_3\sin\theta_3-c_{T1}t\right)=\\&k_4\left(-x_2\cos\theta_4+x_3\sin\theta_4-c_L' t\right).\end{aligned} \tag{35a}$$

At the interface we get from Eq. (35a)

$$\begin{aligned}&k_0\left(x_3\sin\theta_0-c_L t\right)=k_1\left(x_3\sin\theta_1-c_{L1}t\right)=k_2\left(x_3\sin\theta_2-c_T t\right)=k_3\left(x_3\sin\theta_3-c_{T1}t\right)=\\&k_4\left(x_3\sin\theta_4-c_L' t\right)\end{aligned} \tag{35b}$$

which gives

$$k_0 c_L=k_1 c_{L1}=k_2 c_T=k_3 c_{T1}=k_4 c_L'=\kappa, \tag{36}$$

and

$$k_0\sin\theta_0=k_1\sin\theta_1=k_2\sin\theta_2=k_3\sin\theta_3=k_4\sin\theta_4=\omega \tag{37}$$

where k and ω are the apparent wave number and circular frequency, respectively. Displacement and traction are continuous along the interface $x_2=0$. We must relax the condition for continuity of displacement for qSH waves as the upper medium has no effect of SH wave. The only boundary condition of the traction may be considered. This traction condition leads to A_3/A_0 form Eq. (34), as:

$$\frac{A_3}{A_0}=\frac{-1}{e_3}\left[1+e_1\frac{A_1}{A_0}+e_2\frac{A_2}{A_0}\right] \tag{38}$$

where

$$e_i = \frac{\bar{R}_i k_i}{\bar{R}_0 k_0}, \bar{R}_i = \frac{R_i}{C_{44}}, \text{ for } i = 1,2,3. \tag{39}$$

Now substituting Eq. (38) in Eqs. (31), (32) and (33) we get

$$\left[1 - \frac{1}{e_3}\frac{d_2^{(3)}}{d_2^{(0)}}\right] + \frac{A_1}{A_0}\left[\frac{d_2^{(1)}}{d_2^{(0)}} - \frac{e_1}{e_3}\frac{d_2^{(3)}}{d_2^{(0)}}\right] + \frac{A_2}{A_0}\left[\frac{d_2^{(2)}}{d_2^{(0)}} - \frac{e_2}{e_3}\frac{d_2^{(3)}}{d_2^{(0)}}\right] = \frac{A_4}{A_0}\frac{d_2^{(4)}}{d_2^{(0)}} \tag{40}$$

$$\left[1 - \frac{1}{e_3}\frac{d_3^{(3)}}{d_3^{(0)}}\right] + \frac{A_1}{A_0}\left[\frac{d_3^{(1)}}{d_3^{(0)}} - \frac{e_1}{e_3}\frac{d_3^{(3)}}{d_3^{(0)}}\right] + \frac{A_2}{A_0}\left[\frac{d_3^{(2)}}{d_3^{(0)}} - \frac{e_2}{e_3}\frac{d_3^{(3)}}{d_3^{(0)}}\right] = \frac{A_4}{A_0}\frac{d_3^{(4)}}{d_3^{(0)}} \tag{41}$$

and

$$\left[1 - \frac{f_3}{e_3}\right] + \frac{A_1}{A_0}\left[f_1 - f_3\frac{e_1}{e_3}\right] + \frac{A_2}{A_0}\left[f_2 - f_3\frac{e_2}{e_3}\right] = f_4\frac{A_4}{A_0}. \tag{42}$$

where $f_i = \frac{\bar{Q}_i k_i}{\bar{Q}_0 k_0}, \bar{Q}_i = \frac{Q_i}{C_{44}}$, for i=1,2,3,4, and the values of e_i can be obtained from Eq. (39).

The amplitude ratios of reflected *qP, qSV* and *qSV* waves and refracted (transmitted) *P* waves are denoted by $\frac{A_1}{A_0}, \frac{A_2}{A_0}, \frac{A_3}{A_0}$ and $\frac{A_4}{A_0}\cdot$ Solving the Eqs. (40) to (42), we get

$$\frac{A_i}{A_0} = \frac{D_i}{D_0}, (i = 1,2,4) \tag{43}$$

where

$$D_0 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{44}$$

and the determinant $D_i \left(i = 1,2,4\right)$ can be obtained by replacing all three elements of the i^{th} column of D_0 by a_0, b_0 and c_0 respectively.

The values of a_i, b_i, c_i (for $i = 0,1,2,3$) $e_i, \bar{R}_i$ (for $i = 1,2,3$) and $f_i, \bar{Q}_i$ (for $i = 1,2,3,4$) are given in Appendix B.

The amplitude ratio A_3 / A_0 (for the reflected *SH* wave) may be obtained from Eq. (38).

5 PARTICULAR CASES

Case 1

When the lower medium is isotropic, the elastic parameters are taken as λ and μ and the density is ρ. For the isotropic medium the polarization vectors and phase velocities are given in the following Table 1.

Table 1
Polarization vectors and phase velocities of incident and reflected waves.

Wave type	Polarization vector	Phase velocity
For incident *P* wave (*n*=0)	$d_1^{(0)}=0, d_2^{(0)}=-\cos\theta_0, d_3^{(0)}=\sin\theta_0$	$c_0=c_L=\sqrt{\frac{\lambda+2\mu}{\rho}}$
For reflected *P* wave (*n*=1)	$d_1^{(1)}=0, d_2^{(1)}=\cos\theta_1, d_3^{(1)}=\sin\theta_1$	$c_1=c_{L1}=\sqrt{\frac{\lambda+2\mu}{\rho}}$
For reflected *SV* wave (*n*=2)	$d_1^{(2)}=0, d_2^{(2)}=\cos\theta_2, d_3^{(2)}=\sin\theta_2$	$c_2=c_T=\sqrt{\frac{\mu}{\rho}}$
For reflected *SH* waves (*n*=3)	$d_1^{(3)}=0, d_2^{(3)}=\cos\theta_3, d_3^{(3)}=\sin\theta_3$	$c_3=c_{T1}=\sqrt{\frac{\mu}{\rho}}$

Hence from Eqs. (36) and (37), we get $k_0=k_1, k_2=k_3$ and $\theta_0=\theta_1, \theta_2=\theta_3$.

For incident *qP* wave (*n*=0): $p_2^{(0)}=-\cos\theta_0, p_3^{(0)}=\sin\theta_0, c_0=c_L$.

For reflected *qP* wave (*n*=1): $p_2^{(1)}=\cos\theta_1, p_3^{(1)}=\sin\theta_1, c_1=c_{L1}$.

For reflected *qSV* wave (*n*=2): $p_2^{(2)}=\cos\theta_2, p_3^{(2)}=\sin\theta_2, c_2=c_T$.

For reflected *qSH* wave (*n*=3): $p_2^{(3)}=\cos\theta_3, p_3^{(3)}=\sin\theta_3, c_3=c_{T1}$.

For refracted *P* wave (*n*=4): $d_2^{(4)}=-\cos\theta_4, d_3^{(4)}=\sin\theta_4, p_2^{(4)}=-\cos\theta_4, p_3^{(4)}=\sin\theta_4, c_4=c_L{}'=\sqrt{\frac{\lambda'}{\rho'}}$.

In this case, *qSH* and *qSV* waves in the lower medium coincide with *SV* wave. Thus incident wave will generate reflected *P* wave, reflected *SV* wave and refracted *P* wave only. So, by using Eqs. (40), (41) and (42) and above values of $p_1^{(i)}, p_2^{(i)}, p_3^{(i)}$ and $d_1^{(i)}, d_2^{(i)}, d_3^{(i)}$ for *i*=0,1,2,3,4 and considering $\bar{Q}_4=Q_4$, for liquid medium, we get the following:

$$\begin{aligned}
&(A_0-A_1)\left\{\cos\theta_0-\frac{c_{T1}}{c_L}\cos\theta_3\frac{\sin 2\theta_0}{\sin 2\theta_3}\right\}-A_4\cos\theta_4=0,\\
&(A_0+A_1)\left\{\sin\theta_0-\frac{c_{T1}}{c_L}\sin\theta_3\frac{\sin 2\theta_0}{\sin 2\theta_3}\right\}-A_4\sin\theta_4=0,\\
&\left[1+\frac{\sin 2\theta_0}{\sin 2\theta_3}\frac{\lambda+2\mu\cos^2\theta_3}{\lambda+2\mu\cos^2\theta_0}\right]A_0+A_1\left[1-\frac{\sin 2\theta_0}{\sin 2\theta_3}\frac{\lambda+2\mu\cos^2\theta_3}{\lambda+2\mu\cos^2\theta_0}\right]-\frac{c_L}{c_L'}\frac{\lambda'\mu}{\lambda+2\mu\cos^2\theta_0}A_4=0.
\end{aligned} \tag{45}$$

where

$$e_i=\frac{\bar{R}_i k_i}{\bar{R}_0 k_0}, \bar{R}_i=\frac{R_i}{C_{44}}, \text{ for } i=1,2,3, f_i=\frac{\bar{Q}_i k_i}{\bar{Q}_0 k_0}, \bar{Q}_i=\frac{Q_i}{C_{44}}, \text{ for } i=1,2,3,4 \text{ and } \bar{R}_0=\frac{R_0}{C_{44}}, \bar{Q}_0=\frac{Q_0}{C_{44}}.$$

Eq. (45) may be written as:

$$\begin{bmatrix}
-\left(\cos\theta_0-\frac{c_{T1}}{c_L}\cos\theta_3\frac{\sin 2\theta_0}{\sin 2\theta_3}\right) & 0 & -\cos\theta_4\\
\left(\sin\theta_0-\frac{c_{T1}}{c_L}\sin\theta_3\frac{\sin 2\theta_0}{\sin 2\theta_3}\right) & 0 & -\sin\theta_4\\
\left(1-\frac{\sin 2\theta_0}{\sin 2\theta_3}\frac{\lambda+2\mu\cos^2\theta_3}{\lambda+2\mu\cos^2\theta_0}\right) & 0 & -\frac{c_L}{c_L'}\frac{\lambda'\mu}{\lambda+2\mu\cos^2\theta_0}
\end{bmatrix}
\begin{bmatrix} A_1\\ A_2\\ A_4\end{bmatrix}=A_0
\begin{bmatrix}
-\left(\cos\theta_0-\frac{c_{T1}}{c_L}\cos\theta_3\frac{\sin 2\theta_0}{\sin 2\theta_3}\right)\\
-\left(\sin\theta_0-\frac{c_{T1}}{c_L}\sin\theta_3\frac{\sin 2\theta_0}{\sin 2\theta_3}\right)\\
-\left(1+\frac{\sin 2\theta_0}{\sin 2\theta_3}\frac{\lambda+2\mu\cos^2\theta_3}{\lambda+2\mu\cos^2\theta_0}\right)
\end{bmatrix} \tag{46}$$

Case 2

Consider the case of normal incidence, which is defined by $\theta_0 = 0.$

Thus from Eq. (46), we get

$$\begin{bmatrix} -1 & 0 & -\cos\theta_4 \\ 0 & 0 & -\sin\theta_4 \\ 1 & 0 & -\dfrac{\lambda'\mu}{\lambda+2\mu}\dfrac{c_L}{c_L'} \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = A_0\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \qquad A_2 = 0.$$

Then from the remaining equations we get

$$\frac{A_4}{A_0} = \frac{2}{\cos\theta_4 + \dfrac{\lambda'\mu}{\lambda+2\mu}\dfrac{c_L}{c_L'}}, \quad \frac{A_1}{A_0} = \frac{\dfrac{\lambda'\mu}{\lambda+2\mu}\dfrac{c_L}{c_L'} - \cos\theta_4}{\cos\theta_4 + \dfrac{\lambda'\mu}{\lambda+2\mu}\dfrac{c_L}{c_L'}}, \quad \frac{A_3}{A_0} = -\left[1 + e_1\frac{A_1}{A_0} + e_2\frac{A_2}{A_0}\right]\Big/ e_3 . \tag{47}$$

6 NUMERICAL EXAMPLES AND DISCUSSION

Numerical calculations have been performed using data of Vosges sandstone material which exhibits triclinic anisotropy [1]. The elastic constants of Vosges sandstone are:

C_{11} = 16.248 *GPa*, C_{22} = 11.88 *GPa*, C_{33} = 12.216 *GPa*, C_{12} = 1.48 *GPa*, C_{13} = 2.4 *GPa*, C_{14} = −1.152 *GPa*, C_{15} = 0.0 *GPa*, C_{16} = −0.561 *GPa*, C_{23} = 1.032 *GPa*, C_{24} = 0.912 *GPa*, C_{25} = 1.608 *GPa*, C_{26} = 1.248 *GPa*, C_{34} = −0.6724 *GPa*, C_{35} = 0.216 *GPa*, C_{36} = −0.216 *GPa*, C_{44} = 5.64 *GPa*, C_{45} = 2.16 *GPa*, C_{46} = 0.0 *GPa*, C_{55} = 5.88 *GPa*, C_{66} = 6.912 *GPa*, C_{56} = 0.0 *GPa*, ρ = 2.40 g/cm^3.

The upper layer is isotropic homogeneous liquid medium having Lame's constants $\lambda = 1.5,\ \mu = 0$ and $\rho = 1.23\ g/cm^3$. The amplitude ratios for the reflected *qP, qSV, qSH* waves and refracted *qP* wave have been computed and depicted by means of graphs. Figs. 2 to 4 represent the variation in reflection coefficients of *qP, qSV* and *qSH* waves with respect to incident angle ranging from 0° to 60°, whereas the variation in refraction coefficient with respect to incident angle is represented by Fig. 5.

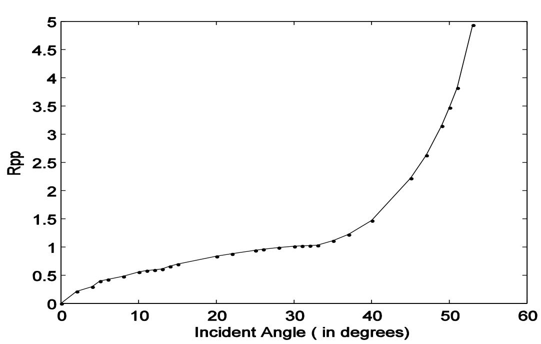

Fig.2
Variation of A_1/A_0 with the angle of incidence due to incident *qP* waves.

Fig. 2 shows the variation of the reflection coefficient ($R_{PP} = A_1/A_0$) of an incident *qP* wave reflected as another *qP* wave. The coefficient *R*pp increase with increment in incidence angle. A significant increment in *Rpp* can be observed when the angle increases from 35^0 to 52^0.

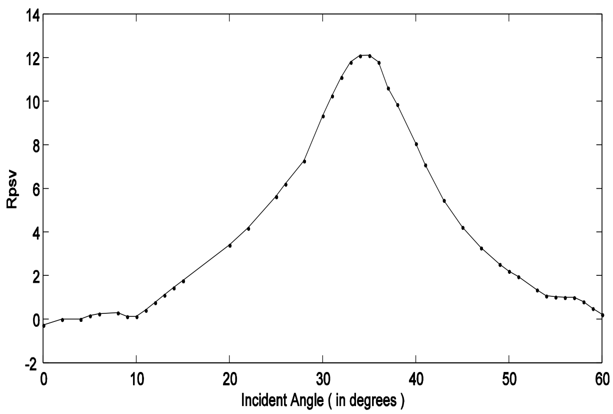

Fig.3
Variation of A_2 / A_0 with the angle of incidence due to incident *qP* waves.

Fig.3 represents the variation of the reflection coefficient ($R_{PSV} = A_2 / A_0$) of an incident *qP* wave reflected as *qSV* wave. The coefficient R_{PSV} increases for the incidence angle from 0^0 to 35^0 but decreases for 35^0 and onwards. Particularly the increment from 11^0 to 35^0 is very rapid in compare to the initial increment. The incident angle 35^0 may be considered as the critical point of the graph as the nature of curve reverses after this value.

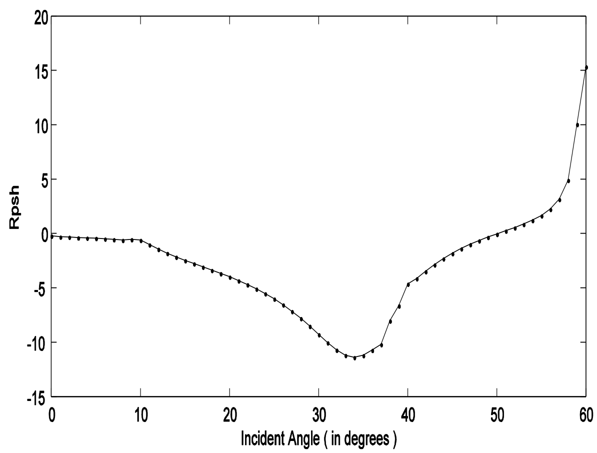

Fig.4
Variation of A_3 / A_0 with the angle of incidence due to incident *qP* waves.

Fig. 4 shows the variation of the reflection coefficient ($R_{PSH} = A_3 / A_0$) of an incident *qP* wave reflected as *qSH* wave. Very small change in R_{PSH} can be observed when the angle of incidence (θ) lies between 0^0 and 10^0. The value of R_{PSH} decreases continuously from 11^0 to 34^0 and then increases sharply from 35^0 to 40^0. R_{PSH} increases gradually from 40^0 to 57^0 but after 57^0 a rapid increase can be observed.

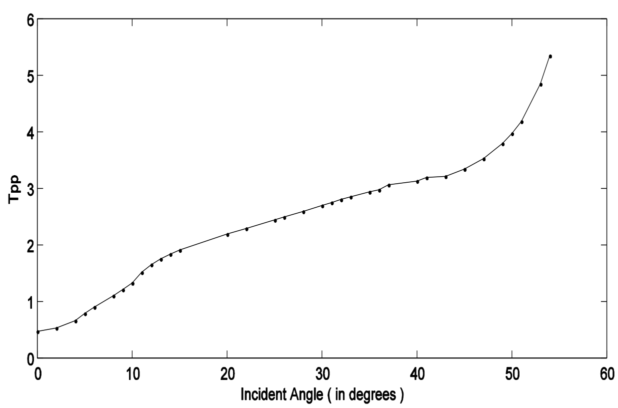

Fig.5
Variation of A_4 / A_0 with the angle of incidence due to incident *qP* waves.

Fig. 5 shows the variation of the refraction coefficient ($T_{PP} = A_4 / A_0$) of an incident *qP* wave refracted as *qP* wave. The values of the refracted coefficient *Tpp* increases steadily as the angles of incidence (θ) lie between 0^0 and 13^0. The values increase sharply from 13^0 to 15^0. The refracted coefficient *Tpp* increases uniformly as the incidence angle increases between 15^0 and 43^0 but from 43^0 and onwards the values increases at a high rate.

7 CONCLUSIONS

The reflection and refraction phenomenon of plane wave at the interface of an isotropic liquid medium and a triclinic (anisotropic) half-space has been studied. Phase velocities have been obtained for all three waves generated by incident plane *qP* wave. Expressions for reflection coefficients of *qP, qSV, qSH* waves and refracted *qP* wave have been obtained. More precisely the outcomes of the present study may be concluded as:

At the smooth interface between triclinic and homogeneous media, the displacement components and stress components, aligned to the direction of wave propagation are consistent.

The amplitude ratios of reflected *qP* and refracted *P* waves have incremental variation with respect to incident angle.

The variation in amplitude ratios of reflected *qSV* wave and reflected *qSH* wave are almost opposite to each other. In particular the amplitude ratio of reflected *qSV* waves increases up to a certain value of incident angle and decreases onward, whereas the amplitude ratio of reflected *qSH* waves decreases initially and then rises afterwards.

APPENDIX A

$$\bar{c}^2 = \frac{\rho c_n^2}{C_{44}}, \bar{C}_{ij} = \frac{C_{ij}}{C_{44}},$$

$$P = \bar{C}_{35}\left\{p_3^{(n)}\right\}^2 + \left(\bar{C}_{36} + \bar{C}_{45}\right) p_2^{(n)} p_3^{(n)} + \bar{C}_{46}\left\{p_2^{(n)}\right\}^2,$$

$$Q = \bar{C}_{44}\left\{p_3^{(n)}\right\}^2 + 2\bar{C}_{24} p_2^{(n)} p_3^{(n)} + \bar{C}_{22}\left\{p_2^{(n)}\right\}^2,$$

$$R = \bar{C}_{34}\left\{p_3^{(n)}\right\}^2 + \left(\bar{C}_{23} + \bar{C}_{44}\right) p_2^{(n)} p_3^{(n)} + \bar{C}_{24}\left\{p_2^{(n)}\right\}^2,$$

$$S = \bar{C}_{55}\left\{p_3^{(n)}\right\}^2 + 2\bar{C}_{56} p_2^{(n)} p_3^{(n)} + \bar{C}_{66}\left\{p_2^{(n)}\right\}^2,$$

$$T = \bar{C}_{45}\left\{p_3^{(n)}\right\}^2 + \left(\bar{C}_{46} + \bar{C}_{25}\right) p_2^{(n)} p_3^{(n)} + \bar{C}_{26}\left\{p_2^{(n)}\right\}^2,$$

$$W = \bar{C}_{33}\left\{p_3^{(n)}\right\}^2 + 2\bar{C}_{34} p_2^{(n)} p_3^{(n)} + \bar{C}_{44}\left\{p_2^{(n)}\right\}^2.$$

APPENDIX B

$$a_0 = -\left(1 - \frac{1}{e_3}\frac{d_2^{(3)}}{d_2^{(0)}}\right),\quad a_1 = \left(\frac{d_2^{(1)}}{d_2^{(0)}} - \frac{e_1}{e_3}\frac{d_2^{(3)}}{d_2^{(0)}}\right),\quad a_2 = \left(\frac{d_2^{(2)}}{d_2^{(0)}} - \frac{e_2}{e_3}\frac{d_2^{(3)}}{d_2^{(0)}}\right),\quad a_3 = -\frac{d_2^{(4)}}{d_2^{(0)}},$$

$$b_0 = -\left(1 - \frac{1}{e_3}\frac{d_3^{(3)}}{d_3^{(0)}}\right),\quad b_1 = \left(\frac{d_3^{(1)}}{d_3^{(0)}} - \frac{e_1}{e_3}\frac{d_3^{(3)}}{d_3^{(0)}}\right),\quad b_2 = \left(\frac{d_3^{(2)}}{d_3^{(0)}} - \frac{e_2}{e_3}\frac{d_3^{(3)}}{d_3^{(0)}}\right),\quad b_3 = -\frac{d_3^{(4)}}{d_3^{(0)}},$$

$$c_0 = -\left(1 - \frac{f_3}{e_3}\right),\quad c_1 = \left(f_1 - f_3\frac{e_1}{e_3}\right), c_2 = \left(f_2 - f_3\frac{e_2}{e_3}\right) \text{ and } c_3 = -f_4.$$

$$e_i = \frac{\bar{R}_i k_i}{\bar{R}_0 k_0}, \bar{R}_i = \frac{R_i}{C_{44}}, f_i = \frac{\bar{Q}_i k_i}{\bar{Q}_0 k_0}, \bar{Q}_i = \frac{Q_i}{C_{44}} \qquad for \qquad i = 1, 2, 3, 4.$$

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