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Analysis of Nonlinear Vibrations of Slightly Curved Tripled-Walled Carbon Nanotubes Resting on Elastic Foundations in a Magneto-Thermal Environment

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ABSTRACT

In this work, nonlocal elasticity theory is applied to analyze nonlinear free vibrations of slightly curved multi-walled carbon nanotubes resting on nonlinear Winkler and Pasternak foundations in a thermal and magnetic environment. With the aid of Galerkin decomposition method, the systems of nonlinear partial differential equations are transformed into systems of nonlinear ordinary differential equations that are solved using homotopic perturbation method. The influences of elastic foundations, magnetic field, temperature rise, interlayer forces, small scale parameter and boundary conditions on the frequency ratio are investigated. It is observed form the results that the frequency ratio for all boundary conditions decreases as the number of walls increases. In addition, it is established that the frequency ratio is highest for clamped-simple supported and lowest for clamped-clamped supported. Further investigations on the controlling parameters of the phenomena reveal that the frequency ratio decreases with increase in the value of spring constant (k_1) temperature and magnetic field strength. It is hoped that this work will enhance the applications of carbon nanotubes in structural, electrical, mechanical and biological applications especially in a thermal and magnetic environment.

Keywords: Multi-walled carbon nanotubes; Magnetic and thermal environment; Nonlocal elastic theory; Small-scale effects; Winkler and Pasternak foundations.

1 INTRODUCTION

IIJIMA [1] discovered the novel nanostructure materials and due to their promising applications of the nanomaterials in Nano devices, Nano electronics, and nanocomposites, this novel discovery has led to

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considerable number of studies on carbon nanotubes (CNTs). In fact, it could be said that the excellent mechanical, electrical, structural and thermal properties coupled with high strength to weight ratio property of carbon nanotubes have continuously and tremendously expanded their applications in various industrial, engineering, physical and natural sciences processes. Indisputably, the nanostructures have merits when applied to the function ability of transistors and diodes. However, carbon nanotubes are capable of undergoing large deformations within the elastic limit and vibrate at frequency in the order of GHz and THz. Consequently, logical investigations and analysis of carbon nanotube have been a subject of interest such as the vibrations of a micro-resonator that is excited by electrostatic and piezoelectric actuations. Various studies have been carried out on beams, CNT, nano-wires, nano-rods and nano-beam so as to specifically understand and achieve their area of best fit [2-13]. In achieving this, the well know beam models were employed and dynamic ranges were obtained in the scope of the structures. In such studies, Liew et al. [5], Pantano et al. [6,7], Qian et al. [8] and Salvetat et al. [9] examined the mechanics of single and multiwalled carbon nanotubes. Sears and Batra [10] analyzed carbon nanotubes buckling under the influence of axial compression. Yoon et al. [11] and Wang and Cai [12] investigated the impacts of initial stress on multiwall carbon nanotube with a focus on non-coaxial resonance. Wang et al. [13] explored the dynamic response of multi-walled carbon nanotubes using Timoshenko beam model. Zhang et al. [14] scrutinized the influence of compressive axial load on the transverse dynamic behaviour of double-walled carbon nanotubes. Elishakoff and Pentaras [15] presented another study on the vibration of double-walled carbon nanotubes. In addition, studies on nonlinear vibration of nanomechanical resonator, nanotube and nanowire-based electromechanical systems have been carried out by Buks and Yurke [16] and Postma et al. [17] while Fu et al. [18] examined nonlinear vibration analysis of embedded carbon nanotubes. In the same year, Xu et al. [19] considered the dynamic response of a double-walled carbon nanotube under the influence of nonlinear inter-tube van der Waals forces. The vibration of carbon nanotube-based switches with focus on static and dynamic responses was analyzed by Dequesnes et al. [20]. Few years later, Ouakad and Younis [21] investigated the nonlinear vibration of electrically actuated carbon nanotube resonators. In an earlier work, Zamanian et al. [22] presented the non-linear vibrations analysis of a micro resonator subjected to piezoelectric and electrostatic actuations. As a continuation of the tremendous work, Abdel-Rahman, Hawwa, Hajnayeb, and Belhadj [23-26] performed a vibration and instability studies of a DWCNT using a nonlinear model and considering an electrostatic actuation as an external excitation agent. In their work, a DWCNT was situated and conditioned to a direct and alternating voltage and different behaviors of the nanotubes were recorded as the exciting agent is varied. They went further to determine the bifurcation point of the DWCNT and concluded that both walls have the same frequency of vibration under the two resonant conditions considered. Belhadj et al. [26] carried out the vibration analysis of a pinned-pinned supported SWCNT employing nonlocal theory of elasticity and obtained natural frequency up to third mode. The authors also put forward an explanation on the advantages of the high frequency obtained in their work to optical applications. Lei et al. [27] studied the dynamic behaviour of DWCNT by employing the well-known Timoshenko theory of beam. The nonlinear governing equations generated by Sharabiani and Yazdi [28] derived relations in the application to Nano beams that are graded and have surface roughness. Wang [29] generated a close form model for the aforementioned surface roughness effect for an unforced fluid conveying nanotubes and beams based on nonlocal theory of elasticity and ascertained the significance of the study for reasonably small thickness of the tube considered. Interesting foundation studies have been considered after modelling of CNTs as structures resting or embedded on elastic foundations such as Winkler, Pasternak and Visco-Pasternak medium [30-35]. Other interesting works through modelling and experiment have also been presented to justify the widespread application of SWCNTs [36-41]. However, the development of internal non-coaxial deformation and distort due to van der Wall forces in the MWCNTs has been considered [42,43]. In another study, Ansari and Hemmatnezhad [44] applied variational approach to analyze nonlinear vibrations of embedded MWCNTs while Ghorbanpour et al. [45] utilized averaging method to theoretically investigate the nonlinear vibrations of MWCNTs embedded in an elastic medium. Yoon et al. [46] explored the resonant frequencies and the modes of vibration of embedded in a Winkler elastic medium. Fu et al. [47] adopted continuum mechanics to study the nonlinear free vibration of embedded MWCNTs. With the aids of Timoshenko beam model and differential quadrature method, Wang et al. [48] presented the analysis of a free vibration of MWCNTs. Aydogdu [49] presented the free vibration of MWCNTs applying generalized shear deformation-beam theory.

The dynamic behaviour of SWCNTs, DWCNTs and MWCNTs have been characterized and their dynamic behaviour have been investigated with the aids of experimental measurements, density functional theory, molecular dynamics simulations, and continuum mechanics. However, there are difficulties in performing experiment at the nanoscale level. Consequently, over the years, the classical continuum models (which do not consider the small-scale effects) have been widely applied to the small-scale structures as reviewed in the preceding section. The demerit of such classical continuum theories is witnessed in their scale-free models, as they cannot incorporate the small-scale effects in their formulations. Therefore, in other to correct the inadequacy in the classical continuum models, Eringen [50-53] developed nonlocal continuum mechanics based on nonlocal elasticity theory. The nonlocal elasticity theory considers the stress state at a given point to be a function of the strain field at all points in the body. Therefore, in this work, nonlocal elasticity theory is used to analyze nonlinear vibrations of slightly curved multi-walled resting on nonlinear Winkler and Pasternak foundation in a thermal and magnetic environment. With the aid of van der Waals interlayer interaction, the nested slender multi-walled nanotubes are coupled with each other. Such study on the simultaneous influences of thermal and magnetic field, Winkler and Pasternak foundations on the vibration of multi-walled carbon nanotubes using nonlocal elasticity theory has not been presented in literature. Additionally, the development of analytical expressions for the frequencies, frequency ratio and deflections of the multi-walled carbon nanotubes is shown to be another novel idea of the present study. The analytical solutions are used to investigate the influences of elastic foundations, magnetic field, temperature rise, interlayer forces, small-scale parameter and boundary conditions on the frequency ratio.

2 PROBLEM DESCRIPTION AND THE GOVERNING EQUATIONS

Consider a slightly curved triple-walled carbon nanotube under the influence of stretching effects and resting on Winkler and Pasternak foundations in a thermal and magnetic environment as depicted in Fig. 1. We applied the Eringen's nonlocal elasticity theory, Euler-Bernoulli beam theory, Hamilton's principle and followed Sobamowo [54], the governing equation becomes

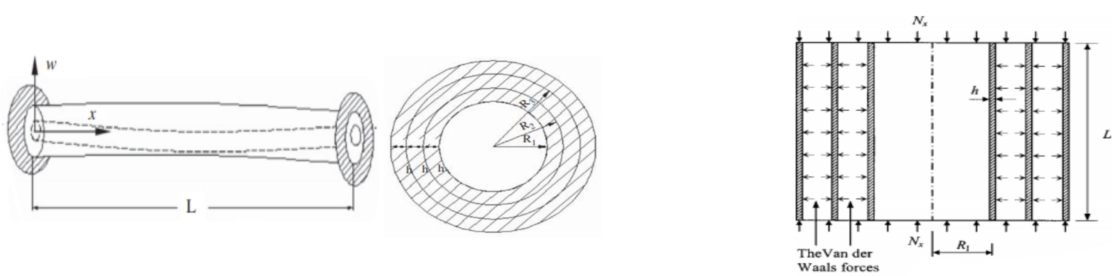


Fig.1
The embedded slightly curved clamped TWCNT on two-parameter elastic foundation. [55, 56, 60]

$$\begin{aligned}
&EI\frac{\partial^4 w_i}{\partial x^4}+m_c\frac{\partial^2 w_i}{\partial t^2}+k_1 w_i+k_3 w_i^3-\left(\frac{EA}{2L}\int_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_i}{\partial x}+\left(\frac{\partial w_i}{\partial x}\right)^2\right)dx\right)\left(\frac{\partial^2 Z_o}{\partial x^2}+\frac{\partial^2 w_i}{\partial x^2}\right)\\
&-\left(EA\alpha_x T+\eta A H_x^2+k_p\right)\frac{\partial^2 w_i}{\partial x^2}+\left(e_o a\right)^2\left(\begin{array}{l}
m_c\dfrac{\partial^4 w_i}{\partial x^2\partial t^2}+k_1\dfrac{\partial^2 w_i}{\partial x^2}+6k_3 w_i\left(\dfrac{\partial w_i}{\partial x}\right)^2+3k_3 w_i^2\left(\dfrac{\partial w_i}{\partial x}\right)\\
-\left(\dfrac{EA}{2L}\displaystyle\int_0^L\left(\dfrac{\partial Z_o}{\partial x}\dfrac{\partial w_i}{\partial x}+\left(\dfrac{\partial w_i}{\partial x}\right)^2\right)dx\right)\left(\dfrac{\partial^4 Z_o}{\partial x^4}+\dfrac{\partial^4 w_i}{\partial x^4}\right)\\
-\left(EA\alpha_x T+\eta A H_x^2+k_p\right)\dfrac{\partial^4 w_i}{\partial x^4}
\end{array}\right)=F_{i,i-1}
\end{aligned} \tag{1}$$

where i=1,2 and 3 and $k_w w=k_1 w+k_3 w^3$, $w(x,t)$ is the bending deflection of the tube, t is the time coordinate, EI is the bending rigidity, m_c is the mass of tube per unit length and Z_o is the initial curvature of the tube. The term $EA\alpha_x T$ denotes the constant axial force due to thermal effects and the term $\eta A H_x^2$ is the magnetic force per unit length due to Lorentz force exerted on the tube in z-direction. In addition, A is the cross-sectional area of the tube, α_x is the coefficient of thermal expansion and T is the change in temperature. In addition, the term η is the magnetic field permeability and H_x is the magnetic field strength. Further works on the SWCNTs can be found in Sobamowo [57, 58] and Sobamowo et al. [59]. In order to incorporate the interlayer interactions for the TWCNTs with 3 layers, it is established that the pressure at any point between any two adjacent tubes depends on the difference of their deflections at that point. Therefore, one can express the linear form of the van der Waals forces as:

$$F_i = c_i(\mathrm{w}_i - w_{i-1}) \tag{2}$$

where F_i is the van der Waals force between the *ith* tube and the (i-1)*th* tube, c_i is the coefficient of the van der Waals force between the *ith* tube and the (i-1)*th* tube. Assuming that all nested individual tubes of the TWCNT vibrate in the same plane. Using the van der Waals forces in Eq. (2), the developed nonlinear governing equations of vibration for the embedded TWCNT in a thermal and magnetic environment with N layers are given as:

$$EI_1\frac{\partial^4 w_1}{\partial x^4}+m_{c1}\frac{\partial^2 w_1}{\partial t^2}-\left(\frac{EA_1}{2L}\int_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_1}{\partial x}+\left(\frac{\partial w_1}{\partial x}\right)^2\right)dx\right)\left(\frac{\partial^2 Z_o}{\partial x^2}+\frac{\partial^2 w_1}{\partial x^2}\right)-\left(EA_1\alpha_x T+\eta A_1 H_x^{\,2}+k_p\right)\frac{\partial^2 w_1}{\partial x^2}$$
$$-(e_o a)^2\left(\begin{array}{l} m_{c1}\dfrac{\partial^4 w_1}{\partial x^2\partial t^2}-\left(\dfrac{EA_1}{2L}\displaystyle\int_0^L\left(\dfrac{\partial Z_o}{\partial x}\dfrac{\partial w_1}{\partial x}+\left(\dfrac{\partial w_1}{\partial x}\right)^2\right)dx\right)\left(\dfrac{\partial^4 Z_o}{\partial x^4}+\dfrac{\partial^4 w_1}{\partial x^4}\right)\\ -\left(EA_1\alpha_x T+\eta A_1 H_x^{\,2}+k_p\right)\dfrac{\partial^4 w_1}{\partial x^4}+c_1\left(\dfrac{\partial^2 w_2}{\partial x^2}-\dfrac{\partial^2 w_1}{\partial x^2}\right)\end{array}\right)=-c_1(w_2-w_1) \tag{3}$$

$$EI_2\frac{\partial^4 w_2}{\partial x^4}+m_{c2}\frac{\partial^2 w_2}{\partial t^2}-\left(\frac{EA_2}{2L}\int_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_2}{\partial x}+\left(\frac{\partial w_2}{\partial x}\right)^2\right)dx\right)\left(\frac{\partial^2 Z_o}{\partial x^2}+\frac{\partial^2 w_2}{\partial x^2}\right)-\left(EA_2\alpha_x T+\eta A_2 H_x^{\,2}+k_p\right)\frac{\partial^2 w_2}{\partial x^2}$$
$$-(e_o a)^2\left(\begin{array}{l} m_{c2}\dfrac{\partial^4 w_2}{\partial x^2\partial t^2}-\left(\dfrac{EA_2}{2L}\displaystyle\int_0^L\left(\dfrac{\partial Z_o}{\partial x}\dfrac{\partial w_2}{\partial x}+\left(\dfrac{\partial w_2}{\partial x}\right)^2\right)dx\right)\left(\dfrac{\partial^4 Z_o}{\partial x^4}+\dfrac{\partial^4 w_2}{\partial x^4}\right)\\ -\left(EA_2\alpha_x T+\eta A_2 H_x^{\,2}+k_p\right)\dfrac{\partial^4 w_2}{\partial x^4}-c_2\left(\dfrac{\partial^2 w_2}{\partial x^2}-\dfrac{\partial^2 w_1}{\partial x^2}\right)+c_2\left(\dfrac{\partial^2 w_3}{\partial x^2}-\dfrac{\partial^2 w_2}{\partial x^2}\right)\end{array}\right)=c_2(w_2-w_1)-c_2(w_3-w_2) \tag{4}$$

$$EI_3\frac{\partial^4 w_3}{\partial x^4}+m_{c3}\frac{\partial^2 w_3}{\partial t^2}-\left(\frac{EA_3}{2L}\int_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_3}{\partial x}+\left(\frac{\partial w_3}{\partial x}\right)^2\right)dx\right)\left(\frac{\partial^2 Z_o}{\partial x^2}+\frac{\partial^2 w_3}{\partial x^2}\right)-\left(EA_3\alpha_x T+\eta A_3 H_x^{\,2}+k_p\right)\frac{\partial^2 w_3}{\partial x^2}$$
$$-(e_o a)^2\left(\begin{array}{l} m_{c3}\dfrac{\partial^4 w_3}{\partial x^2\partial t^2}-\left(\dfrac{EA_3}{2L}\displaystyle\int_0^L\left(\dfrac{\partial Z_o}{\partial x}\dfrac{\partial w_3}{\partial x}+\left(\dfrac{\partial w_3}{\partial x}\right)^2\right)dx\right)\left(\dfrac{\partial^4 Z_o}{\partial x^4}+\dfrac{\partial^4 w_3}{\partial x^4}\right)\\ -\left(EA_3\alpha_x T+\eta A_3 H_x^{\,2}+k_p\right)\dfrac{\partial^4 w_3}{\partial x^4}-c_3\left(\dfrac{\partial^2 w_3}{\partial x^2}-\dfrac{\partial^2 w_2}{\partial x^2}\right)+c_3\left(\dfrac{\partial^2 w_4}{\partial x^2}-\dfrac{\partial^2 w_3}{\partial x^2}\right)\end{array}\right)=c_3(w_3-w_2)-c_3(w_4-w_3) \tag{5}$$

It should be noted that the k_1 and k_3 would not enter into the equations of the inner tubes since only the outer tube interacts with the elastic medium. The displacements of the nanotubes are subjected to the following boundary conditions:

For simply supported (S-S) nanotube,

$$w_i(0,t)=0,\quad \frac{\partial^2 w_i(0,t)}{\partial^2 x}=0,\quad w_i(L,t)=0,\quad \frac{\partial^2 w_i(L,t))}{\partial^2 x}=0. \tag{6}$$

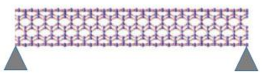
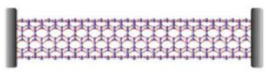
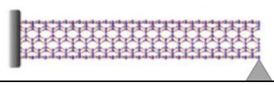
For clamped-clamped supported (C-C) nanotube,

$$w_i(0,t)=0,\quad \frac{\partial w_i(0,t)}{\partial x}=0,\quad w_i((L,t)=0,\quad \frac{\partial w_i(L,t)}{\partial x}=0. \tag{7}$$

For a clamped-simply supported (C-S) nanotube,

$$w_i(0,t)=0,\quad \frac{\partial w_i(0,t)}{\partial x}=0,\quad w_i(L,t)=0, \frac{\partial^2 w_i((L,t)}{\partial^2 x}=0. \tag{8}$$

The basic functions corresponding to the above boundary conditions [37].

Cases	Mode shape, $\phi(x)$	Value of β
Simply support 	$sin\left(\frac{\beta x}{L}\right)$	π
Clamped-Clamped support 	$\left(\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right)-\left(\frac{\sinh\beta+\sin\beta}{\cosh\beta-\cos\beta}\right)\left(\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right)$	4.730041
Clamped-Simply support 	$\left(\cosh\left(\frac{\beta x}{L}\right)-\cos\left(\frac{\beta x}{L}\right)\right)-\left(\frac{\cosh\beta-\cos\beta}{\sinh\beta-\sin\beta}\right)\left(\sinh\left(\frac{\beta x}{L}\right)-\sin\left(\frac{\beta x}{L}\right)\right)$	3.926602

3 SOLUTION METHODOLOGY

Using the Galerkin's decomposition procedure to separate the spatial and temporal parts of the lateral displacement functions,

$$w_i(x,t)=\phi(x)\mathrm{W}_i(t) \qquad i=1,2,3,...,N \tag{9}$$

where $w_i(x,t)$ is the lateral displacement functions, $\mathrm{W}_i(t)$ is the time-dependent parameter or time-dependent maximum amplitude of oscillation of the i_{th} layer of the nanotube and $\phi(x)$ is a trial/comparison function that will satisfy both the geometric and natural boundary conditions.

Applying one-parameter Galerkin, we have

$$\int_0^L R_i(x,t)\phi(x)dx=0 \tag{10}$$

where $R(x,t)$ is the equation of motion for each wall. For the outer wall of multi wall carbon nanotubes,

$$\begin{aligned}
&R(x,t)=EI_N\frac{\partial^4 w_N}{\partial x^4}+m_{cN}\frac{\partial^2 w_N}{\partial t^2}+k_1 w_N+k_3 w_N{}^3-\left(\frac{EA_N}{2L}\int_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_N}{\partial x}+\left(\frac{\partial w_N}{\partial x}\right)^2\right)dx\right)\left(\frac{\partial^2 Z_o}{\partial x^2}+\frac{\partial^2 w_N}{\partial x^2}\right)\\
&-\left(EA_N\alpha_x T+\eta A_N H_x{}^2+k_p\right)\frac{\partial^2 w_N}{\partial x^2}\\
&-(e_o a)^2\left(\begin{aligned}&m_{cN}\frac{\partial^4 w_N}{\partial x^2\partial t^2}+k_1\frac{\partial^2 w_N}{\partial x^2}+6k_3 w_N\left(\frac{\partial w_N}{\partial x}\right)^2+3k_3 w_N{}^2\left(\frac{\partial w_N}{\partial x}\right)\\
&-\left(\frac{EA_N}{2L}\int_0^L\left(\frac{\partial Z_o}{\partial x}\frac{\partial w_N}{\partial x}+\left(\frac{\partial w_N}{\partial x}\right)^2\right)dx\right)\left(\frac{\partial^4 Z_o}{\partial x^4}+\frac{\partial^4 w_N}{\partial x^4}\right)\\
&-\left(EA_N\alpha_x T+\eta A_N H_x{}^2+k_p\right)\frac{\partial^4 w_N}{\partial x^4}+c_{N-1}\left(\frac{\partial^2 w_N}{\partial x^2}-\frac{\partial^2 w_{N-1}}{\partial x^2}\right)\end{aligned}\right)=-c_{N-1}(w_N-w_{N-1})
\end{aligned} \tag{11}$$

One arrives at

$$\alpha_1 EI_N W_N + \alpha_2 m_{CN}\frac{d^2W_N}{dt^2} + \alpha_2 k_1 W_N + \alpha_3 k_3 W_N^{\ 3} - \alpha_4 \frac{EA_N}{2L} W_N^{\ 3} - \left(a_0 a\right)^2 \alpha_5 m_{cN}\frac{d^2W_N}{dt^2} - \left(a_0 a\right)^2 \alpha_5 k_1 W_N$$
$$-6\alpha_6\left(a_0 a\right)^2 k_3 W_N^{\ 3} - 3\alpha_7\left(a_0 a\right)^2 k_3 W_N^{\ 3} + \alpha_8\left(a_0 a\right)^2 \frac{EA_N}{2L} W_N^{\ 3} + \alpha_1\left(a_0 a\right)^2\left(EA_N \alpha_x T_t + \eta A_N H_x^{\ 2}\right)W_N \tag{12}$$
$$-\alpha_5\left(a_0 a\right)^2 C_{N-1}(W_N - W_{N-1}) - \alpha_5(N_{tN} + Q_N)W_N + \alpha_2\left(a_0 a\right)^2 C_{N-1}(W_N - W_{N-1}) = 0$$

After collecting like terms, we have

$$\frac{d^2W_N}{dt^2} + \left(\frac{\alpha_1 EI_1 + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5)\left(EA_N \alpha_x T_t + \eta A_N H_x^{\ 2} + k_p\right)}{\left(\alpha_2 - \alpha_5\mu\right)\rho A_N}\right) + \left(\frac{(\alpha_2 - \alpha_5\mu)c_{N-1}}{\left(\alpha_2 - \alpha_5\mu\right)\rho A_N}\right)(W_N - W_{N-1}) +$$
$$\left(\frac{\alpha_3 k_3 - \alpha_4\dfrac{EA_N}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu\dfrac{EA_N}{2L}}{\left(\alpha_2 - \alpha_5\mu\right)\rho A_N}\right)W_N^{\ 3} = 0 \tag{13}$$

where

$$\alpha_1 = \int_0^L \phi(x)\frac{d^4\phi(x)}{dx^4}dx \ , \quad \alpha_2 = \int_0^L \phi^2(x)dx \ , \qquad \alpha_3 = \int_0^L \phi^4(x)dx$$
$$\alpha_4 = \int_0^L\left(\left(\int_0^L \phi(x)\left(\frac{\partial Z_o}{\partial x}\frac{\partial\phi(x)}{\partial x} + \left(\frac{\partial\phi(x)}{\partial x}\right)^2\right)dx\right)\left(\frac{\partial^2 Z_o}{\partial x^2} + \frac{\partial^2\phi(x)}{\partial x^2}\right)\right)dx \ , \tag{14}$$
$$\alpha_5 = \int_0^L \phi(x)\frac{d^2\phi(x)}{dx^2}dx \ , \quad \alpha_6 = \int_0^L \phi^2(x)\left(\frac{d\phi(x)}{dx}\right)^2 dx$$

$$\alpha_7 = \int_0^L \phi^3(x)\frac{d\phi(x)}{dx}dx \ , \ \alpha_4 = \int_0^L\left(\left(\int_0^L \phi(x)\left(\frac{\partial Z_o}{\partial x}\frac{\partial\phi(x)}{\partial x} + \left(\frac{\partial\phi(x)}{\partial x}\right)^2\right)dx\right)\left(\frac{\partial^4 Z_o}{\partial x^4} + \frac{\partial^4\phi(x)}{\partial x^4}\right)\right)dx \ , \tag{15}$$
$$\mu = \left(e_0 a\right)^2 , \ m_{cN} = \rho A_N \ ,$$

Similarly, the same procedure is applied to other inner walls appropriately. Therefore, the general governing equations of motion for nonlinear vibrations of embedded MWCNTs in a thermal and magnetic environment in ODE form is obtained as:

$$\frac{d^2W_1}{dt^2} + \left(\frac{\alpha_1 EI_1 + (\alpha_1\mu - \alpha_5)\left(EA_N \alpha_x T_t + \eta A_N H_x^{\ 2} + k_p\right)}{\left(\alpha_2 - \alpha_5\mu\right)\rho A_1}\right)W_1 - \frac{c_1}{\rho A_1}(W_2 - W_1) + \left(\frac{(\alpha_8\mu - \alpha_4)\dfrac{EA_1}{2L}}{\left(\alpha_2 - \alpha_5\mu\right)\rho A_1}\right)W_1^{\ 3} = 0 \tag{16}$$

$$\frac{d^2W_2}{dt^2} + \left(\frac{\alpha_1 EI_2 + (\alpha_1\mu - \alpha_5)\left(EA_N \alpha_x T_t + \eta A_N H_x^{\ 2} + k_p\right)}{\left(\alpha_2 - \alpha_5\mu\right)\rho A_2}\right)W_2 + \frac{c_1}{\rho A_2}(W_2 - W_1) - \frac{c_2}{\rho A_2}(W_3 - W_2) + \left(\frac{(\alpha_8\mu - \alpha_4)\dfrac{EA_2}{2L}}{\left(\alpha_2 - \alpha_5\mu\right)\rho A_2}\right)W_2^{\ 3} = 0 \tag{17}$$

$$\frac{d^2W_3}{dt^2} + \left(\frac{\alpha_1 EI_3 + (\alpha_1\mu - \alpha_5)\left(EA_N \alpha_x T_t + \eta A_N H_x^{\ 2} + k_p\right)}{\left(\alpha_2 - \alpha_5\mu\right)\rho A_3}\right)W_3 + \frac{c_2}{\rho A_3}(W_3 - W_2) - \frac{c_3}{\rho A_3}(W_4 - W_3) + \left(\frac{(\alpha_8\mu - \alpha_4)\dfrac{EA_3}{2L}}{\left(\alpha_2 - \alpha_5\mu\right)\rho A_3}\right)W_3^{\ 3} = 0 \tag{18}$$

and the initial conditions are

$$W_i(0) = X \tag{19}$$

and

$$\frac{dW_i(0)}{dt} = 0 \qquad i=1,2,3,\ldots,N \tag{20}$$

3.1 Homotopic perturbation method

The nonlinear terms in Eqs. (16)- (18) make the development of exact analytical solution tedious. Therefore, for the purpose of generating a symbolic solution for the nonlinear equations, we made a recourse to homotopic perturbation method. The principle and the procedures of the method can be found in our previous works [60, 61].

Substituting the following dimensionless parameters

$$r = \sqrt{\frac{I_1}{A_1}}, \quad a_1 = \frac{W_1}{r}, \quad a_2 = \frac{W_2}{r}, \quad a_3 = \frac{W_3}{r}, \quad \tau = \omega_0 t \tag{21}$$

These equations are transformed to the following dimensionless nonlinear system of equations using the dimensionless parameters in Eq. (21)

$$\omega_0^2 \frac{d^2 a_1}{d\tau^2} + f_1 a_1 + f_2 a_1^3 - f_3 a_2 = 0 \tag{22}$$

$$\omega_0^2 \frac{d^2 a_2}{d\tau^2} + g_1 a_2 + g_2 a_2^3 - g_3 a_1 - g_4 a_3 = 0. \tag{23}$$

$$\omega_0^2 \frac{d^2 a_3}{d\tau^2} + h_1 a_3 + h_2 a_3^3 - h_3 a_2 = 0 \tag{24}$$

where,

$$\begin{gathered}
f_1 = \frac{\alpha_1 EI_1 + (\alpha_1\mu - \alpha_5)\left(EA_N \alpha_x T_t + \eta A_N H_x^{\,2} + k_p\right)}{(\alpha_2 - \alpha_5\mu)\rho A_1} + \frac{c_1}{\rho A_1}, \quad f_2 = \frac{(\alpha_8\mu - \alpha_4)EI_1}{2L(\alpha_2 - \alpha_5\mu)\rho A_1}, \quad f_3 = \frac{c_1}{\rho A_1} \\
g_1 = \left(\frac{\alpha_1 EI_2 + (\alpha_1\mu - \alpha_5)\left(EA_N \alpha_x T_t + \eta A_N H_x^{\,2} + k_p\right)}{(\alpha_2 - \alpha_5\mu)\rho A_2}\right) + \frac{c_1}{\rho A_2} + \frac{c_2}{\rho A_2}, \quad g_2 = \frac{(\alpha_8\mu - \alpha_4)EI_1}{2L(\alpha_2 - \alpha_5\mu)\rho A_1} \\
g_3 = \frac{c_1}{\rho A_2}, \quad g_4 = \frac{c_2}{\rho A_2}, \quad h_1 = \left(\frac{\alpha_1 EI_3 + \alpha_2 k_1 - \alpha_5\mu k_1 + (\alpha_1\mu - \alpha_5)\left(EA_N \alpha_x T_t + \eta A_N H_x^{\,2} + k_p\right)}{(\alpha_2 - \alpha_5\mu)\rho A_3}\right) + \frac{c_2}{\rho A_3} \\
h_2 = \frac{\left(\alpha_3 k_3 - \alpha_4 \frac{EA_3}{2L} - 6\alpha_6\mu k_3 - 3\alpha_7\mu k_3 + \alpha_8\mu \frac{EA_3}{2L}\right)}{(\alpha_2 - \alpha_5\mu)\rho A_3}\left(\frac{I_1}{A_1}\right), \quad h_3 = \frac{c_2}{\rho A_3}
\end{gathered} \tag{25}$$

We construct a homotopy on Eqs. (22-24) as follows:

$$\left(1-p\right)\left\{\omega_0^{\ 2}\left(\frac{d^2a_1}{d\tau^2}+a_1\right)\right\}+p\left\{\omega_0^{\ 2}\frac{d^2a_1}{d\tau^2}+f_1a_1+f_2a_1^3-f_3a_2\right\}=0 \tag{26}$$

$$\left(1-p\right)\left\{\omega_0^{\ 2}\left(\frac{d^2a_2}{d\tau^2}+a_2\right)\right\}+p\left\{\omega_0^{\ 2}\frac{d^2a_2}{d\tau^2}+g_1a_2+g_2a_2^3-g_3a_1\right\}=0 \tag{27}$$

$$\left(1-p\right)\left\{\omega_0^{\ 2}\left(\frac{d^2a_3}{d\tau^2}+a_3\right)\right\}+p\left\{\omega_0^{\ 2}\frac{d^2a_3}{d\tau^2}+h_1a_3+h_2a_3^3-h_3a_2\right\}=0 \tag{28}$$

Assuming the solution of Eqs. (26), (27) and (28) to be in the following form

$$a_1\left(\tau\right)=a_{10}\left(\tau\right)+pa_{11}\left(\tau\right)+p^2a_{12}\left(\tau\right)+p^3a_{13}\left(\tau\right)+..., \tag{29}$$

$$a_2\left(\tau\right)=a_{20}\left(\tau\right)+pa_{21}\left(\tau\right)+p^2a_{22}\left(\tau\right)+p^3a_{23}\left(\tau\right)+..., \tag{30}$$

$$a_3\left(\tau\right)=a_{30}\left(\tau\right)+pa_{31}\left(\tau\right)+p^2a_{32}\left(\tau\right)+p^3a_{33}\left(\tau\right)+..., \tag{31}$$

$$\omega=\omega_0+p\omega_1+p^2\omega_2+p^3\omega_3+.... \tag{32}$$

Substituting Eqs. (29-32) into the homotopic in Eqs. (29)- (32), collecting and rearranging the coefficients of the terms with identical powers of p, we have a series of linear differential equations

$$p^0:\begin{cases}\dfrac{d^2a_{10}}{d\tau^2}+a_{10}=0, & a_{10}\left(0\right)=X_1, \quad \dfrac{da_{10}\left(0\right)}{d\tau}=0,\\ \dfrac{d^2a_{20}}{d\tau^2}+a_{20}=0, & a_{20}\left(0\right)=X_1, \quad \dfrac{da_{20}\left(0\right)}{d\tau}=0,\\ \dfrac{d^2a_{30}}{d\tau^2}+a_{30}=0, & a_{30}\left(0\right)=X_1, \quad \dfrac{da_{30}\left(0\right)}{d\tau}=0\end{cases} \tag{33}$$

$$p^1:\begin{cases}\omega_0^2\left\{\dfrac{d^2a_{11}}{d\tau^2}+a_{11}\right\}-\omega_0^2a_{10}+f_1a_{10}+f_2a_{10}^3-f_3a_{20}=0, & a_{11}\left(0\right)=0 \quad \dfrac{da_{11}\left(0\right)}{d\tau}=0,\\ \omega_0^2\left\{\dfrac{d^2a_{21}}{d\tau^2}+a_{21}\right\}-\omega_0^2a_{20}+g_1a_{20}+g_2a_{20}^3-g_3a_{10}=0, & a_{21}\left(0\right)=0, \quad \dfrac{da_{21}\left(0\right)}{d\tau}=0,\\ \omega_0^2\left\{\dfrac{d^2a_{31}}{d\tau^2}+a_{31}\right\}-\omega_0^2a_{30}+h_1a_{30}+h_2a_{30}^3-h_3a_{20}=0, & a_{31}\left(0\right)=0 \quad \dfrac{da_{31}\left(0\right)}{d\tau}=0,\end{cases} \tag{34}$$

$$p^2:\begin{cases}\omega_0^2\left\{\dfrac{d^2a_{12}}{d\tau^2}+a_{12}\right\}-\omega_0^2a_{11}+2\omega_0\omega_1\dfrac{d^2a_{10}}{d\tau^2}+f_1a_{11}+3f_2a_{10}^2a_{11}-f_3a_{21}=0, & a_{12}\left(0\right)=0 \quad \dfrac{da_{12}\left(0\right)}{d\tau}=0,\\ \omega_0^2\left\{\dfrac{d^2a_{22}}{d\tau^2}+a_{22}\right\}-\omega_0^2a_{21}+2\omega_0\omega_1\dfrac{d^2a_{20}}{d\tau^2}+g_1a_{21}+3g_2a_{20}^2a_{21}-g_3a_{11}=0, & a_{22}\left(0\right)=0, \quad \dfrac{da_{22}\left(0\right)}{d\tau}=0,\\ \omega_0^2\left\{\dfrac{d^2a_{32}}{d\tau^2}+a_{12}\right\}-\omega_0^2a_{31}+2\omega_0\omega_1\dfrac{d^2a_{30}}{d\tau^2}+h_1a_{31}+3h_2a_{30}^2a_{31}-h_3a_{21}=0, & a_{32}\left(0\right)=0 \quad \dfrac{da_{32}\left(0\right)}{d\tau}=0,\end{cases} \tag{35}$$

In order to calculate the nonlinear natural frequencies for TWCNT, we assumed initial zeroth approximations given as:

$$a_{10}\left(\tau\right)=X_1 cos\,\tau \tag{36}$$

$$a_{20}\left(\tau\right)=X_2 cos\,\tau \tag{37}$$

$$a_{30}\left(\tau\right)=X_3 cos\,\tau \tag{38}$$

Substituting Eqs. (33-35) into Eqs. (22-24), we have the following nonlinear system of equations

$$-X_1\omega_0^2+f_1X_1+\frac{3}{4}f_2X_1^3-f_3X_2=0 \tag{39}$$

$$-X_2\omega_0^2+g_1X_2+\frac{3}{4}g_2X_2^3-g_3X_1-g_4X_3=0 \tag{40}$$

$$-X_1\omega_0^2+h_1X_3+\frac{3}{4}h_2X_3^3-h_3X_2=0 \tag{41}$$

From Eq. (39)

$$X_2=\frac{-X_1\omega^2{}_0+f_1X_1+\frac{3}{4}f_2X_1{}^3}{f_3} \tag{42}$$

Substituting and making X_3 the subject to obtain,

$$X_3=\frac{-\omega^2{}_0\left(\frac{-X_1\omega^2{}_0+f_1X_1+\frac{3}{4}f_2X_1{}^3}{f_3}\right)+g_1\left(\frac{-X_1\omega^2{}_0+f_1X_1+\frac{3}{4}f_2X_1{}^3}{f_3}\right)+\frac{3}{4}g_2\left(\frac{-X_1\omega^2{}_0+f_1X_1+\frac{3}{4}f_2X_1{}^3}{f_3}\right)^3-g_3X_1}{g_4} \tag{43}$$

Substituting again, we arrived at

$$\begin{aligned}&-\omega_0{}^2\left(\frac{-\omega^2{}_0\left(\frac{-X_1\omega^2{}_0+f_1X_1+\frac{3}{4}f_2X_1{}^3}{f_3}\right)+g_1\left(\frac{-X_1\omega^2{}_0+f_1X_1+\frac{3}{4}f_2X_1{}^3}{f_3}\right)+\frac{3}{4}g_2\left(\frac{-X_1\omega^2{}_0+f_1X_1+\frac{3}{4}f_2X_1{}^3}{f_3}\right)^3-g_3X_1}{g_4}\right)\\&+h_1\left(\frac{-\omega^2{}_0\left(\frac{-X_1\omega^2{}_0+f_1X_1+\frac{3}{4}f_2X_1{}^3}{f_3}\right)+g_1\left(\frac{-X_1\omega^2{}_0+f_1X_1+\frac{3}{4}f_2X_1{}^3}{f_3}\right)+\frac{3}{4}g_2\left(\frac{-X_1\omega^2{}_0+f_1X_1+\frac{3}{4}f_2X_1{}^3}{f_3}\right)^3-g_3X_1}{g_4}\right)\end{aligned} \tag{44}$$

$$+\frac{3}{4}h_1\left(\frac{\left(-\omega^2{}_0\left(\frac{-X_1\omega^2{}_0+f_1X_1+\frac{3}{4}f_2X_1{}^3}{f\,3}\right)+g_1\left(\frac{-X_1\omega^2{}_0+f_1X_1+\frac{3}{4}f_2X_1{}^3}{f\,3}\right)+\frac{3}{4}g_2\left(\frac{-X_1\omega^2{}_0+f_1X_1+\frac{3}{4}f_2X_1{}^3}{f\,3}\right)^3-g_3X_1{}^3\right)}{g_4}\right)^3-h_3\left(\frac{-X_1\omega^2{}_0+f_1X_1+\frac{3}{4}f_2X_1{}^3}{f_3}\right) \tag{44}$$

The above Eq. (44) gives a nonlinear algebraic equation of degree 18. Although, it is very difficult to develop analytical expressions for such equation, recourse is made to numerical method using Newton-Raphson to solve the equation by finding the roots of solving for ω_0 in the equation. The smallest real value of ω_0 obtained from the solutions is the nonlinear natural frequency for TWCNT.

To calculate the linear natural frequencies for TWCNT, substitute

$$a_{10}(\tau)=X_1cos\,\omega\tau \tag{45}$$

$$a_{20}(\tau)=X_2cos\,\omega\tau \tag{46}$$

$$a_{30}(\tau)=X_3cos\,\omega\tau \tag{47}$$

Into Eqs. (39-41) and neglecting the nonlinear terms give

$$-X_1\omega^2+f_1X_1-f_3X_2=0 \tag{48}$$

$$-X_2\omega^2+g_1X_2-g_3X_1=0 \tag{49}$$

$$-X_1\omega^2+h_1X_2-h_3X_1=0 \tag{50}$$

These equations can be written in matrix form as:

$$\begin{bmatrix} -\omega^2+f\,1 & -f\,3 & 0 \\ -g\,3 & -\omega^2+g\,1 & g\,4 \\ 0 & h\,3 & -\omega^2+h\,1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{51}$$

Since, $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ cannot be equal to zero, for nontrivial case, then

$$\begin{bmatrix} -\omega^2+f_1 & -f_3 & 0 \\ -g_3 & -\omega^2+g_1 & g_4 \\ 0 & h_3 & -\omega^2+h_1 \end{bmatrix}=0 \tag{52}$$

By equating the determinant of the matrix of Eq. (52) to zero, the characteristic equation is obtained as:

$$\omega^6-(f_1+g_1+h_1)\omega^4+(2f_1h_1-f_1g_1-f_3g_3-h_3g_4)\omega^2-(f_1g_1h_1-f_1h_3g_4-f_3h_1g_3)=0 \tag{53}$$

Eq. (53) can be written as:

$$\omega^6+\gamma_1\omega^4+\gamma_2\omega^2+\gamma_3=0 \tag{54}$$

where

$$\gamma_1=-\left(f_1+g_1+h_1\right)$$
$$\gamma_2=\left(2f_1h_1-f_1g_1-f_3g_3-h_3g_4\right)$$
$$\gamma_3=-\left(f_1g_1h_1-f_1h_3g_4-f_3h_1g_3\right)$$

The roots of the sextic equations are

$$\omega_1=\sqrt{\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)+\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}+\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)-\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}-\frac{\gamma_1}{3}} \tag{55}$$

$$\omega_2=-\sqrt{\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)+\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}+\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)-\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}-\frac{\gamma_1}{3}} \tag{56}$$

$$\omega_3=\sqrt{\frac{-1}{2\lambda_1}\left[\sqrt{\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)+\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}+\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)-\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}}\right]+\frac{\sqrt{-3}}{2\lambda_1}\left[\sqrt{\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)+\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}-\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)-\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}-\frac{\gamma_1}{3}}\right]-\frac{\gamma_1}{3}} \tag{57}$$

$$\omega_4=-\sqrt{\frac{-1}{2\lambda_1}\left[\sqrt{\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)+\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}+\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)-\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}}\right]+\frac{\sqrt{-3}}{2\lambda_1}\left[\sqrt{\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)+\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}-\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)-\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}-\frac{\gamma_1}{3}}\right]-\frac{\gamma_1}{3}} \tag{58}$$

$$\omega_5=\sqrt{\begin{matrix}\dfrac{-1}{2\lambda_1}\left[\left[\sqrt{\begin{matrix}\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)+\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}\\+\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)-\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}\end{matrix}}\right]\right]\\ \dfrac{-\sqrt{-3}}{2\lambda_1}\left[\left[\sqrt{\begin{matrix}\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)+\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}\\-\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)-\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}\end{matrix}}\right]\right]-\dfrac{\gamma_1}{3}\end{matrix}} \tag{59}$$

$$\omega_6=\sqrt{\begin{matrix}\dfrac{-1}{2\lambda_1}\left[\left[\sqrt{\begin{matrix}\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)+\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}\\+\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)-\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}\end{matrix}}\right]\right]\\ \dfrac{-\sqrt{-3}}{2\lambda_1}\left[\left[\sqrt{\begin{matrix}\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)+\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}\\-\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)-\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}\end{matrix}}\right]\right]-\dfrac{\gamma_1}{3}\end{matrix}} \tag{60}$$

The fundamental linear vibration frequency of TWNT is the lowest root which gives.

$$\omega=\sqrt{\begin{matrix}\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)+\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}\\+\sqrt[3]{\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)-\sqrt{\left(\frac{\gamma_2}{3}-\frac{\gamma_1^2}{9}\right)^3+\left(\frac{-\gamma_1^3}{27}+\frac{\gamma_1\gamma_3}{6}-\frac{\gamma_3}{2}\right)^2}}-\frac{\gamma_1}{3}\end{matrix}} \tag{61}$$

4 RESULTS AND DISCUSSION

Using the material and geometric parameters of the carbon nanotubes, E = 1.1 *TPa*, ρ =1300 kg/m^3, l =45 *nm*, the outer diameter do= 3 *nm*, and the thickness of each layer, h= 0.68 *nm*, the frequency ratio against non-dimensional maximum amplitude for the nonlinear vibrations of SWCNTs and DWCNTs in a thermal and magnetic environment are presented. The results of the simulation and the effects of various parameters on the frequency ratio of nonlinear vibrations of embedded single- and double-walled carbon nanotubes in a thermal and magnetic environment are presented and discussed.

4.1 Effects of boundary conditions on the frequency ratio of the carbon nanotubes

Fig. 2 shows the effects of boundary conditions on the frequency ratio for the nonlinear vibrations of CNTs in thermal and magnetic environment, ($k_1=10^7 N/m^2$, $k_3=10^8 N/m^2$, T=40K, $H_x=10^7 A/m$, $e_oa=1.5\times10^{-9}$ and $c_1=c_2=c_3=0.3\times10^{12} N/m^2$). As it is depicted in the figures, the frequency ratio for all boundary conditions decreases as the number of wall increases. This is due to the fact that carbon nanotubes generally have weak shear interactions between adjacent tubes and become more predominant as the number of walls increases. It could therefore be inferred that in an application where linear vibration is preferred for system stability, MWCNTs will perform better than SWCNTs of the same geometry and size. In addition, the figures show that for both the SWCNTs and MWCNTs, the frequency ratio is highest for clamped simple supported and least for clamped-clamped supported. This establishes that the clamped-clamped supported system provided the best grip (support) for the nanotubes and this can be used to control

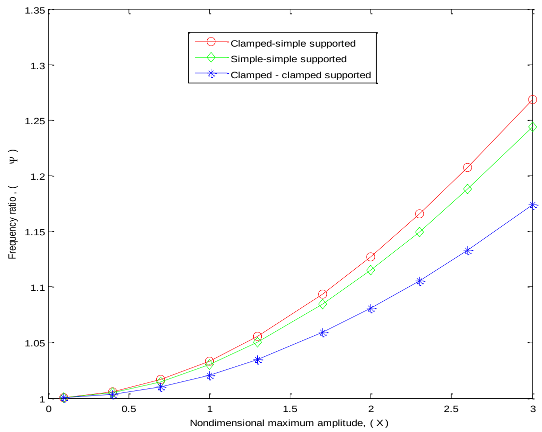


Fig.2
Frequency ratio versus non-dimensional amplitude for TWCNT under various boundary conditions.

4.2 Effects of spring stiffness (k_1) on the frequency ratio of the carbon nanotubes

The impacts of the spring stiffness (k_1) on the dimensionless frequency ratio of the carbon nanotubes in thermal and magnetic environment is shown in Fig 3. It is depicted that he frequency ratio decreases with increases in the value of spring constant (k_1) for CNTs. This is because, the linear frequency increases as the value spring constant increases. At large value of k_1 ($k_1 = 10^{10} N/m^2$), the vibration of the system becomes stable and this can be used as good measure in controlling nonlinear vibration of the system.

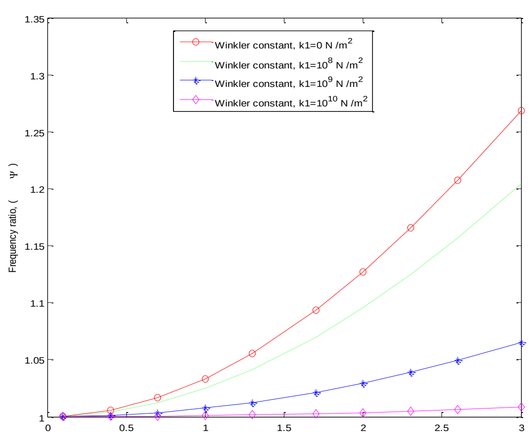


Fig.3
Effect of Winkler constant (k_1) on amplitude-frequency ratio curve for TWCNTs.

4.3 Effects of nonlinear spring stiffness (k_3) on the frequency ratio of the carbon nanotubes

Fig. 4 shows the effect of nonlinear spring stiffness (k_3) on the frequency ratio of outer walled of embedded DWCNTs in a thermal and magnetic environment. It could be seen that the frequency ratio increases with increases in the value of the nonlinear spring constant. This is because the nonlinear frequency increases as the value of the nonlinear spring constant increases without producing any effect on the linear frequency. The value of nonlinear spring constant should be kept as low as possible since it makes the vibration of the system to be nonlinear and this can lead to the instability of the system.

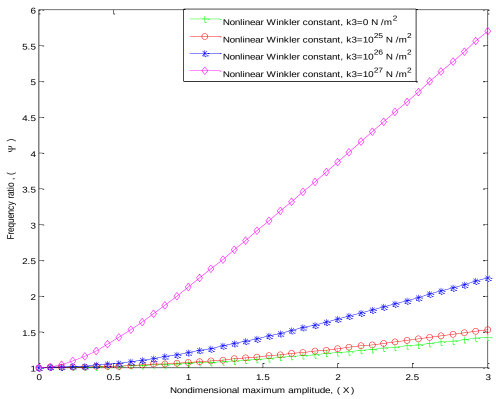


Fig.4
Effect of nonlinear spring constant constant (k_3) on amplitude-frequency ratio curve for TWCNTs.

4.4 Effects of Van der Waal force on the frequency ratio of the carbon nanotubes

Fig. 5 presents the effects of Van der Waal force on the frequency ratio of the SWCNTs and DWCNTs, respectively, in a thermal and magnetic environment. It can be seen that when the coefficient of the van der Waals forces is zero i.e. c=0 N/m^2, it means a single-walled carbon nanotube with the same dimension and geometry with double-walled carbon nanotubes. The results reveals that the frequency ratio decreases as the number of walls increases. Increasing the number of walls can be used as a good parameter to control the nonlinear vibration of the system.

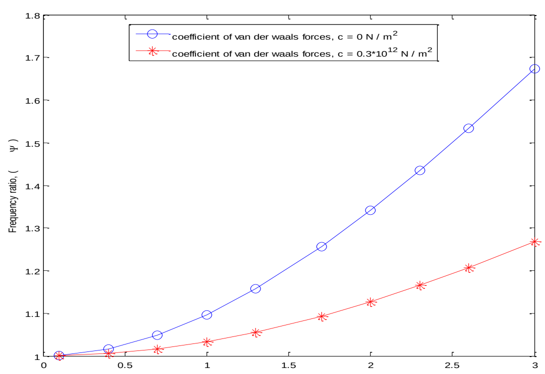


Fig.5
Effect of Van der waals force on amplitude-frequency ratio curve for TWCNTs.

4.5 Effects of Temperature on the frequency ratio of outer wall of TWCNTs

Fig. 6 illustrates the influence of temperature on the frequency ratio the outer wall of DWCNTs in a thermal and magnetic environment. The result presents that as the temperature increases, the frequency ratio decreases. This shows that increase in temperature leads to increase in the value of linear natural frequency of the system.

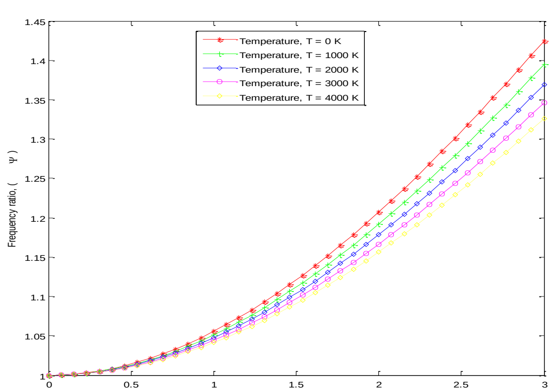


Fig.6
Effect of magnetic force strength on amplitude-frequency ratio curve on outerwall of TWCNTs.

4.6 Effects of magnetic force strength on the frequency ratio of outer wall of TWCNTs

Fig. 7 presents the impact of magnetic force strength on the dimensionless frequency ratio. From the figure, it is established that as the magnetic field strength increases, the vibration of the system approaches linear vibration and become linear at higher value of magnetic force strength, $H=10^9$ A/m.

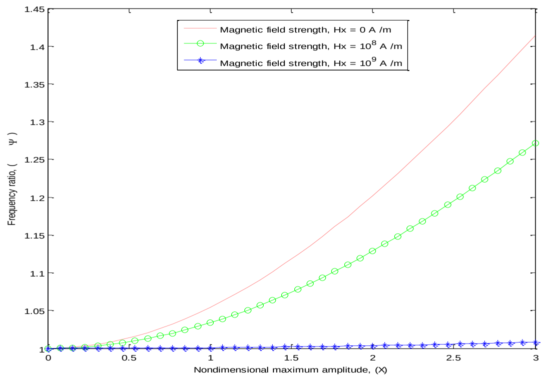


Fig.7
Effect of magnetic force strength on amplitude-frequency ratio curve on outerwall of TWCNTs.

4.7 The linear and nonlinear vibration deflection-time curve of outer wall of TWCNTs

Fig. 8 shows the comparison of the linear vibration with nonlinear vibration of the TWCNT. It could be seen in the figure that the discrepancy between the linear and nonlinear amplitudes increases as the vibration time progresses.

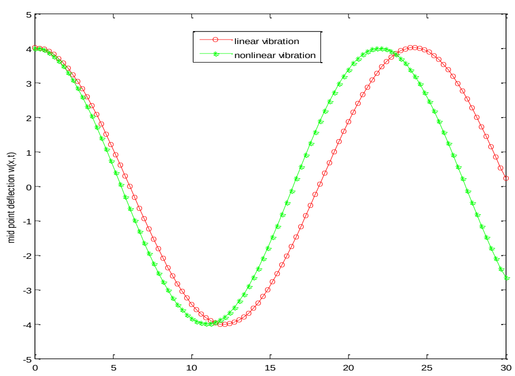


Fig.8
The linear and nonlinear vibration deflection-time curve of outer wall of TWCNTs.

4.8 Verification of the results

In order to verify the results of the developed approximate analytical solutions, the results are compared from the used scheme with the results of the numerical solutions in previous work of Pentaras and Elishakoff [60] where Bubnov–Galerkin and Petrov–Galerkin methods were used. Table 1, shows the comparison of results between the homotopy perturbation method and the numerical method for the linearized models as presented [60]. From, the table, it is established that there is an excellent agreement between the numerical solutions and the approximate analytical solutions computed in the present study.

Table 1
Comparison of Natural frequencies for the first 5 vibrational modes and L/d = 10 (simply supported TWCNTs at both ends).

Mode number, n	$\omega_{n(1)}$ [60]	$\omega_{n(1)}$ (Present)	$\omega_{n(2)}$ [60]	$\omega_{n(2)}$ (Present)	$\omega_{n(3)}$ [60]	$\omega_{n(3)}$ (Present)
1	0.647954	0.647955	5.953946	5.953945	9.642204	9.642207
2	2.531134	2.531136	6.302394	6.302397	9.879866	9.879863
3	5.038042	5.038047	7.915761	7.915765	10.965936	10.965932
4	7.131562	7.131565	11.299882	11.299882	14.266972	14.266975
5	12.069053	12.069051	15.452939	15.452935	20.559854	20.559858

5 CONCLUSION

In this work, nonlocal elasticity theory has been used to analyze the nonlinear vibrations of slightly curved multi-walled carbon nanotubes resting on Winkler and Pasternak foundations in a thermal and magnetic environment. The effects of the various controlling parameters on the nonlinear to linear frequency ratio have been analyzed and discussed. The results established that the frequency ratio for all boundary conditions decreases as the number of wall increases. In addition, it is established that the frequency ratio is highest for clamped-simple supported and least for clamped-clamped supported. Additionally, the results revealed that the frequency ratio decreases with increase in the value of spring constant (k_1), temperature and magnetic field strength. This work will enhance the applications of carbon nanotubes in structural, electrical, mechanical and biological applications especially those nano-devices in a thermal and magnetic environment.

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