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A Mathematical Modeling for Plastic Analysis of Planar Frames by Linear Programming and Genetic Algorithm

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Abstract

In this paper, a mathematical modeling is developed for plastic analysis of planar frames. To this end, the researcher tried to design an optimization model in linear format in order to solve large scale samples. The computational result of CPU time requirement is shown for different samples to prove efficiency of this method for large scale models.
The fundamental concept of this model is obtained from moment distribution method which is a safe theorem based method, so in this mathematical modeling, the objective is finding the largest load which ensures equilibrium and yield conditions. Contrary to moment distribution method, calculation of load factor and the value of moments in the elements are completely automatic and not to need user decision. As the objective function and constraints of this model are linear so it can be solved by linear programming (LP) software such as LINGO that is shown in this paper and also the model is solved by genetic algorithm (GA) to compare two solutions.
Keywords: optimization; Mathematical Modeling; Plastic Analysis; Linear Programming; Genetic Algorithm; Planar Frames

1. Introduction

The minimum and maximum principles are the basis of all general analytical methods for plastic analysis and design. In the safe analysis or minimum principle, when the full plastic moments of the members are known, an analysis can be carried out to determine the maximum load factor. According to this definition, it can be concluded that the main attempt in safe analysis is finding maximum collapse load factor between all possible safe solutions [1-4].

In this paper, a mathematical modeling is designed for finding collapse load factor according to safe theorem. The fundamental concept of this model is obtained from moment distribution method. The proposed model is solved by linear programming (LP) software such as LINGO and also genetic algorithm (GA) and two solutions are compared [5-7].

After introduction, the basic concept for plastic analysis is reviewed and the assumption for making this model is explained. Then, the application of moment distribution method for a planer frame is shown through an example to clarify the concept of the proposed mathematical model. The formulation of proposed model, object function, design variable and constraints are shown in section 4 and then this formulation is executed for a planer frame with 2-bay and 2-story. Also, an example is solved for a frame whose elements have a different plastic moment. Computational result of CPU time requirement is shown for different problems from 1bay-1floor frame up to 100bay-100 floor frame. In section 7, the model of genetic algorithm is described and the later examples are solved by this tool again to compare the results [8-10].

2. The Concept of Plastic Analysis and Design of Structures

In Plastic design, the target is finding the proper section for members while the loading and geometry are determined, but in plastic analysis the collapse load factor shall be calculated and the moment plastic of members are specified.
The following criteria shall be considered in the plastic analysis of structures:

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1. equilibrium equations
2. yield conditions
3. mechanism conditions

According to these criteria, the plastic analysis and design methods can be based on two main theorems: safe theorem and unsafe theorem.

In the safe analysis, when the full plastic moments of the members are known, an analysis can be carried out to determine the maximum load factor.

In this paper, the moment distribution method which is based on safe theorem is used for designing proposed mathematical modeling as described in following section.

In order to cast this mathematical modeling, the following assumptions are made:

1. The equilibrium equations are referred to the unreformed geometry of the structure.
2. The point loading is considered in the middle of frame span and the plastic hinges can be occurred in two ends or middle of members.
3. The loads are assumed to increase proportionally.
4. Effects of axial forces are neglected.

3. The Application of Moment Distribution Method for Plastic Analysis of a Planar Frame

The application of moment distribution method for plastic analysis of a planar frame will be illustrated by a 2-bay 1-story portal frame with fixed ends. The plastic moment of elements section is 30 Kn.m and effect of axial forces are neglected. The geometry and loading of frame is shown in Figure 1.

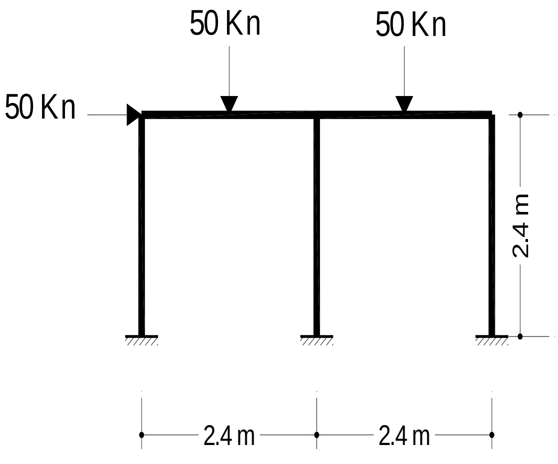


Fig. 1. The frame geometry and loading

At first, the end points of beams are fixed to determine local collapse moment in beams and then the assumed moments are considered at the end of columns to satisfying equilibrium of sway mechanism. For 50Kn horizontal load, these moments are 20Kn.m as shown in Figure 2.

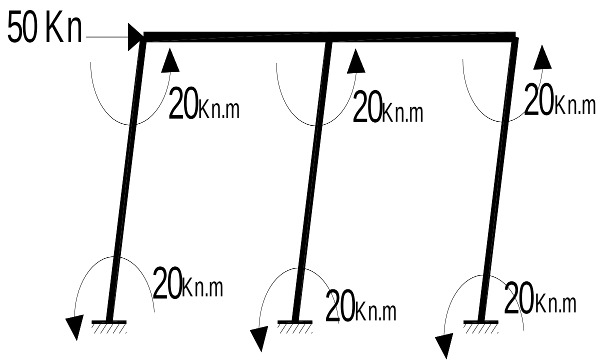


Fig. 2. Assumed moments for satisfying equilibrium of sway mechanism

In beams, balanced moment is transferred to middle of span so the equilibrium for vertical loads is satisfied but for satisfying equilibrium under horizontal loading, the sum of moment at two ends of column shall be zero. This summation is shown in column S as illustrated in Figure 3.

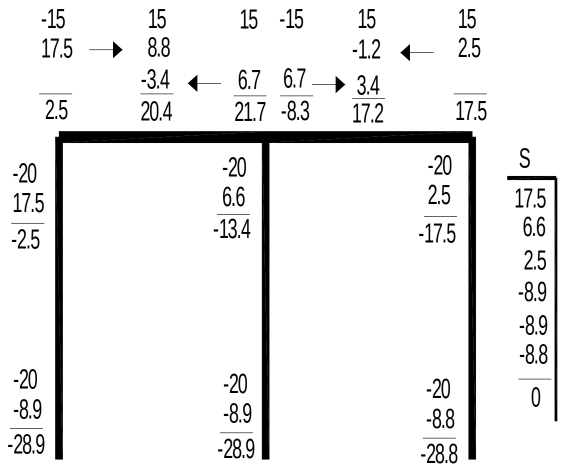


Fig. 3. Assumed moments for satisfying equilibrium under vertical & horizontal loading

In fig. 3, the 2.5, 6.6 and 17.5 Kn.m moments are balanced moments. By adding these moments in S column the total value is 266 Kn.m which is distributed by negative sign to end of column equally. So, the equilibrium will be satisfied. For this type of moment distribution, the maximum moment in frame is 28.9 Kn.m and the collapse load factor will be $\lambda_c \geq \frac{30}{28.9} = 1.04$.

Better collapse load factor will be obtained by changing the way of moment distribution. The repeated process is the same and results are shown in Figure 4 for 3 steps.

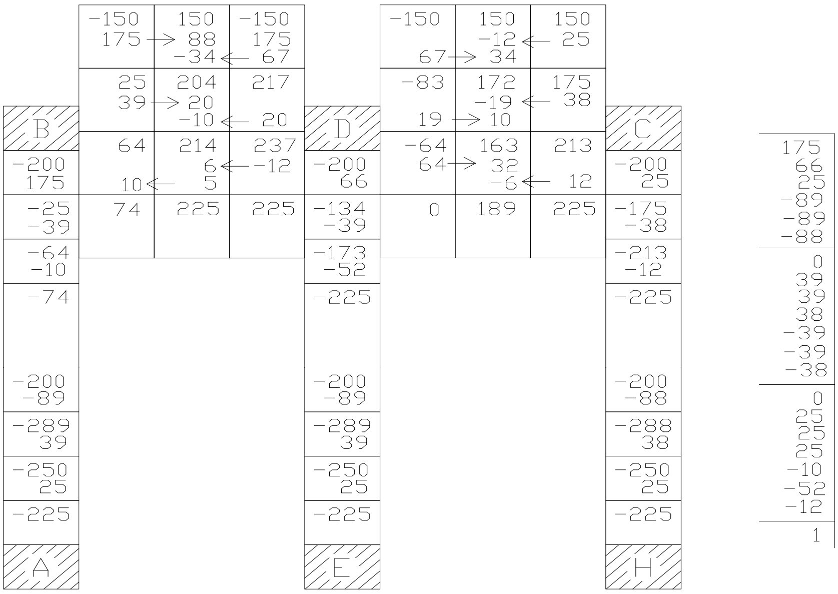


Fig. 4. Three different models for distribution of moment

According to the final results, the collapse load factor is $\lambda_c = \frac{30}{22.5} = 1.333$. It's collapse mechanism is shown in Figure 5.

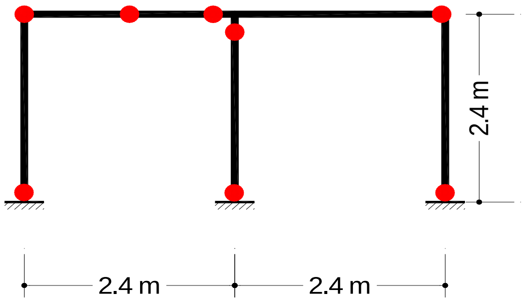


Fig. 5. Collapse mechanism

As this example illustrates, the moment distribution method is not automatic and it depends on user experience.

In this paper, a mathematical model is designed the concepts of which are derived from this method. This model is solved by linear programming for finding maximum allowable collapse load factor and also genetic algorithm as a heuristic method is applied to solve this mathematical model.

4. Formulating the Optimization Problem for Moment Plastic Distribution Method

First, ending nodes of beams are fixed against the rotation and the bending moments are calculated for local collapse through the beams.

In addition, balance in sway mechanism is satisfied under horizontal load condition by imposing imaginary moment at both top and bottom of columns. These values are constant and can be calculated based on geometrical structure and loading.

In this model, design variables are balanced moment. Balanced moment of beam and column are shown by two different variables (x, y) each having three Andes. As a result, each design variable seems a whole form of x_{ijk} or y_{ijk}.

i is the beam number or column number and j shows that balanced moment is located in which part of beam or column. For example, in each beam, balanced moment can be for beginning or end point of beam so j can be 1 or 2, respectively. K is the floor numbers which that beam or column belonged to.

In order to show the relation which will be described in the following section, columns and beams numbering is considered similar to Figure. 6.

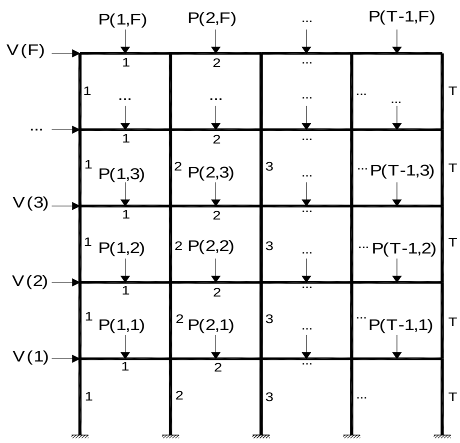


Fig. 6. Elements ad loading numbering

4.1. Objective function

In order to moment distribution method which is based on safety theorem, the target is achieving maximum collapse load factor. When complete plastic moment is constant for all members of frame, the collapse coefficient is calculated by dividing complete plastic moment by maximum bending moment in frame.

In this case, for achieving the maximum amount of λ the objective function can be defined as minimizing the maximum moment in the frame.

$$objective \rightarrow \min: Z$$

$$Z = \max\left(\left|m_{ijk} + x_{ijk}\right|, \left|m_{mnk} + y_{mnk}\right|\right)$$
$$\forall\ i \in \{1,2,\dots,T-1\}$$
$$\forall\ j \in \{1,2,3\}$$
$$\forall\ m \in \{1,2,\dots,T\}$$
$$\forall\ n \in \{1,2\}$$
$$\forall\ k \in \{1,2,3,\dots,F\}$$

(1)

For showing Z as the maximum moment in the frame the constraint is defined as follow.

$$\left|m_{ijk} + x_{ijk}\right| \le Z \quad \forall i, j, k$$
$$\left|m_{mnk} + y_{mnk}\right| \le Z \quad \forall m, n, k$$

It is important to say, M_{ijk} is that moment caused by fixed ends of the beams and M_{mnk} is the imposing imaginary moment for balance in sway mechanism under horizontal load, which these two values are constant and can be calculated at the first based on loading and geometry of structure.

When Mp is different for each elements of frame, recognizing the objective function and finding mathematic formulation is more complicated. However, in this paper, attempts are made to model the objective function as a linear format which is illustrated in the following lines.

When the plastic moment of members are different, the collapse load factor for each element of frame shall be calculated by dividing its Mp by the minimum bending moment in that part; this is called load coefficient of that special part. After calculation of this coefficient for all members of the frame, the minimum coefficient is the collapse load factor of the frame that in this model based on safe theorem, the target is maximizing this coefficient.

The mathematical modeling of this definition is illustrated in the following way.

$$\max \lambda$$
$$\lambda = \min \lambda_i$$
$$\lambda_i = \frac{Mp_i}{\min M_i}$$

Based on current information, the objective function will be as follows.

$$Max: \quad \min\left(\frac{Mp_i}{\min M_i}\right)$$

This format for objective function needs extra variables to explain minimum function which is used in objective function. That causes the Mathematical model of objective function turns to non- linear format.

For avoiding this problem, a technique is used to change the maximum objective function in to minimizing one, In order to make the objective function with linear format.

The objective function for maximization of λ is equal to the objective function for minimization of $\frac{1}{\lambda}$.

$$\max \lambda \text{ € } \min \frac{1}{\lambda}$$

And based on the former relations $\frac{1}{\lambda}$ is equal to:

$$\frac{1}{\lambda} = \max\left(\frac{1}{\lambda_i}\right)$$
$$\frac{1}{\lambda_i} = \frac{\min M_i}{Mp_i}$$
$$\frac{1}{\lambda} = \max\left(\frac{\min M_i}{Mp_i}\right)$$

So the new objective function will be:

$$Min: \frac{1}{\lambda}$$

This equals

$$Min: \quad \max\left(\frac{\min M_i}{Mp_i}\right)$$

As the $\frac{1}{\lambda}$ is the largest amount among all $\frac{\min M_i}{Mp_i}$, so it shall be larger or equal to each $\frac{\min M_i}{Mp_i}$ and because the objective function is the smallest amount of $\frac{1}{\lambda}$ by writing unequvalence $\frac{1}{\lambda} \ge \frac{\min M_i}{Mp_i}$, surely $\frac{1}{\lambda}$ will be the largest amount among all $\frac{\min M_i}{Mp_i}$. On the other hand if $\frac{1}{\lambda}$ is larger or equal to minimum amount of $\frac{M_i}{Mp_i}$, it will be larger than each amount of $\frac{M_i}{Mp_i}$. As the result, the unequvalance can be simplified as follow:

$$\frac{1}{\lambda} \ge \frac{M_i}{Mp_i}$$

Finally, the simplified objective function is shown below with a unequvalant constraint.

$$Min: \frac{1}{\lambda}$$
$$\frac{1}{\lambda} \ge \frac{M_i}{Mp_i}$$

Based on X and Y variables which are defined for this, explanation of the former mathematic objective function is like this:

$$Z = \frac{1}{\lambda}$$

Min: Z

$$Z \geq \frac{\left| m_{ijk} + x_{ijk} \right|}{BMp_{ik}} \quad \forall i, j, k$$

$$Z \geq \frac{\left| m_{mnk} + y_{mnk} \right|}{CMP_{mk}} \quad \forall m, n, k$$

In this formula, BMp_{ik} is the complete plastic moment of the i th beam in k th level and CMp_{mk} is the complete plastic moment of m th column in k th level [11,12].

4.2. Beam balance constraint

To keep the balance in each beam, the beginning, middle and the end of balanced moment should be determined like satisfying the following formula.

$$x_{i1k} - 2x_{i2k} - x_{i3k} = 0$$
$$\forall\ i \in \{1, 2, ..., T-1\} \qquad (2)$$
$$\forall\ k \in \{1, 2, 3, ..., F\}$$

This formula is obtained from the free diagram of beam which is shown in Figure 7.

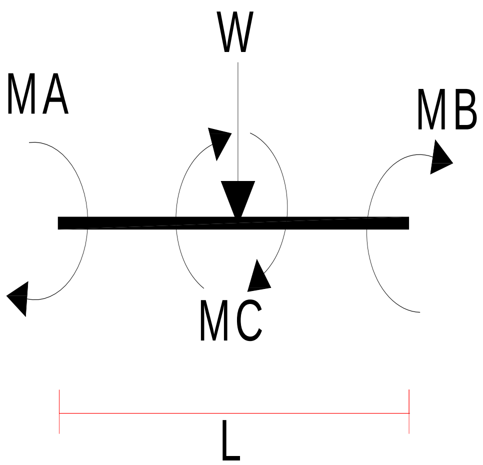


Fig. 7. Beam free diagram

$$\frac{WL}{4} = \frac{M_B - M_A}{2} + M_C$$
$$M_A - M_B - 2M_C = \frac{-WL}{2}$$
$$\Delta M_A - \Delta M_B - 2\Delta M_C = 0$$
$$if\ \ \Delta M_B = 0\ \ then\ \ \Delta M_c = \frac{1}{2}\Delta M_A$$
$$if\ \ \Delta M_A = 0\ \ then\ \ \Delta M_c = \frac{-1}{2}\Delta M_A$$
$$if\ \ \Delta M_C = 0\ \ then\ \ \Delta M_A = \frac{1}{2}\Delta M_B \qquad (3)$$

Beam balancing constraint which was explained at the beginning of this section is a general form of the above formula.

4.3. Balance constraint for sway mechanism

Similar to section 3 where the balance in sway mechanism is controlled by formation column S, the mathematical definition of this procedure is as follows.

$$\sum_{i=1}^{T}\sum_{j=1}^{2} y_{ijk} = 0 \qquad (4)$$
$$\forall\ k \in \{1, 2, ..., F\}$$

4.4. Node balance constraint

In addition to the above constraints, the sum of moments in each node should be equal to zero in order to keep the node balance. The mathematical definition of this constraint is considered below:

$$y_{(i+1)2k} - \frac{\sum_{m=k}^{F} V_m \times h_m}{2T} + y_{(i+1)|(k+1)} - \frac{\sum_{m=k+1}^{F} V_m \times h_m}{2T} + x_{(i+1)1k} - \frac{P_{(i+1)k} \times L}{8} + x_{i3k} + \frac{P_{ik} \times L}{8} = 0$$
$$\forall\ 1 \leq i \leq T-2$$
$$\forall\ 1 \leq k \leq F$$

(5) Inter-nodes balance relationship

The above formula is related to inter nodes which is connected to the beam form two sides and connected to the column from the two other sides.

In the special case for side nodes and up level nodes, the mentioned formula changes to the following forms.

The first nodes on the left side:

$$y_{12K} - \frac{\sum_{m=k}^{F} V_m \times h_m}{2T} + y_{11(K+1)} - \frac{\sum_{m=k+1}^{F} V_m \times h_m}{2T} + X_{11K} - \frac{P_{1K}\ L_1}{8} = 0 \ (6)$$
$$\forall\ k \leq F-1$$

Ending nodes on the right side:

$$y_{12K} - \frac{\sum_{m=k}^{F} V_m \times h_m}{2T} + y_{T1(K+1)} - \frac{\sum_{m=k+1}^{F} V_m \times h_m}{2T} + x_{(T-1)3k} + \frac{P_{(T-1)K}\ L_{(T-1)}}{8} = 0 \qquad (7)$$
$$\forall\ k \leq F-1$$

First nodes on the left side of the up level:

$$y_{T2F} - \frac{V_F \times h_F}{2T} + x_{11F} - \frac{P_{1F}\ L_1}{8} = 0 \qquad (8)$$

First nodes on the right side of the up level:

$$y_{T\,2F} - \frac{V_F \times h_F}{2T} + x_{(T-1)3F} - \frac{P_{(T-1)F}\, L_{(T-1)}}{8} = 0 \quad (9)$$

Inter nodes of the up level:

$$y_{(i+1)2F} - \frac{V_F \times h_F}{2T} + x_{(i+1)1F} - \frac{P_{(i+1)F} \times L}{8} + x_{i\,3F} + \frac{P_{iF} \times L}{8} = 0 \quad (10)$$
$$\forall\ 1 \le i \le T-2$$

4.5. Yield constraint

To satisfy the yield constraint, the moment shouldn't be more than complete plastic moment of member in every section.
In case where plastic moment is constant for all members, for satisfying this constraint, the objective function (z) as the maximum moment in the frame shall be smaller than the complete plastic moment. Therefore, this constraint can be defined as follows:

$$|Z| \le m_p \quad (11)$$

But in the case where plastic moments are different in each members of frame, Yield condition should be checked for each member by following formula:

$$|m_{i\,j\,k} + x_{i\,j\,k}| \le BMp_{ik} \quad \forall i, j, k$$
$$|m_{i\,j\,k} + x_{i\,j\,k}| \le CMp_{ik} \quad \forall i, j, k \quad (12)$$

5. Example Solution and Numerical Results

According to the achieved constraint in the previous part and objective function, the proposed model is in linear programming format, so the LINGO software is used to solve the different examples.
In the first example, according to objective function and constraints formulation which are explained in the former part, for a 2-floor and 2-bays frame, the objective function and constraints are shown and then the result is expressed but in other examples only the final results are provided.

Example 1: the collapse load factor for the frame in Figure 8 is calculated. The plastic moment of the members is constant and equal to 10 Kn.m.

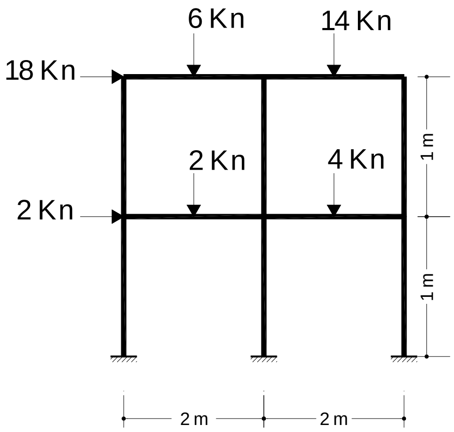


Fig. 8. The geometry & loading of frame example 1

$$\lambda = 2.6$$
$$M\ \max = 3.85$$

By means of formula (2), balancing constraints is defined as the following:

$$X\ (1,1,1) - 2X\ (1,2,1) - X\ (1,3,1) = 0$$
$$X\ (1,1,2) - 2X\ (1,2,2) - X\ (1,3,2) = 0$$
$$X\ (2,1,1) - 2X\ (2,2,1) - X\ (2,3,1) = 0$$
$$X\ (2,1,2) - 2X\ (2,2,2) - X\ (2,3,2) = 0$$

Balance constraints in sway mechanism are explained with formula 4 as below:
First level:

$$Y\ (1,1,1) + Y\ (1,2,1) + Y\ (2,1,1) + Y\ (2,2,1) + Y\ (3,1,1) + Y\ (3,2,1) = 0$$

Second level:

$$Y\ (1,1,2) + Y\ (1,2,2) + Y\ (2,1,2) + Y\ (2,2,2) + Y\ (3,1,2) + Y\ (3,2,2) = 0$$

Balance node constraints are mentioned with formulas of section (4-4) as below.
Inter-node:

$$Y\ (2,1,2) + Y\ (2,2,1) + X\ (1,3,1) + X\ (2,1,1) = .333333$$

First nodes on the left:

$$Y\ (1,1,2) + Y\ (1,2,1) + X\ (11,1) + X\ (1,1,1) = 6.833333$$

First nodes on the right:

$$Y\ (3,1,2) + Y\ (3,2,1) + X\ (2,3,1) = 5.333333$$

First nodes on the left in the up level:

$$Y\ (1,2,2) + X\ (1,1,2) = 4.5$$

End nodes on the right in the up level:

$$Y\ (3,2,2) + X\ (2,3,2) = -.5$$

Inter-nodes in the up level:

$$Y\ (2,2,2) + X\ (1,3,2) + X\ (2,1,2) = 3$$

According to geometry of structure and its loading which is shown in figure 9, the beam fixed moments and imposed imaginary moments in columns for causing balance in sway mechanism are as follow.
For beams:

$M\ (1,1,1) = -2*2/8 = -0.5$
$M\ (1,2,1) = 2*2/8 = 0.5$
$M\ (1,3,1) = 2*2/8 = 0.5$
$M\ (2,1,1) = -4*2/8 = -1$
$M\ (2,2,1) = 4*2/8 = 1$
$M\ (2,3,1) = 4*2/8 = 1$
$M\ (1,1,2) = -6*2/8 = -1.5$
$M\ (1,2,2) = 6*2/8 = 1.5$
$M\ (1,3,2) = 6*2/8 = 1.5$

For columns:

$M\ (1,1,1) = -(18+2)*1/6 = -3.333$
$M\ (1,2,1) = -(18+2)*1/6 = -3.333$
$M\ (2,1,1) = -(18+2)*1/6 = -3.333$
$M\ (2,2,1) = -(18+2)*1/6 = -3.333$
$M\ (3,1,1) = -(18+2)*1/6 = -3.333$
$M\ (3,2,1) = -(18+2)*1/6 = -3.333$
$M\ (1,1,2) = -18*1/6 = -3$ And
$M\ (1,2,2) = -18*1/6 = -3$
$M\ (2,1,2) = -18*1/6 = -3$
$M\ (2,2,2) = -18*1/6 = -3$

After executing the model in LINGO 8.0 software collapse load factor equals $\lambda = 2.6$ and the final mechanism is as figure 9.

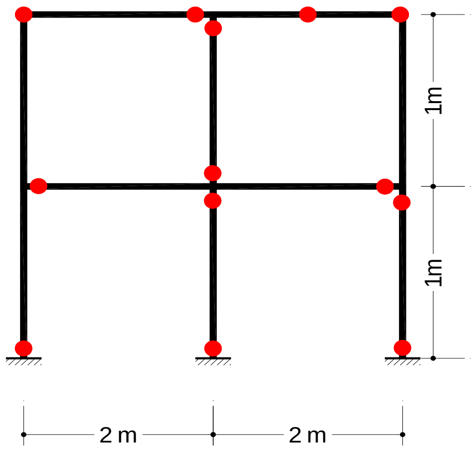


Fig. 9. The final mechanism

Example 2: For a four-floor four-bay frame with the loading is shown in fig. 10, the mentioned mathematical modeling is used, Units are Kn.m.

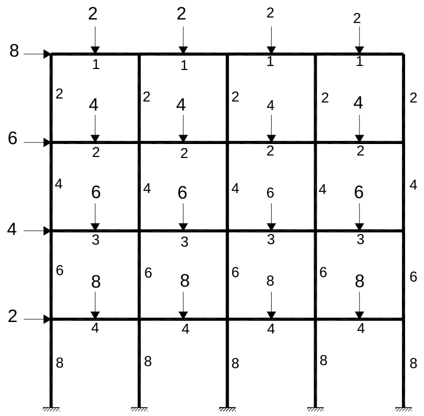


Fig. 10. The loading and plastic moment of frame example 2

Collapse load coefficient equals to $\lambda = 0.6511$ and final mechanism is similar to fig. 11.

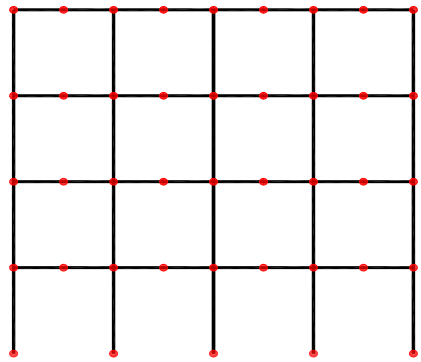


Fig. 11. Final collapse mechanism of frame example 2

6. Large Scale Examples and the Computational Required Time

In this section different examples, from simple case (1-bay 1-floor frame) up to large scale model (100–bay 100-floor frame), are analyzed by proposed method, results are shown in the below chart.

In order to draw elapsed time diagram for different moods (number of variables), it starts from a small example in the form of one-level one-bay frame and the number of bays and levels is increased 10 numbers in each step to achieve a curve with enough accuracy, Units are Kn.m.

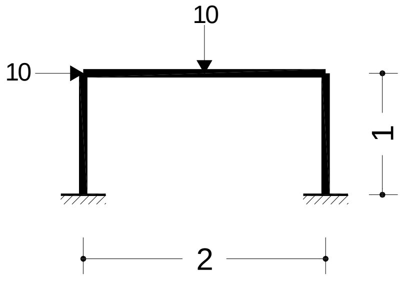


Fig. 12. 1-bay 1-floor frame example

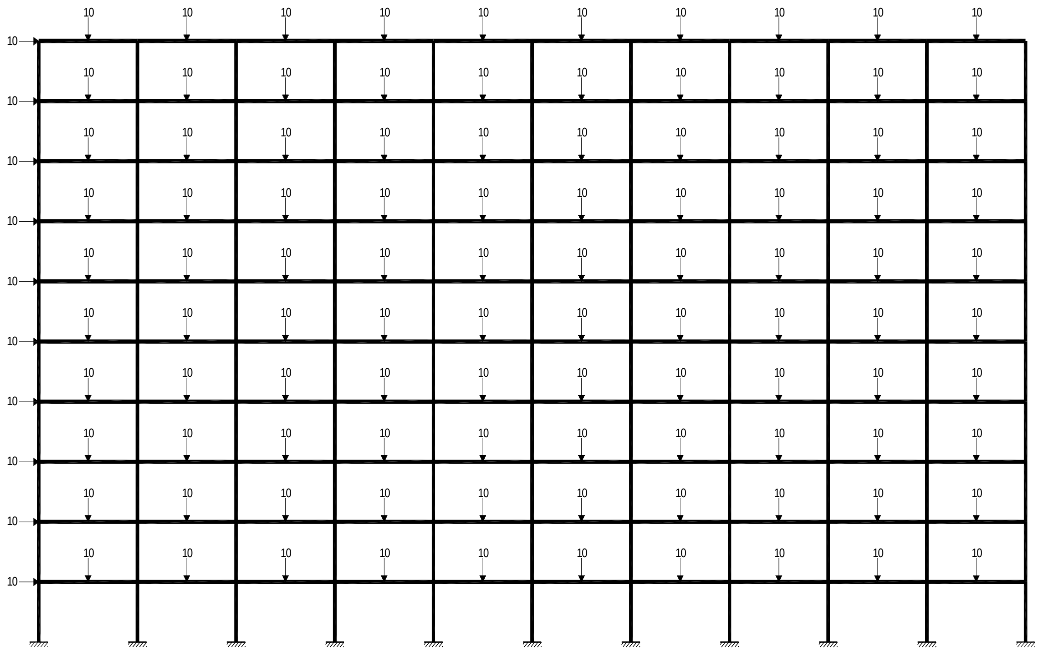


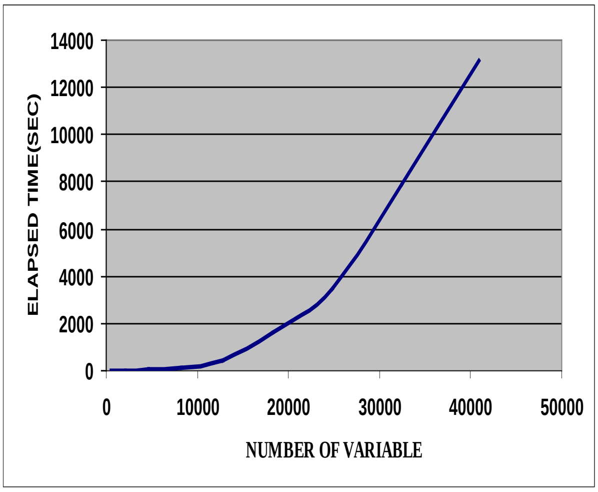
Fig. 13. 10-bay 10-floor frame example

In figure 12, example of one-bay one-floor frame and in figure 13, the example of 10-bay 10-level frame are shown. The procedure of designing of other examples is similar to these figures. The results are mentioned in below table.

Table1. Number of Variables & Elapsed Time Table

number of Floor number of Bay	Elapsed Time (hh:mm:ss)	Elapsed Time (sec)	Number of Variables	Number of Constraints
1	00:00:00	0	11	33
10	00:00:00	0	551	2301
20	00:00:11	11	2101	9001
30	00:00:34	34	4651	20101
40	00:02:18	138	8201	35601
50	00:07:03	423	12751	55501
60	00:31:19	1879	18301	79801
70	00:58:32	3512	24851	108501
80	02:11:21	7881	32401	141601
90	03:39:05	13145	40951	179101

As it's illustrated in this table, the model works for about 50000 variables in acceptable time. Below elapsed time – variables curve is shown.



Curve1. Elapsed time – variables curve

7. Genetic Algorithm Model

For comparing results, genetic algorithm as meta-heuristic method is applied [13]. The concepts of making genetic models are:

7.1. Population

In this model, 20 participants were taken in to account for the initial sampling for convergence purposes.

7.2. Chromosome

In this model, the optimization variables were selected directly with a deformed shape as the chromosome length; so that the objective function is a function of X and Y variables. In the meantime, each of the variables X, and Y were selected in the form of three-dimensional matrices with the dimension $i \times j \times k$, $m \times n \times k$ respectively. In order to, a chromosome with a length of $i \times j \times k + m \times n \times k$ in a cluster format was selected for measuring the objective function.

7.3. Cross-section

According to the nature of the problem, the cross-section operator is considered as two-point cross-section.

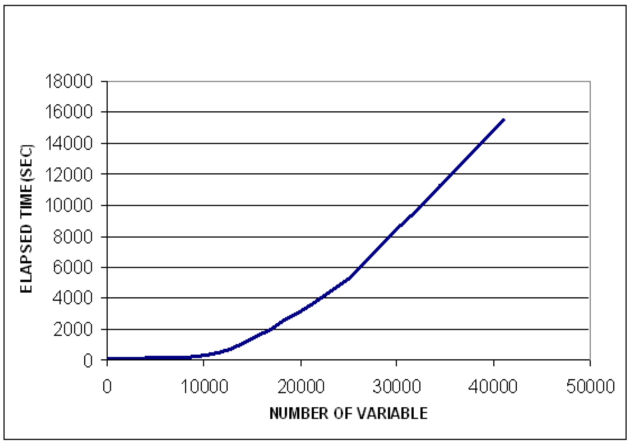
7.4. Mutation

In this model, the mutation is considered as a mutation adopts fissile point, because this will cause a non-linear constrained solution for the problem.

The examples of section 6 are repeated by genetic algorithm model. The table and diagram illustrating the results are provided below.

Table2. Number of Variables & Elapsed Time Table

number of Floor number of Bay	Elapsed Time (hh:mm:ss)	Elapsed Time (sec)	Number of Variables	Number of Constraints
1	00:00:05	5	11	33
10	00:00:40	40	551	2301
20	00:01:11	71	2101	9001
30	00:02:14	134	4651	20101
40	00:03:35	215	8201	35601
50	00:12:03	723	12751	55501
60	00:42:19	2539	18301	79801
70	01:28:12	5292	24851	108501
80	02:45:28	9928	32401	141601
90	04:19:05	15545	40951	179101



Curve2. Elapsed time – variables curve

8. Concluding Remarks

The present method for plastic analysis of frames and finding collapse load factor is based on moment distribution method as one of the safe theorem methods.
Here, only point loads are considered, however, distributed loads can conveniently be replaced by equivalent concentrated loads as discussed in classical books on plastic analysis and design.
The main differences between the present method and the existing algorithms can be summarized as follows:

1. The moment distribution method is not automatic and it depends on user experience, but in the proposed method, the moments are distributed automatically between members such as collapse load factor is the maximum allowable value.
2. As the objective function and constraints of this model are designed in linear format, so it can be solved by linear programming (LP) software that causes the exact solution and requires less computational time.
3. For this proposed model, we tried to design an optimization model in linear format in order to solve large scale samples as it is shown in the paper for 100bay-100story frame with 50,000 variables.
4. The proposed model has been solved with two methods, genetic algorithm (GA) and linear programming (LP), and the results have been compared. Since, in this model, objective function and constrains are all linear, solution with linear programming method is more accurate and less time consuming.Also the number of initial population in genetic algorithm method is equal to 20, which causes the method needs more time to yield appropriate results.

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