Free Vibration Analysis of Thick Functionally Graded Rectangular Plates Using Variable Refined Plate Theory

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Abstract: In this paper, free vibration of functionally graded (FG) rectangular simply supported thick plates based on two variable-refined plate theory is presented. According to a power-law distribution, the mass density and elasticity modulus of the plate are considered to vary while Poisson's ratio is constant. In order to extract the five constitutive equations of motion, Hamilton principle is employed. The high accuracy of this theory is investigated by comparing the exact results reported by higher order shear deformation theory. Furthermore, parametric study of non-dimensional natural frequencies is carried out and the influences of geometrical parameters such as aspect ratios of the plate on these frequencies are studied.

Keywords: Free Vibration, Functionally Graded Rectangular Plates, Small Deformation, Refined Plate Theory

1. Introduction

Functionally graded materials (FGMs) are a novel class of composite structures which have great application for engineering design and manufacture. First, they are a new generation of composite structures presented by a group of Japanese scientists [1, 2]. These materials have desirable properties for specific applications under stress concentration, particularly for aircrafts, mechanical, electronic and other engineering structures. In order to solve plate problems, the choice of the plate theory and the type of solution method are necessary. Through the study of comprehensive survey of literature, it is found that many researches have been carried out on free vibration of the FG plates and most of them emphasized numerical methods [3-7]. Free vibration of simply supported and clamped rectangular thin FG plates was presented by Abrate [6] based on the classical plate theory.

In another work, Abrate [7] also investigated free vibration, buckling and static deflections of different shapes of FG, such as square, circular and skew plates with several boundary conditions on the basis of the classical plate theory, first order shear deformation theory and third order shear deformation theory. Zhao et al. [8] have presented a free vibration analysis for FG square and skew plates with different boundary conditions using the element-free *kp*-Ritz method on the basis of the FSDT. Kant and Mallikarjuna [9] presented a higher-order theory for free vibration analysis of un-symmetrically laminated multilayered plates in such a way that the transverse shear strains through the thickness of the plate and rotary inertia effects have been considered. Matsunaga [10] presented natural frequencies and buckling stresses of simply supported FG plates based on 2D higher-order approximate plate theory. Hosseini Hashemi et al. [11] analyzed a new exact closed-form procedure to solve free vibration analysis of functionally graded rectangular thick plates using the third-order shear deformation plate theory (TSDT) for a plate with two opposite simply supported edges.

Hosseini-Hashemi et al. [12] also presented an analytical method for free vibration analysis of moderately thick rectangular FG plates which has been supported by either Winkler or Pasternak elastic foundations. Hosseini-Hashemi et al. [13] studied exact free vibration of Reissner–Mindlin FG plates with the help of the Levy-type solutions related to the cases where two opposite edges are simply supported and the other edges of the plate can have different boundary

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conditions. Seung-Eock et al. [14] developed the two variable refined plate theory (RPT2) for plates which are under the action of the transverse and in-plane forces and obtained the stiffness and mass matrices. They compared the non-dimensional deflection obtained by various theories namely the classical laminate plate theory, the first order shear deformation theory, the higher order shear deformation theory and the refined plate theory. Seung-Eock et al. [15] also carried out buckling analysis of isotropic and orthotropic plates using the foregoing theory.

The purpose of this paper is to develop the RPT2 for FG plates. The present theory satisfies equilibrium conditions at the top and bottom faces of the plate without considering the shear correction factors. The equations of motion of the plate are obtained with the help of calculus of variations. To illustrate the accuracy of the present theory, the obtained results are compared with three dimensional elasticity solutions obtained by the first-order and the higher-order theories.

2. Basic formulation

2.1. Material properties of P-FGM plates

A flat and moderately thick functionally rectangular plate with length, width and uniform thickness equal to *a*, *b* and *h* respectively is shown in Fig. 1. The plate is assumed to have simply supported boundaries. Considering of the volume fraction of the constituent material law [16], the young's modulus, $E(z)$ and density, $\rho(z)$ of FG plates can be concluded as functions of thickness coordinates, z, as below:

$$E(z) = E_m + (E_c - E_m)\left(\frac{1}{2} + \frac{z}{h}\right)^p \qquad (1)$$

$$\rho(z) = \rho_m + (\rho_c - \rho_m)\left(\frac{1}{2} + \frac{z}{h}\right)^p$$

The subscripts m and c show the metallic and ceramic constituent, respectively. *p* is the power law index which takes positive values. The variation of Young's modulus in the P-FGM plates is shown in Fig. 2. Other types of FGM materials such as S-FGM and E-FGM can be found in [17].

2.2. Refined plate theory

A rectangular plate with length, width and thickness equal to a, b and h respectively is considered. In order to proceed with the formulation of the problem using the two-variable refined plate theory (RPT2), it is assumed that the displacements (u, v, w) of the plate are small in comparison with the thickness of the plate; hence the strains involved are considered to be infinitesimal. On the other hand, the transverse normal stress in the z-direction, σ_z, is very small in comparison with the in-plane stresses, σ_x and σ_y. So, with above definitions, the stress – strain relations can be reduced from a 6×6 matrix to a 5×5 matrix which can reduce the complexity of the problem. The total displacement of the plate in the z-direction (W) is assumed to be consisting of three components of extension w^a, bending w^b and shear w^s terms which are functions of x, y and the time. The displacements in the x, y and z-directions are also defined as [14]:

$$U(x,y,z,t) = u(x,y,t) - z\frac{\partial w^b(x,y,t)}{\partial x}$$

$$+\left[\frac{1}{4}z - \frac{5}{3h^2}z^3\right]\frac{\partial w^s(x,y,t)}{\partial x} \qquad (2)$$

$$V(x,y,z,t) = v(x,y,t) - z\frac{\partial w^b(x,y,t)}{\partial y}$$

$$+\left[\frac{1}{4}z - \frac{5}{3h^2}z^3\right]\frac{\partial w^s(x,y,t)}{\partial y}$$

$$W(x,y,z,t) = w^a(x,y,t) + w^b(x,y,t) + w^s(x,y,t)$$

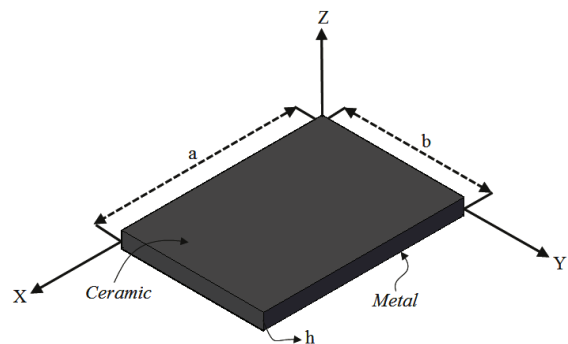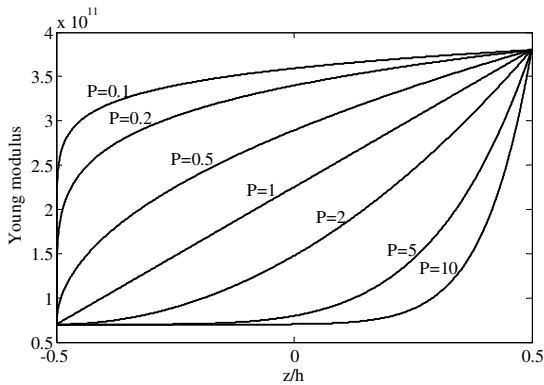**Fig. 1. CAD model of FGM rectangular plate.**

Fig. 2. Variation of Young modulus in a P-FGM plate.

Where u, v, w^a, w^b and w^s are the mid-plane displacements and t is the time. The strain displacement relations are given as:

$$\varepsilon_x = \frac{\partial U}{\partial x}, \varepsilon_y = \frac{\partial V}{\partial y}, \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right),$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right), \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right) \quad (3)$$

Based on Hooke's law, the stress displacement relations are defined as:

$$\sigma_x = \frac{E(z)}{1-v^2}\left[\varepsilon_x + v\varepsilon_y\right], \sigma_y = \frac{E(z)}{1-v^2}\left[\varepsilon_y + v\varepsilon_x\right], \sigma_{xy} = 2\left(\frac{E(z)}{2(1+v)}\right)\varepsilon_{xy}$$

$$\sigma_{xz} = 2\left(\frac{E(z)}{2(1+v)}\right)\varepsilon_{xz}, \sigma_{yz} = 2\left(\frac{E(z)}{2(1+v)}\right)\varepsilon_{yz} \quad (4)$$

3. Governing equation

It can be verified that the first variation of the Lagrangian, $L = T - U$ (i.e., Hamilton's principle) leads to the equations of motion. In this study, U_{plate} denotes the total strain energy and T is the kinetic energy of the plate can be written as:

$$T = \frac{\delta}{2}\int_0^t \left(\int_{-h/2}^{h/2}\int_0^b\int_0^a \rho(z)\left(\dot{U}^2 + \dot{V}^2 + \dot{W}^2\right)dxdydz\right)dt \quad (5)$$

$$U_{plate} = \int_0^t\int_{-h/2}^{h/2}\int_0^b\int_0^a \left(\begin{array}{c}\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + 2\sigma_{xy}\delta\varepsilon_{xy} \\ + 2\sigma_{xz}\delta\varepsilon_{xz} + 2\sigma_{yz}\delta\varepsilon_{yz}\end{array}\right)dxdydzdt$$

Substituting Eqs. (2) and (3) into Eq. (5) and collecting the coefficients of $\delta u, \delta v, \delta w^a, \delta w^b$, and δw^s,

the following equations of motion for FG plate are given as [14]:

$$\delta u \rightarrow \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_1\ddot{u} \quad (6)$$

$$\delta v \rightarrow \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = I_1\ddot{v}$$

$$\delta w^b \rightarrow \frac{\partial^2 M_x^b}{\partial x^2} + \frac{\partial^2 M_y^b}{\partial y^2} + 2\frac{\partial^2 M_{xy}^b}{\partial x\partial y} = I_1\ddot{W} - I_3\nabla^2\ddot{w}^b$$

$$\delta w^s \rightarrow \frac{\partial^2 M_x^s}{\partial x^2} + \frac{\partial^2 M_y^s}{\partial y^2} + 2\frac{\partial^2 M_{xy}^s}{\partial x\partial y} + \frac{\partial Q_{xz}^s}{\partial x} + \frac{\partial Q_{yz}^s}{\partial y} = I_1\ddot{W} - \frac{I_3}{84}\nabla^2\ddot{w}^s$$

$$\delta w^a \rightarrow \frac{\partial Q_{xz}^a}{\partial x} + \frac{\partial Q_{yz}^a}{\partial y} = I_1\ddot{W}$$

Where I_1, I_3 are the inertia terms and define as below:

$$(I_1, I_3) = \int_{-h/2}^{h/2}\rho(z)\left(1, z^2\right)dz \quad (7)$$

Where dot-operator convention shows the differentiation with respect to the time. The stress resultants N_i, M_i^b, M_i^s, Q_i^a and Q_i^s are given as [14]:

$$\left(N_x, N_y, N_{xy}\right) = \int_{-h/2}^{h/2}\left(\sigma_x, \sigma_y, \sigma_{xy}\right)dz \quad (8)$$

$$\left(M_x^b, M_y^b, M_{xy}^b\right) = \int_{-h/2}^{h/2}\left(\sigma_x, \sigma_y, \sigma_{xy}\right)zdz$$

$$\left(M_x^s, M_y^s, M_{xy}^s\right) = \int_{-h/2}^{h/2}\left(\sigma_x, \sigma_y, \sigma_{xy}\right)\left(\frac{-z}{4} + \frac{5z^3}{3h^2}\right)dz$$

$$\left(Q_{xz}^a, Q_{yz}^a\right) = \int_{-h/2}^{h/2}\left(\sigma_{xz}, \sigma_{yz}\right)dz$$

$$\left(Q_{xz}^s, Q_{yz}^s\right) = \int_{-h/2}^{h/2}\left(\sigma_{xz}, \sigma_{yz}\right)\left(\frac{5}{4} - \frac{5z^2}{h^2}\right)dz$$

Where *N*, *M* and *Q* are the stress resultants and defined in the Appendix. Inserting Eq. (A1) into Eq. (6), the governing equation of FG plate can be written as:

$$A_{11}\frac{\partial^2 u}{\partial x^2} + A_{66}\frac{\partial^2 u}{\partial y^2} + (A_{12} + A_{66})\frac{\partial^2 v}{\partial x\partial y}$$

$$-\left(B_{11}\frac{\partial^3 w^b}{\partial x^3} + (B_{12} + 2B_{66})\frac{\partial^3 w^b}{\partial x\partial y^2}\right) - \quad (9)$$

$$\left(B_{11}^s\frac{\partial^3 w^s}{\partial x^3} + (B_{12}^s + 2B_{66}^s)\frac{\partial^3 w^s}{\partial x\partial y^2}\right) = I_1\ddot{u}$$

$$A_{22}\frac{\partial^2 v}{\partial y^2} + A_{66}\frac{\partial^2 v}{\partial x^2} + (A_{12}+A_{66})\frac{\partial^2 u}{\partial x \partial y}$$

$$-\left(B_{22}\frac{\partial^3 w^b}{\partial y^3} + (B_{12}+2B_{66})\frac{\partial^3 w^b}{\partial x^2 \partial y}\right)-$$

$$\left(B_{22}^s\frac{\partial^3 w^s}{\partial y^3} + (B_{12}^s+2B_{66}^s)\frac{\partial^3 w^s}{\partial x^2 \partial y}\right) = I_1\ddot{v}$$

$$B_{11}\frac{\partial^3 u}{\partial x^3} + B_{22}\frac{\partial^3 v}{\partial y^3} + (B_{12}+2B_{66})\left(\frac{\partial^3 u}{\partial x \partial y^2}+\frac{\partial^3 v}{\partial x^2 \partial y}\right)-$$

$$\left(D_{11}\frac{\partial^4 w^b}{\partial x^4} + D_{22}\frac{\partial^4 w^b}{\partial y^4} + 2(D_{12}+2D_{66})\frac{\partial^4 w^b}{\partial x^2 \partial y^2}\right)-$$

$$\left(D_{11}^s\frac{\partial^4 w^s}{\partial x^4} + D_{22}^s\frac{\partial^4 w^s}{\partial y^4} + 2(D_{12}^s+2D_{66}^s)\frac{\partial^4 w^s}{\partial x^2 \partial y^2}\right) = I_1\ddot{W} + I_3\nabla^2\ddot{w}^b$$

$$B_{11}^s\frac{\partial^3 u}{\partial x^3} + B_{22}^s\frac{\partial^3 v}{\partial y^3} + (B_{12}^s+2B_{66}^s)\left(\frac{\partial^3 u}{\partial x \partial y^2}+\frac{\partial^3 v}{\partial x^2 \partial y}\right)-$$

$$\left(D_{11}^s\frac{\partial^4 w^b}{\partial x^4} + D_{22}^s\frac{\partial^4 w^b}{\partial y^4} + 2(D_{12}^s+2D_{66}^s)\frac{\partial^4 w^b}{\partial x^2 \partial y^2}\right)-$$

$$\left(H_{11}^s\frac{\partial^4 w^s}{\partial x^4} + H_{22}^s\frac{\partial^4 w^s}{\partial y^4} + 2(H_{12}^s+2H_{66}^s)\frac{\partial^4 w^s}{\partial x^2 \partial y^2}\right) + A_{55}^a\frac{\partial^2 w^a}{\partial x^2} +$$

$$A_{44}^a\frac{\partial^2 w^a}{\partial y^2} + A_{55}^s\frac{\partial^2 w^s}{\partial x^2} + A_{44}^s\frac{\partial^2 w^s}{\partial y^2} = I_1\ddot{W} + \frac{I_3}{84}\nabla^2\ddot{w}^s$$

$$A_{55}\frac{\partial^2 w^a}{\partial x^2} + A_{44}\frac{\partial^2 w^a}{\partial y^2} + A_{55}^a\frac{\partial^2 w^s}{\partial x^2} + A_{44}^a\frac{\partial^2 w^s}{\partial y^2} = I_1\ddot{W}$$

4. Boundary condition

Here, boundary conditions for FG plates with simply supported boundary at all edges are considered as:

For x=0,a $\qquad\qquad$ (10)

$$v = w^a = w^b = w^s = \frac{\partial w^a}{\partial y} = \frac{\partial w^b}{\partial y} = \frac{\partial w^s}{\partial y} = N_x = M_x^b = M_x^s = 0$$

For y=0,b

$$u = w^a = w^b = w^s = \frac{\partial w^a}{\partial x} = \frac{\partial w^b}{\partial x} = \frac{\partial w^s}{\partial x} = N_y = M_y^b = M_y^s = 0$$

To solve this problem with the given boundary conditions, Navier solution procedure is employed for satisfying the following expansion:

$$\begin{Bmatrix} u \\ v \\ w_b \\ w_s \\ w_a \end{Bmatrix} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{Bmatrix} U_{mn}\cos\alpha_m x \sin\beta_n y \\ V_{mn}\sin\alpha_m x \cos\beta_n y \\ W_{bmn}\sin\alpha_m x \sin\beta_n y \\ W_{smn}\sin\alpha_m x \sin\beta_n y \\ W_{amn}\sin\alpha_m x \sin\beta_n y \end{Bmatrix} \qquad (11)$$

Where $\alpha = m\pi/a$, $\beta = n\pi/b$, U_{mn}, V_{mn}, W_{bmn}, W_{smn} and W_{amn} are coefficients. Finally, the Eignvalue problem for vibration of the plate is obtained as:

$$\left([K]-[M]\,\omega^2\right)\{\lambda\} = \{0\} \qquad (12)$$

Where $[K], [M]$, ω and λ are the stiffness, mass matrices, natural frequency and the vector of unknown coefficients respectively.

For convenience, the fundamental frequency parameter (Ω) and the non-dimensional natural frequency parameter $(\hat{\Omega})$ of the plate are given respectively as [13]:

$$\Omega = \omega\frac{a^2}{h}\sqrt{\frac{\rho_c}{E_c}}, \quad \hat{\Omega} = \omega h\sqrt{\frac{\rho_c}{E_c}} \qquad (13)$$

5. Numerical results and discussion

This section illustrates the efficiency and stability of proposed theory. Using the previously developed analytical solutions based on the FSDT and HSDT, numerical parametric studies based on RPT2 are presented. In this study, the FG plate is taken to be made of two types of material (i.e., Al/Al_2O_3 and Al/ZrO_2) which the material properties are in Table 1. In order to investigate the efficiency and accuracy of the presented theory, the results are compared with isotropic FG plates. Here, for generality and convenience, two non-dimensional terms, thickness ratio (τ) and aspect ratio (η) are defined as h/a and b/a respectively. As a matter of fact, when the power law index (p) approaches to zero, the plate is isotropic fully ceramic and when this index approaches to ∞, the plate is isotropic and fully metallic.

To examine free vibrations of the plate, the values of in-plane distributed loads are set to zero. Table 2 shows the comparison of the fundamental frequency parameter for the homogenous isotropic Al/Al_2O_3 square plates for $\tau = 0.1$ and $\tau = 0.2$. The results are compared with exact higher order shear deformation theory (HSDT) [13,18] which shows a very good agreement. Table 3 shows the comparison of the non-dimensional natural frequency parameter $(\hat{\Omega})$ for Al/Al_2O_3 square plate with those results obtained based on HSDT [13, 19] (Exact solution) and FSDT [20] for $p = 0, 0.5, 1, 4, 10$ and ∞.

This table shows that there is a very good agreement between the results of RPT2 and those of exact analytical solution using higher order shear deformation plate theory [13,19]. In this Table, *m* and *n* are the

corresponding mode shapes which are defined as numbers of half waves in X and Y-directions, respectively. The first three natural frequency parameters for six different values of the power law index (i.e., $p = 0, 0.5, 1, 2, 5, 10$) are obtained and the numerical results are compared with exact closed form solutions [13] in Table 4, whereas the considered plate is made of Al / Al_2o_3. Considering the obtained results, all natural frequencies decrease when the power law index increases. The reason is the fact that, the natural frequency parameters of FG plate are directly dependent on Young's modulus. Therefore, by increasing the power law index, the elastic modulus of FG plate decreases. Moreover, as can be seen in this table, all the natural frequency parameters rise with an increase in the thickness ratio of the plate.

This is undoubtedly due to increasing the rigidity of the plate. As depicted in Table 5, the results of non-dimensional natural frequency $(\hat{\Omega})$ for SSSS Al / ZrO_2 square thin and moderately thick plate are compared with those obtained by HSDT [13, 19], 3D theory by employing the power series method [21] and finite element HSDT method [22].

The influence of the aspect ratio on the fundamental frequency parameter (Ω) for constant values of thickness ratio $(\tau = 0.2)$ and power law index $(p = 1)$ of rectangular plate Al / Zro_2 is presented in Table 6. It is observed that by increasing the aspect ratio, the fundamental frequency parameter decreases.

Table 1. Material properties of the used FG plate

Material	Properties		
	$E(GPa)$	ν	$\rho(kg/m^3)$
Aluminium (Al)	70	0.3	2702
Alumina (Al_2O_3)	380	0.3	3800
Zirconia (ZrO_2)	200	0.3	5700

Table 2. Comparison of fundamental frequency parameter (Ω), for *Al/Al₂O₃* **square plates** $(\eta = 1)$

Material	Method	$\tau = h/a$	
		0.1	0.2
Fully ceramic	Proposed	5.7694	5.2813
	HSDT $(p = 10^{-6})$ [13]	5.7694	5.2813
	HSDT[18]	5.7694	5.2813
Fully metallic	Proposed	2.9376	2.6891
	HSDT $(p = 10^5)$ [13]	2.9376	2.6891
	HSDT[18]	2.9376	2.6891

Table 3. Comparative results of the non-dimensional natural frequency $(\hat{\Omega})$ for *Al/Al₂O₃* **square plates** $(\eta = 1)$

$\tau = h/a$	(m,n)	Method	Power law index (p)					
			0	0.5	1	4	10	∞
0.05	(1,1)	Proposed	0.0148	0.0125	0.0113	0.0098	0.0094	-
		Exact HSDT[13]	0.0148	0.0125	0.0113	0.0098	0.0094	-
		FSDT[17]	0.0146	0.0124	0.0112	0.0097	0.0093	-
0.1	(1,1)	Proposed	0.0577	0.0490	0.0442	0.0380	0.0364	0.0293
		Exact HSDT[13]	0.0577	0.0490	0.0442	0.0381	0.0364	0.0293
		HSDT[19]	0.0577	0.0492	0.0443	0.0381	0.0364	0.0293
		FSDT[17]	0.0568	0.0482	0.0435	0.0376	0.0359	-
	(1,2)	Proposed	0.1377	0.1173	0.1058	0.0901	0.0856	0.0701
		Exact HSDT[13]	0.1377	0.1174	0.1059	0.0903	0.0856	0.0701
		HSDT[19]	0.1381	0.1180	0.1063	0.0904	0.0859	0.0701
		FSDT[17]	0.1354	0.1154	0.1042	-	0.0850	-
	(2,2)	Proposed	0.2113	0.1806	0.1628	0.1375	0.1299	0.1073
		Exact HSDT[13]	0.2113	0.1807	0.1631	0.1378	0.1301	0.1076
		HSDT[19]	0.2121	0.1819	0.1640	0.1383	0.1306	0.1077
		FSDT[17]	0.2063	0.1764	0.1594	-	0.1289	-
0.2	(1,1)	Proposed	0.2113	0.1806	0.1628	0.1375	0.1299	0.1075
		Exact HSDT[13]	0.2113	0.1807	0.1631	0.1378	0.1301	0.1076
		HSDT[19]	0.2121	0.1819	0.1640	0.1383	0.1306	0.1077
		FSDT[17]	0.2055	0.1757	0.1587	0.1356	0.1284	-
	(1,2)	Proposed	0.4622	0.3982	0.3595	0.2969	0.2767	0.2352
		Exact HSDT[13]	0.4623	0.3989	0.3607	0.2980	0.2771	0.2355
		HSDT[19]	0.4658	0.4040	0.3644	0.3000	0.2790	0.2365
	(2,2)	Proposed	0.6688	0.5790	0.5234	0.4268	0.3943	0.3401
		Exact HSDT[13]	0.6688	0.5803	0.5254	0.4284	0.3948	0.3407
		HSDT[19]	0.6753	0.5891	0.5444	0.4362	0.3981	0.3429

Table 4. First three natural frequency parameters (Ω) for *Al/Al₂O₃* rectangular plates, $(\eta = 2)$

| $\tau = h/a$	(m,n)	Method	Power law index (p)					
			0	0.5	1	2	5	10
0.05	(1,1)	Proposed	3.7123	3.1456	2.8350	2.5768	2.4400	2.3617
		Exact FSDT[13]	3.7123	3.1456	2.8352	2.5777	2.4425	2.3642
	(1,2)	Proposed	5.9199	5.0176	4.5221	4.1091	3.8875	3.7618
		Exact FSDT[13]	5.9198	5.0175	4.5228	4.1115	3.8939	3.7681
	(2,1)	Proposed	12.4562	10.5661	9.5231	8.6470	8.1600	7.8892
		Exact FSDT[13]	12.4560	10.5660	9.5261	8.6572	8.1875	7.9166
0.1	(1,1)	Proposed	3.6518	3.0984	2.7926	2.5351	2.3904	2.3104
		Exact FSDT[13]	3.6518	3.0983	2.7937	2.5386	2.3998	2.3197
	(1,2)	Proposed	5.7694	4.9000	4.4167	4.0057	3.7653	3.6554
		Exact FSDT[13]	5.7693	4.8997	4.4192	4.0142	3.7881	3.6580
	(2,1)	Proposed	11.8315	10.0754	9.0832	8.2184	7.6622	7.3770
		Exact FSDT[13]	11.8310	10.0740	9.0928	8.2515	7.7505	7.4639
0.2	(1,1)	Proposed	3.4412	2.9328	2.6442	2.3907	2.2236	2.1390
		Exact FSDT[13]	3.4409	2.9322	2.6473	2.4017	2.2528	2.1677
	(1,2)	Proposed	5.2813	4.5139	4.0707	3.6716	3.3865	3.2481
		Exact FSDT[13]	5.2802	4.5122	4.0773	3.6953	3.4492	3.3094
	(2,1)	Proposed	10.1100	8.7010	7.8532	7.0487	6.3883	6.0875
		Exact FSDT[13]	9.7416	8.6926	7.8711	7.1189	6.5749	5.7518
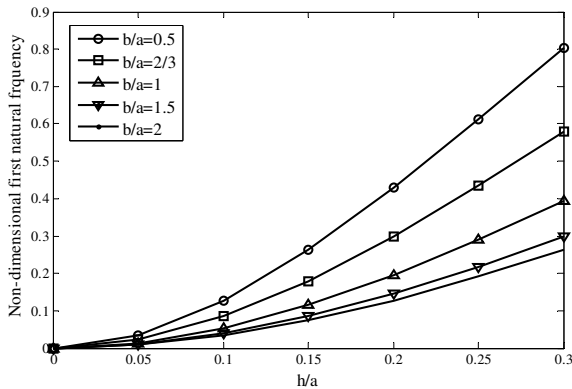
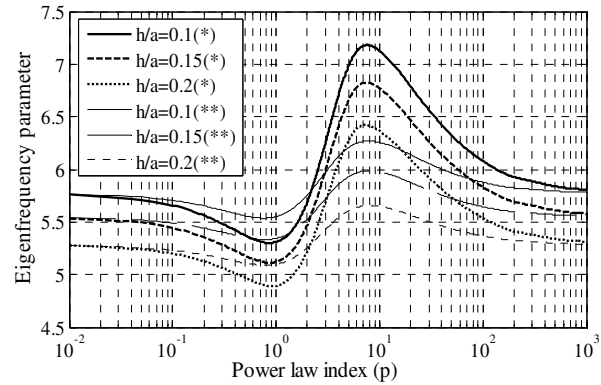**Fig. 3. Variation of Young modulus in a rectangular P-FGM plate.**

Fig. 4. Variation of the eigen-frequency parameter ($\overline{\beta}$) versus the *p* for the FG square plate ($\eta = 1$), *Al/Al₂O₃* (*), *Al/ZrO₂* (**).

Table 5. Comparative results of the non-dimensional natural frequency $(\hat{\Omega})$ for *Al/ZrO₂* square plates

Method	$p = 0$		$p = 1$			$\tau = 0.2$		
	$\tau = 1/\sqrt{10}$	$\tau = 0.1$	$\tau = 0.05$	$\tau = 0.1$	$\tau = 0.2$	$p = 2$	$p = 3$	$p = 5$
Proposed	0.4622	0.0577	0.0158	0.0618	0.2271	0.2249	0.2256	0.2267
HSDT[13]	0.4623	0.0577	0.0158	0.0619	0.2276	0.2256	0.2263	0.2272
HSDT[19]	0.4658	0.0577	0.0158	0.0619	0.2285	0.2264	0.2270	0.2281
3-D [21]	0.4648	0.0577	0.0153	0.0596	0.2192	0.2197	0.2211	0.2225
FSDT[22]	0.4619	0.0577	0.0162	0.0633	0.2323	0.2325	0.2334	0.2334

Table 6. Variation of fundamental frequencies with different values of aspect ratios for $\tau = 0.2$, $(p = 1)$, *Al/ZrO₂*

Method	$(\eta = b/a)$				
	0.5	2/3	1	1.5	2
Proposed	10.7214	7.4741	4.8782	3.6280	3.1737
Exact HSDT[13]	9.3216	7.5005	4.8909	3.6354	3.1796
Error (%)	-0.1502	0.0035	0.0026	0.0020	0.0019

The influence of thickness ratio on free vibration of the rectangular plates with various aspect ratios is shown in Fig. 3. As expected, increasing the aspect ratio leads to decreasing the stiffness of the plate, and consequently decreasing the non-dimensional first natural frequency. Here, a non-dimensional parameter $\left(\overline{\Omega} = \omega a^2/h\sqrt{\rho/E}\right)$ called the eigen-frequency parameter is

defined in which $\bar{\rho}$ and $\bar{E}$ are considered as mass density and Young's modulus of the mid-plane $(z=0)$, respectively. This parameter was defined by [13] and has its own special characteristics. Fig. 4 shows the variation of power law index ($p=0$ to 1000) on the eigen-frequency parameter $\left(\bar{\beta}=\omega a^2/h\sqrt{\bar{\rho}/\bar{E}}\right)$ for FG rectangular plates with different thickness ratios. First three eigenfrequencies are for Al/Al_2O_3 and the next three ones are for Al/ZrO_2.

It can be seen that the power law index has a great effect on $\bar{\Omega}$. As a matter of fact, without considering the materials, $\bar{\Omega}$ has the minimum and maximum values at the point around ($p=1$) and ($p=8$), respectively. The Al/Al_2O_3 FG plate material has more influences than Al/ZrO_2 on the eigen-frequency parameter. As mentioned earlier, the reason is that the frequency parameters are dependant on elasticity modulus. Therefore, by increasing the power law index, the elastic modulus of FG plate decreases due to increasing the rigidity of the plate.

6. Conclusions

Free vibration analysis of thick functionally graded rectangular plates was addressed. A two- variable refined plate theory was developed for functionally graded material plates. The theory gives parabolic distribution of the transverse shear strains, and no shear correction factor is needed. The numerical results have proved the efficiency and accuracy of this theory. The results were compared with the exact solution obtained by higher order shear deformation plate theory. The effects of different aspect ratio and thickness ratio on free vibration of FG rectangular plates with two different materials were also investigated. The main conclusions are listed as:

1- The results of RPT2 are very good agreement with exact analytical solution using higher order shear deformation plate theory.

2- All natural frequencies decrease when the power law index increases.

3- Because of increasing the rigidity of the FG plate, all the natural frequency parameters rise with an increase in the thickness ratio of the plate.

4- By increasing the aspect ratio, the fundamental frequency parameter decreases.

5- It is obvious that the power law index is an effective factor on eigen-frequency parameter.

6- The Al/Al_2O_3 FG plate material has more effects on the eigen-frequency parameter in comparison with Al/ZrO_2 FG plate.

Nomenclature

w_a	Displacement term in extension in the z-direction
w_b	Displacement term in bending in the z-direction
w_s	Displacement term in shear in the z-direction
σ_{ij}	Stress in the local coordinate
ε_{ij}	Strain in the local coordinate
v	Poisson's ratio

Appendix

$$
\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x^b \\ M_y^b \\ M_{xy}^b \\ M_x^s \\ M_y^s \\ M_{xy}^s \end{Bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & B_{11}^s & B_{12}^s & B_{16}^s \\
 & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & B_{12}^s & B_{22}^s & B_{26}^s \\
 & & A_{66} & B_{16} & B_{26} & B_{66} & B_{16}^s & B_{26}^s & B_{66}^s \\
 & & & D_{11} & D_{12} & D_{16} & D_{11}^s & D_{12}^s & D_{16}^s \\
 & & & & D_{22} & D_{26} & D_{12}^s & D_{22}^s & D_{26}^s \\
 & & & & & D_{66} & D_{16}^s & D_{26}^s & D_{66}^s \\
 & & & & & & H_{11}^s & H_{12}^s & H_{16}^s \\
 & & & & & & & H_{22}^s & H_{26}^s \\
Sym. & & & & & & & & H_{66}^s
\end{bmatrix}
$$

$$
\begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \\ -\partial^2 w^b/\partial x^2 \\ -\partial^2 w^b/\partial y^2 \\ -2\partial^2 w^b/\partial x \partial y \\ -\partial^2 w^s/\partial x^2 \\ -\partial^2 w^s/\partial y^2 \\ -2\partial^2 w^s/\partial x \partial y \end{Bmatrix}
\tag{A1}
$$

$$
\begin{Bmatrix} Q_{yz}^a \\ Q_{xz}^a \\ Q_{yz}^s \\ Q_{xz}^s \end{Bmatrix} =
\begin{bmatrix}
A_{44} & A_{45} & A_{44}^a & A_{45}^a \\
 & A_{55} & A_{45}^a & A_{55}^a \\
 & & A_{44}^s & A_{45}^s \\
Sym. & & & A_{55}^s
\end{bmatrix}
\begin{Bmatrix} \partial w^a/\partial y \\ \partial w^a/\partial x \\ \partial w^s/\partial y \\ \partial w^s/\partial x \end{Bmatrix}
$$

$$(A_{11}, A_{22}, A_{12}, A_{16}, A_{26}, A_{66}) = A\left(1,1,v,0,0,\frac{1-v}{2}\right)$$

$$(B_{11}, B_{22}, B_{12}, B_{16}, B_{26}, B_{66}) = B\left(1,1,v,0,0,\frac{1-v}{2}\right)$$

$$(D_{11}, D_{22}, D_{12}, D_{16}, D_{26}, D_{66}) = D\left(1,1,v,0,0,\frac{1-v}{2}\right)$$

$$\left(B_{11}^s, B_{22}^s, B_{12}^s, B_{16}^s, B_{26}^s, B_{66}^s\right) = \left(\frac{-B}{4} + \frac{5E}{3h^2}\right)\left(1,1,\nu,0,0,\frac{1-\nu}{2}\right)$$

$$\left(D_{11}^s, D_{22}^s, D_{12}^s, D_{16}^s, D_{26}^s, D_{66}^s\right) = \left(\frac{-D}{4} + \frac{5F}{3h^2}\right)\left(1,1,\nu,0,0,\frac{1-\nu}{2}\right)$$

$$\left(H_{11}^s, H_{22}^s, H_{12}^s, H_{16}^s, H_{26}^s, H_{66}^s\right) = \left(\frac{D}{16} + \frac{25H}{9h^4} - \frac{5F}{6h^2}\right)\left(1,1,\nu,0,0,\frac{1-\nu}{2}\right)$$

$$\left(A_{55}^a, A_{44}^a, A_{45}^a\right) = \left(\frac{5A}{4} - \frac{5D}{h^2}\right)\left(\frac{1-\nu}{2}\right)(1,1,0)$$

$$\left(A_{55}^s, A_{44}^s, A_{45}^s\right) = \left(\frac{25A}{16} - \frac{25D}{2h^2} + \frac{25F}{h^4}\right)\left(\frac{1-\nu}{2}\right)(1,1,0)$$

$$\left(A_{44}, A_{55}, A_{45}\right) = A\left(\frac{1-\nu}{2}\right)(1,1,0)$$

$$\left(A, B, D, E, F, H\right) = \int_{-h/2}^{h/2} \frac{E(z)}{1-\nu^2}\left(1, z, z^2, z^3, z^4, z^6\right) \qquad (A3)$$

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