Journal of Mechanical Research and Application (JMRA),Vol. 13 No.1, 1402(2023),44-52

New technique to short Steps of Calculation of Natural Frequency of Systems with One Degree of Freedom by using the “Theory of square of ratio of displacements”

M. Jabbari

Associate Professor, Tehran South Branch Azad University, Tehran, Iran

A new method for calculation of the natural frequency of one-degree of freedom system is presented in this paper. A new theory to find the equivalent for every type of mass, moment of inertia, linear, and torsional spring is proposed that may be used to obtain the natural frequency of the system.

1-Introduction

There are three major methods to obtain the natural frequency of a one-degree of freedom systems [1 to 11]. The first method is based on the second Newton's Law, which follows three steps as:

1- Drawing the free body diagram.
2- Using the second Newton's Law to write the governing differential equation.
3- Solving the differential equation analytically and finding the equation of motion and the natural frequency.

The second method is based on the energy method, which comes in three steps

1- Calculating the total energy which is summation of potential and kinetic energies.
2- Differentiating the total energy with respect to time and finding the governing differential equation.
3- The third step is the same as the third step of Newton's second Law method.

The third method is Rayleigh method and it is based on the energy method, which comes in three steps

1- Calculating t the potential and kinetic energies.
2- Obtain Effective stiffness and mass (or equivalent Effective stiffness and mass) and calculating the natural frequency.

In this paper a new method is present that can be used to calculate the Natural frequency in one step. This goal can be achieved by a new theory which is called the “Theory of square of ratio of displacements”. This theory is capable to calculate the equivalents of any mass and stiffness of system to every considered place.

2- Theory of square of ratio of displacements

The equivalent of any element of system in a considered place is the amount of element multiple to square of ratio of displacements of old place to new place.

Proof: To proof this theory we equal the energy of the element at old and new places in the following category:

1- Equivalent of linear stiffness at old place to linear stiffness in new place:

$$P.E = \frac{1}{2} k x_{old}^{2} = \frac{1}{2} k_{Eq} x_{new}^{2} \Rightarrow k_{Eq} = k\left(\frac{x_{old}}{x_{new}}\right)^{2} \quad (1)$$

2- Equivalent of linear stiffness at old place to torsional stiffness in new place:

$$P.E = \frac{1}{2} k x_{old}^{2} = \frac{1}{2} k_{TEq} \theta_{new}^{2} \Rightarrow k_{TEq} = k\left(\frac{x_{old}}{\theta_{new}}\right)^{2} \quad (2)$$

3- Equivalent of torsional stiffness at old place to torsional stiffness in new place:

$$P.E = \frac{1}{2} k_{T} \theta_{old}^{2} = \frac{1}{2} k_{TEq} \theta_{new}^{2} \Rightarrow k_{TEq} = k_{TEq}\left(\frac{\theta_{old}}{\theta_{new}}\right)^{2} \quad (3)$$

4- Equivalent of torsional stiffness at old place to linear stiffness in new place:

$$P.E = \frac{1}{2} k_{T} \theta_{old}^{2} = \frac{1}{2} k_{Eq} x_{new}^{2} \Rightarrow k_{Eq} = k_{T}\left(\frac{x_{old}}{\theta_{new}}\right)^{2} \quad (4)$$

5- Equivalent of Lumped Mass at old place to Lumped Mass in new place:

$$K.E=\frac{1}{2}M\dot{x}_{old}^{2}=\frac{1}{2}M\dot{x}_{new}^{2}\Rightarrow M_{Eq}=M(\frac{\dot{x}_{old}}{\dot{x}_{new}})^{2}$$

$$x_{new}=ax_{old}\Rightarrow\dot{x}_{old}=a\dot{x}_{old}\Rightarrow(\frac{\dot{x}_{old}}{\dot{x}_{new}})=(\frac{x_{old}}{x_{new}})\Rightarrow M_{Eq}=M(\frac{x_{old}}{x_{new}})^{2}$$

(5)

6- Equivalent of Mass Moment of inertia at old place to Mass Moment of inertia in new place:

$$K.E=\frac{1}{2}M\dot{x}_{old}^{2}=\frac{1}{2}I_{Eq}\dot{\theta}_{new}^{2}\Rightarrow I_{Eq}=M(\frac{x_{old}}{\theta_{new}})^{2}$$

(6)

7- Equivalent of Lumped Mass at old place to Mass Moment of inertia in new place:

$$K.E=\frac{1}{2}I\dot{\theta}_{old}^{2}=\frac{1}{2}I_{Eq}\dot{\theta}_{new}^{2}\Rightarrow I_{Eq}=I(\frac{\theta_{old}}{\theta_{new}})^{2}$$

(7)

8- Equivalent of Lumped Mass at old place to Lumped Mass in new place:

$$K.E=\frac{1}{2}I\dot{\theta}_{old}^{2}=\frac{1}{2}M_{Eq}\dot{x}_{new}^{2}\Rightarrow M_{Eq}=I(\frac{\theta_{old}}{x_{new}})^{2}$$

(8)

9- Equivalent of Linear Damper at old place to Linear Damper in new place:

$$D.E=\frac{1}{2}C\dot{x}_{old}^{2}=\frac{1}{2}C\dot{x}_{new}^{2}\Rightarrow C_{Eq}=C(\frac{x_{old}}{x_{new}})^{2}$$

(9)

10- Equivalent of Linear Damper at old place to torsional Damper in new place:

$$D.E=\frac{1}{2}C\dot{x}_{old}^{2}=\frac{1}{2}C_{TEq}\dot{\theta}_{new}^{2}\Rightarrow C_{TEq}=C(\frac{x_{old}}{\theta_{new}})^{2}$$

(10)

11- Equivalent of torsional Damper at old place to torsional Damper in new place:

$$D.E=\frac{1}{2}C_{T}\dot{\theta}_{old}^{2}=\frac{1}{2}C_{TEq}\dot{\theta}_{new}^{2}\Rightarrow C_{TEq}=C(\frac{\theta_{old}}{\theta_{new}})^{2}$$

(11)

12- Equivalent of torsional Damper at old place to linear Damper in new place:

$$D.E=\frac{1}{2}C\dot{\theta}_{old}{}^{2}=\frac{1}{2}C_{TEq}\dot{x}_{new}{}^{2}\Rightarrow C_{TEq}=C(\frac{\theta_{old}}{x_{new}})^{2} \quad (12)$$

It is interesting that the linear and torsional parameters directly changed to each other.

3- **Results and examples**

Some illustrated examples are presented in this paper. The first example shows an equivalent of oblique spring in x-direction in Fig. (1)

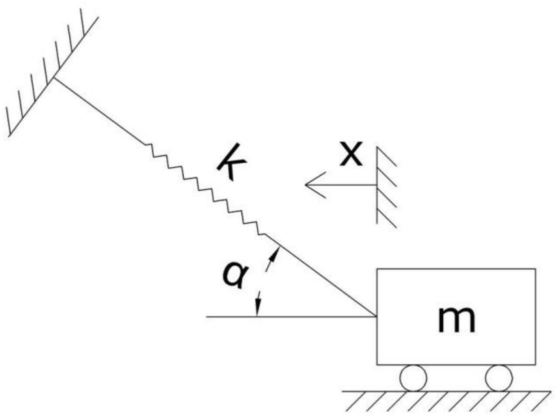

Fig. (1)

$$k_{Eq}=k(\frac{x_{old}}{x_{new}})^{2}\Rightarrow k_{Eq}=k(\frac{x\cos\alpha}{x_{n}})^{2}\Rightarrow k_{Eq}=k\cos^{2}\alpha$$

For the second example equivalents of mass an stiffness at point c in x-direction in Fig. (2) are

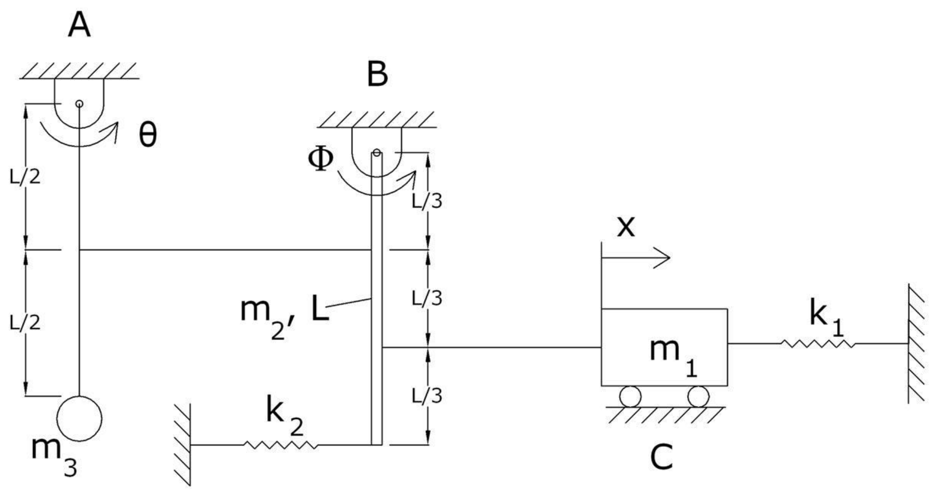

Fig. (2)

Equivalent stiffness is

$$k_{Eq} = k_1 + k_2(\frac{L\phi}{x})^2 + \frac{m_2 gL}{2}(\frac{\phi}{x})^2 + m_3 gL(\frac{\theta}{x})^2$$

In the above equation the torsional stiffness is considered as equivalent of weight as mgL where L is the distance of center of gravity to center of rotation.

$$\phi = \frac{3}{2L}x, \qquad \theta = \frac{1}{L}x$$

$$k_{Eq} = k_1 + \frac{9}{4}k_2 + \frac{9m_2 g}{8L} + \frac{m_3 g}{L}$$

Equivalent mass is

$$m_{Eq} = m_1 + I_B\left(\frac{\phi}{x}\right)^2 + I_c\left(\frac{\theta}{x}\right)^2 = m_1 + \frac{1}{3}m_2L^2\left(\frac{\frac{3}{2L}x}{x}\right)^2 + m_3L^2\left(\frac{\frac{1}{L}x}{x}\right)^2$$

$$m_{Eq} = m_1 + \frac{3}{4}m_2 + m_3$$

And natural frequency is

$$\omega_n = \sqrt{\frac{k_{Eq}}{m_{Eq}}} \Rightarrow \omega_n = \sqrt{\frac{k_1 + \frac{9}{4}k_2 + \frac{9m_2g}{8L} + \frac{m_3g}{L}}{m_1 + \frac{3}{4}m_2 + m_3}}$$

For the third example, two discs with mass M are connected to each other by rod with mass m and they roll without sliding according to Fig. (3). Equivalent parameters are found for the left disc in term of θ_1.

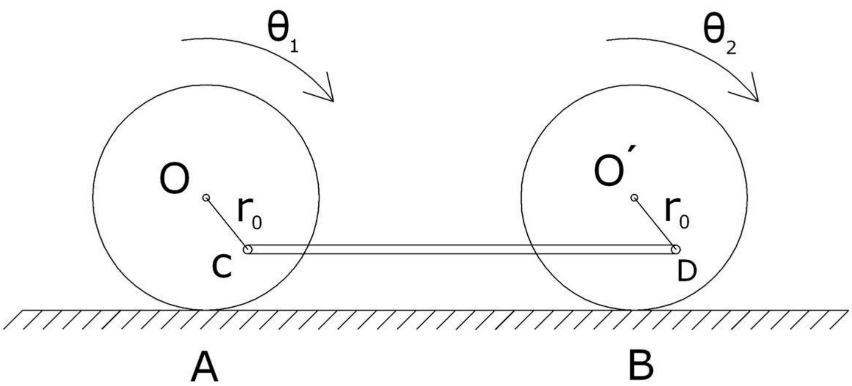

Fig. (3)

The bar CD has curve linear translational motion and since point C rotates around the point O then the vertical displacement of bar CD is $r_0(1-\cos\theta_1)$, so it is equivalent to pendulum with the length r_0 . Equivalent torsional stiffness is

$$k_{TEq} = mgr_0$$

Since the point A is instantaneous zero of velocity of left disc, then the displacement of point C and whole points of bar CD is $(r- r_0)\,\theta_1$. Equivalent moment of inertia around point A is

$$I_{Eq} = I_A + I_B\left(\frac{\theta_2}{\theta_1}\right)^2 + m\left(\frac{(r-r_0)\theta_1}{\theta_1}\right)^2 = \frac{3}{2}Mr^2 + \frac{3}{2}Mr^2(1)^2 + m(r-r_0)^2$$

$$I_{Eq} = 3Mr^2 + m(r-r_0)^2$$

And natural frequency is

$$\omega_n = \sqrt{\frac{K_{TEq}}{I_{Eq}}} \Rightarrow \omega_n = \sqrt{\frac{mgr_0}{3Mr^2 + m(r-r_0)^2}}$$

For the fourth example, two discs with mass m_2 and m_3 and a block m_1 move without sliding according to Fig. (4). Equivalent parameters are found for the block in term of x

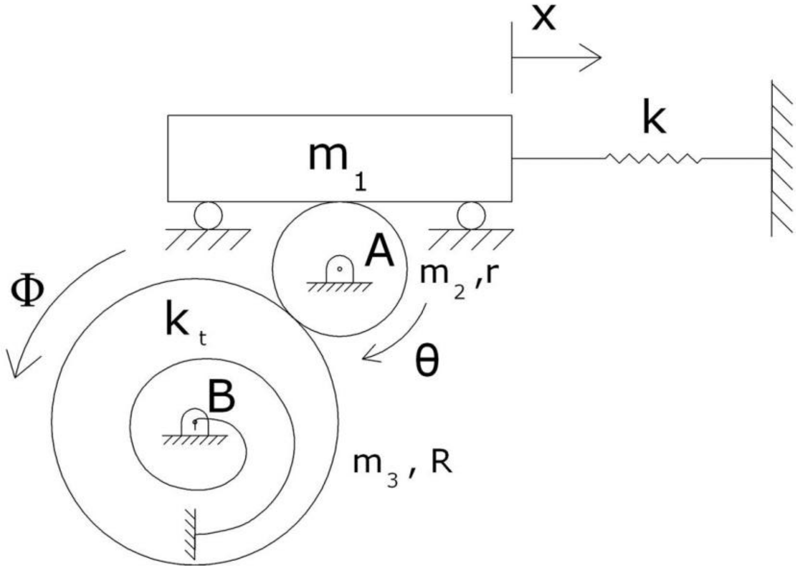

Fig. (4)

$$k_{Eq} = k + k_t \left(\frac{\varphi}{x}\right)^2$$

$$\varphi = \frac{x}{R}x \;, \qquad \theta = \frac{x}{r}$$

$$k_{Eq} = k + \frac{k_t}{R^2}$$

$$m_{Eq} = m_1 + I_A\left(\frac{\theta}{x}\right)^2 + I_B\left(\frac{\varphi}{x}\right)^2 = m_1 + \frac{1}{2}m_2 r^2\left(\frac{1}{r}\right)^2 + \frac{1}{2}m_3 R^2\left(\frac{1}{R}\right)^2$$

$$m_{Eq} = m_1 + \frac{m_2}{2} + \frac{m_3}{2}$$

And natural frequency is

$$\omega_n = \sqrt{\frac{k_{Eq}}{m_{Eq}}} \Rightarrow \omega_n = \sqrt{\frac{k + \frac{k_t}{R^2}}{m_1 + \frac{m_2}{2} + \frac{m_3}{2}}}$$

4-Conclusion

This paper covered the Theory of Square of Ratio of Displacement. This theory is result of energy equations and has capability to find the natural frequency in the shortest way and it decreases unnecessary steps that are used in traditional methods. This method shows that the calculation of potential, kinetic and dissipated energies is unnecessary to find the equivalents of stiffness, mass and damper and they can be obtained directly by Theory of Square of Ratio of Displacement.

REFERENCES

(1) S. S. Rao – Pearson, Mechanical Vibrations, Forth Edition, Prentice Hall (2004)

(2) W. T. Thomson, Theory of Vibration with Applications, Edition, Prentice Hall (1993)

(3) F. S. Tse, I. E Morse, R. T. Hinkle, Mechanical Vibrations: Theory and Applications, CBS Publishers & Distributors, (2004)

(4) Theory and Problems of Mechanical Vibrations, W. W. Seto, Schaum,s Outline Series, McGraw-Hill International Book Company (1983)

(5) S. Timoshenko, Vibration Problems In Engineering, D. Van Nostrand Company Inc. (1956)

(6) L. Meirovitch, Elements of Vibration Analysis, McGraw-Hill Kogakusha, LTD, (1975)

(7) S. G. Kelly, Mechanical Vibrations: Theory and Applications, Publisher, Global Engineering (2012)

(8) S. Bhave, Mechanical Vibrations: Theory and Practice, Dorling Kindersley (2010)

(9) W.G Palm, Mechanical Vibration, John Willey(2007)

(10) J. P. Den Hartog, Mechanical Vibrations, McGraw-Hill, (1956)

(11) R. N. Iyengar, Elements of Vibration Analysis, I.K. International Publishing House, (2010)