Journal of Mechanical Research and Application (JMRA),Vol. 13 No.2, 1402(2023),33-55

Free Vibration Analysis Of Functionally Graded Rectangular Plate Based On Various Shear Deformation Plate Theory

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Abstract

In this paper the free vibration of FGM rectangular plate analyzed by using the Third-order shear deformation theory. By introducing the displacment field according to the Third-order shear deformation plate theory (TSDT), the strain-displacement equations are derived and then by using the Hamilton's principle, dynamic equation for the mentioned plate are achieved and with Navier method whole dynamic equations are converted to an eigen value problem which the natural frequencies of plate can be calculated. Further more, the equations for First-order shear deformation plate theory will be derived and then the results compared with above equation results.

Keywords: *FGM*, Third-order shear deformation plate theory, free vibration, Hamilton's principle.

Nomenclature

a, b length and width of a rectangular plate	h plate thickness
u, v, w displacement in x, y, z direction	σ, ε stress, strain
ϕ_1, ϕ_2 mid-plane rotation	ρ density of plate material
E, G elasticity modulus	υ poison's ratio
N_{ij} total in-plane force	M_{ij} total in-plane moment
A_{ij} extensional stiffeness	D_{ij} bending stiffeness
B_{ij} bending-extensional coupling stiffness	E_{ij}, F_{ij}, H_{ij} high-order stiffeness

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ω frequency	$\overline{\omega}$ natural frequency
q load	P_{ij}, R_i high-order stress
p matreial variation profile through the thickness	T temperature
α termal expansion coefficient	

1. Introduction

In recent years functionally graded materials (FGMs) have gained considerable importance as materials to be used in extremely high temperature environments such as nuclear reactors and high-speed spacecraft industries (Yamanouchi et al., [1]). FGMs were first introduced by a group of scientists in Sendai Japan in 1984 (Koizumi, [2]). FGMs are new inhomogeneous materials, in which the mechanical properties vary smoothly and continuously from one surface to the other. This is achieved by gradually varying the volume fraction of the constituent materials. This continuous change in composition results in the graded properties of FGMs (Reddy and Cheng, [3]). This gradation in properties of the material reduces thermal stresses, residual stresses and stress concentration factors (Reddy et al., [4]). Typically these materials are made from a mixture of ceramic and metal or from a combination of different materials. The ceramic constituent of the material provides the high-temperature resistance due to its low thermal conductivity. The ductile metal constituent on the other hand, prevents fracture caused by stresses due to the high temperature gradient in a very short period of time. Furthermore a mixture of ceramic and metal with a continuously varying volume fraction can be easily manufactured (Fukui, [5]).

Studies on vibration of rectangular plates are extensive. Many of these studies are for isotropic and composite plates. In recent years many researchs about rectangular plates such as stability and vibration plates according to a Higher-Order Shear Deformation Theory [6,7], relationship between vibration frequencies of Reddy and

Kirchhoff Plates With Simply Supported Edges [8], free vibrations of laminated composite plates using second-order shear deformation theory and Layerwise theory [9,10], Analysis of laminated composite plates using HSDT [11], theory of plates and shells [12], buckling an vibration of laminated composite plate using various plate theories [13] has been done.

The First-order Shear Deformation Theory (FSDT) is the simplest plate theory that accounts for transverse shear strains which are represented as constant through the plate thickness, and the theory requires shear correction factors to compute transverse shear forces. In the Third-order Shear Deformation Theory (TSDT) of Reddy, the transverse shear stresses are represented as cubic through the thickness and consequently it isn't require to shear correction factors. The theory also contains the First-order Shear Deformation Theory as a special case. Here we develop the equations of motion of functionally graded plates using TSDT.

In the present work, vibration of functionally graded rectangular plate based on the third order shear deformation theory is studied. The objective is to study the frequency characteristics, the influence of the constituent volume fractions, and the affects of the configurations of the constituent materials on the natural frequencies.

2. Third-order theory of shear deformation plate

Consider a plate of total thickness h and composed of functionally graded material through the thickness. It is assumed that the material is isotropic and the grading is assumed to be only through the thickness. The xy-plane is taken to be the undeformed midplane of the plate with the z-axis positive upward from the midplane. Further, we restrict the formulation to linear elastic material behavior, small strains and displacements, and to the case in which the temperature field is known.

2.1. Displacement field

The Third-order shear deformation theory of Reddy used in the present study is based on the following displacement field [14]:

$$\begin{cases} u(x,y,z) = u_0(x,y) + z\cdot\phi_1(x,y) + z^2\psi_1(x,y) + z^3 u_3(x,y) \\ v(x,y,z) = v_0(x,y) + z\cdot\phi_2(x,y) + z^2\psi_2(x,y) + z^3 v_3(x,y) \\ w(x,y,z) = w_0(x,y) \end{cases} \quad (1)$$

These equations can be reduced by satisfying the stress-free conditions on the top and bottom faces of the plate, which are equivalent to:

$$\begin{cases} u(x,y,z) = u_0(x,y) + z\cdot\phi_1(x,y) - C_1 z^3(\phi_1 + \dfrac{\partial w_0}{\partial x}) \\ v(x,y,z) = v_0(x,y) + z\cdot\phi_2(x,y) - C_1 z^3(\phi_2 + \dfrac{\partial w_0}{\partial y}) \\ w(x,y,z) = w_0(x,y) \end{cases} \quad (2)$$

where(u_0, v_0, w_0) and(ϕ_1, ϕ_2) are displacement and rotation of normal lines on the plane $z = 0$, respectively. Also, $C_1 = 4/(3h^2)$.

2.2. Strain

The linear strain-displacement relations are given by:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} = \begin{Bmatrix} k_{11}^{(\circ)} \\ k_{22}^{(\circ)} \\ k_{12}^{(\circ)} \end{Bmatrix} + z\begin{Bmatrix} k^{(1)}{}_{11} \\ k^{(1)}{}_{22} \\ k^{(1)}{}_{12} \end{Bmatrix} + z^3\begin{Bmatrix} k^{(3)}{}_{11} \\ k^{(3)}{}_{22} \\ k^{(3)}{}_{12} \end{Bmatrix} \quad (3)$$

$$\begin{Bmatrix} \varepsilon_{13} \\ \varepsilon_{23} \end{Bmatrix} = \begin{Bmatrix} \gamma_{13}^{(\circ)} \\ \gamma_{23}^{(\circ)} \end{Bmatrix} + z^2\begin{Bmatrix} \gamma_{13}^{(2)} \\ \gamma_{23}^{(2)} \end{Bmatrix} \quad (4)$$

$$\begin{Bmatrix} k_{11}^{(0)} \\ k_{22}^{(0)} \\ k_{12}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix}, \qquad \begin{Bmatrix} k_{11}^{(1)} \\ k_{22}^{(1)} \\ k_{12}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_1}{\partial x} \\ \frac{\partial \phi_2}{\partial y} \\ \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \end{Bmatrix} \tag{5}$$

$$\begin{Bmatrix} k_{11}^{(3)} \\ k_{22}^{(3)} \\ k_{12}^{(3)} \end{Bmatrix} = -C_1 \begin{Bmatrix} (\frac{\partial \phi_1}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}) \\ (\frac{\partial \phi_2}{\partial y} + \frac{\partial^2 w_0}{\partial y^2}) \\ (\frac{\partial \phi_2}{\partial x} + 2\frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial \phi_1}{\partial y}) \end{Bmatrix}$$

$$\begin{Bmatrix} \gamma_{13}^{(0)} \\ \gamma_{23}^{(0)} \end{Bmatrix} = \begin{Bmatrix} (\phi_1 + \frac{\partial w_0}{\partial x}) \\ (\phi_2 + \frac{\partial w_0}{\partial y}) \end{Bmatrix}, \qquad \begin{Bmatrix} \gamma_{13}^{(2)} \\ \gamma_{23}^{(2)} \end{Bmatrix} = -3C_1 \begin{Bmatrix} (\phi_1 + \frac{\partial w_0}{\partial x}) \\ (\phi_2 + \frac{\partial w_0}{\partial y}) \end{Bmatrix} \tag{6}$$

2.3. Functionally Graded Plates

We assume that the material property gradation is only through the thickness. FGM properties such as density ρ, elasticity modulus E and G are functions of volumetric ratio and the components. The poison's coefficient is considered as constant. If we consider the normal axis of midplane as z, we'll have:

$$\begin{aligned} E(z) &= (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^p + E_m \\ \rho(z) &= (\rho_c - \rho_m)\left(\frac{z}{h} + \frac{1}{2}\right)^p + \rho_m \\ G = G(z) &= (G_c - G_m)\left(\frac{z}{h} + \frac{1}{2}\right)^p + G_m \end{aligned} \tag{8}$$

where E_c, ρ_c, G_c denote the property of top face and E_m, ρ_m, G_m denote bottom face property, h is the total thickness of the plate and p is a parameter that dictates the material variation profile through the thickness.

2.4. Stress-Strain Relations

The stress-Strain relations are similar to isotropic plate's relations, but with this difference that modul E is not constant, but it is according to equation (8).

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{Bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0 & 0 & 0\\ Q_{12} & Q_{22} & 0 & 0 & 0\\ 0 & 0 & Q_{44} & 0 & 0\\ 0 & 0 & 0 & Q_{55} & 0\\ 0 & 0 & 0 & & Q_{66}\end{bmatrix}\cdot\left[\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{23}\\ \varepsilon_{13}\\ \varepsilon_{12}\end{Bmatrix}-\begin{Bmatrix}1\\1\\0\\0\\0\end{Bmatrix}\alpha\cdot\Delta T\right]$$

where:

$$Q_{11}=Q_{22}=\frac{E}{1-\upsilon^2}, \qquad Q_{12}=\upsilon\cdot Q_{11}, \qquad Q_{44}=Q_{55}=Q_{66}=\frac{E}{2(1+\upsilon)} \tag{10}$$

2.5. Equations of motion

The equation of motion based on TSDT by using Hamilton's principle are:

$$\frac{\partial N_{11}}{\partial x}+\frac{\partial N_{12}}{\partial y}=-q_1+I_0\ddot{u}_0+(I_1-C_1I_3)\ddot{\phi}_1-C_1I_3\frac{\partial\ddot{w}_0}{\partial x} \tag{11}$$

$$\frac{\partial N_{22}}{\partial y}+\frac{\partial N_{12}}{\partial x}=-q_2+I_0\ddot{u}_2+(I_1-C_1I_3)\ddot{\phi}_2-C_1I_3\frac{\partial\ddot{w}_0}{\partial y} \tag{12}$$

$$\begin{aligned}&\frac{\partial^2 P_{11}}{\partial x^2}C_1+\frac{\partial^2 P_{22}}{\partial y^2}C_1+2\frac{\partial^2 P_{12}}{\partial x\partial y}C_1+\frac{\partial Q_1}{\partial x}-3C_1\frac{\partial R_{13}}{\partial x}+\frac{\partial Q_2}{\partial y}-3C_1\frac{\partial R_{23}}{\partial y}\\ &\quad=-q_3+I_0\ddot{w}_0-C_1^2I_6\left(\frac{\partial^2\ddot{w}_0}{\partial x^2}+\frac{\partial^2\ddot{w}_0}{\partial y^2}\right)+C_1\left[(I_4-C_1I_6)\left(\frac{\partial\ddot{\phi}_1}{\partial x}+\frac{\partial\ddot{\phi}_2}{\partial y}\right)+I_3\left(\frac{\partial\ddot{u}_0}{\partial x}+\frac{\partial\ddot{v}_0}{\partial y}\right)\right]\end{aligned} \tag{13}$$

$$\begin{aligned}&\frac{\partial M_{11}}{\partial x}-C_1\frac{\partial P_{11}}{\partial x}+\frac{\partial M_{12}}{\partial y}-C_1\frac{\partial P_{12}}{\partial y}+3C_1R_{13}-Q_1=(I_1-C_1I_3)\ddot{u}_0\\ &\quad+(I_2-2C_1I_4+C_1^2I_6)\ddot{\phi}_1-C_1(I_4-C_1I_6)\frac{\partial\ddot{u}_3}{\partial x}\end{aligned} \tag{14}$$

$$\begin{aligned}&\frac{\partial M_{22}}{\partial y}-C_1\frac{\partial P_{22}}{\partial y}+\frac{\partial M_{12}}{\partial x}-C_1\frac{\partial P_{12}}{\partial x}+3C_1R_{23}-Q_2=(I_1-C_1I_3)\ddot{v}_0\\ &\quad+(I_2-2C_1I_4+C_1^2I_6)\ddot{\phi}_2-C_1(I_4-C_1I_6)\frac{\partial\ddot{w}_0}{\partial y}\end{aligned} \tag{15}$$

where:

$$\begin{Bmatrix}N_{11}\\N_{22}\\N_{12}\end{Bmatrix}=\int_{-h/2}^{h/2}\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix}dz, \qquad \begin{Bmatrix}M_{11}\\M_{22}\\M_{12}\end{Bmatrix}=\int_{-h/2}^{h/2}\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix}zdz \tag{16}$$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} dz \tag{17}$$

N_{ij} represent the total in-plane force resultant and M_{ij} the total moment resultants, and

$$\begin{Bmatrix} P_{11} \\ P_{22} \\ P_{12} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} z^3 dz \,, \quad \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} z^2 dz \tag{18}$$

which P_{ij} and R_i show the high-order sress resultants. Also,

$$I_i = \int_{-h/2}^{h/2} \rho(z)^i dz \qquad (i = 0,1,\cdots,6) \tag{19}$$

Consider the following description:

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij}, H_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1, z, z^2, z^3, z^4, z^5, z^6) dz \tag{20}$$

where Q_{ij} comes from (10) and A_{ij} denote extensional stiffeness, D_{ij} the bending stiffeness, B_{ij} the bending-extensional coupling stiffenesses, and E_{ij}, F_{ij} and H_{ij} are high-order stiffeness.

Parameters A_{ij}, D_{ij} are described for $i, j = 1,2,6$, F_{ij} for $i, j = 4,5$ and B_{ij}, E_{ij}, H_{ij} for $i, j = 1,2,6$. It is important to see that E_{ij}, F_{ij} and H_{ij} involve fourth or higher powers of the thickness, so they are expected to affect little to thin and homogeneous plates, Therefore the stress resultants are related to the strain by the relations as following:

$$\begin{Bmatrix} \{N\} \\ \{M\} \\ \{P\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [E] \\ [B] & [D] & [F] \\ [E] & [F] & [H] \end{bmatrix} . \begin{Bmatrix} \{k^{(0)}\} \\ \{k^{(1)}\} \\ \{k^{(3)}\} \end{Bmatrix} \tag{21}$$

$$\begin{Bmatrix} \{Q\} \\ \{R\} \end{Bmatrix} = \begin{bmatrix} [A] & [D] \\ [D] & [F] \end{bmatrix} \begin{Bmatrix} \{\gamma^{(0)}\} \\ \{\gamma^{(2)}\} \end{Bmatrix} \tag{22}$$

2.6. Solutions for a simply supported plate

2.6.1. Boundary conditions

The simply supported boundary conditions for the Third-order shear deformation plate theory are:

$$
\begin{array}{llll}
u_1(x,0,t)=0\ , & \phi_x(x,0,t)=0\ , & u_1(x,b,t)=0\ , & \phi_x(x,b,t)=0 \\
u_2(0,y,t)=0\ , & \phi_y(0,y,t)=0\ , & u_2(a,y,t)=0\ , & \phi_y(a,y,t)=0 \\
u_3(x,0,t)=0\ , & u_3(x,b,t)=0\ , & u_3(0,y,t)=0\ , & u_3(a,y,t)=0 \\
N_{11}(0,y,t)=0\ , & N_{11}(a,y,t)=0\ , & N_{22}(x,0,t)=0\ , & N_{11}(x,b,t)=0 \\
\overline{M}_{11}(0,y,t)=0\ , & \overline{M}_{11}(a,y,t)=0\ , & \overline{M}_{22}(x,0,t)=0\ , & \overline{M}_{11}(x,b,t)=0
\end{array}
\tag{23}
$$

where

$$\overline{M}_{\alpha\beta}=M_{\alpha\beta}-C_1P_{\alpha\beta} \tag{24}$$

2.6.2. The Navier Solutions

The five equations of motion (11)-(15), Solved by Navier solutions for simply supported plates. The boundary conditions in equation (23) are satisfied by the following expansions:

$$u_0(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}U_{mn}(t)\cos\alpha x.\sin\beta y\ , \qquad U_{mn}(t)=Ue^{-i\omega t} \tag{25a}$$

$$v_0(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}V_{mn}(t)\sin\alpha x.\cos\beta y\ , \qquad V_{mn}(t)=Ve^{-i\omega t} \tag{25b}$$

$$w_0(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}w_{mn}(t)\sin\alpha x.\sin\beta y\ , \qquad w_{mn}(t)=We^{-i\omega t} \tag{25c}$$

$$\phi_1(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}(t)\cos\alpha x.\sin\beta y\ , \qquad A_{mn}(t)=Ae^{-i\omega t} \tag{25d}$$

$$\phi_2(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}(t)\sin\alpha x.\cos\beta y\ , \qquad B_{mn}(t)=Be^{-i\omega t} \tag{25e}$$

where $\alpha=m\pi/a$, $\beta=n\pi/b$ and ω is natural frequency. Assume $q_1=q_2=0$, the normal load q_3 can be expanded in double Fourier sin series:

$$q(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q_{mn}(t)\cos\alpha x.\sin\beta y \tag{26}$$

$$q_{mn}(z,t)=\frac{4}{ab}\int_a^b\int_a^b q(x,y,t)\sin\alpha x.\sin\beta y\,dx\,dy \tag{27}$$

2.7. Natural frequency equation of simply supported FGM rectangular plate (TSDT)

Substitution of equations (25a)-(25e) into equations (11)-(15) and simplifying the resultants relations,then we obtain the following equation:

$$[C]\begin{Bmatrix}U_{mn}\\V_{mn}\\W_{mn}\\A_{mn}\\B_{mn}\end{Bmatrix}+[M]\begin{Bmatrix}\ddot{U}_{mn}\\\ddot{V}_{mn}\\\ddot{W}_{mn}\\\ddot{A}_{mn}\\\ddot{B}_{mn}\end{Bmatrix}=\begin{Bmatrix}0\\0\\q_{mn}\\0\\0\end{Bmatrix} \tag{28}$$

or

$$[C]\begin{Bmatrix}U\\V\\W\\A\\B\end{Bmatrix}-\omega^2[M]\begin{Bmatrix}U\\V\\W\\A\\B\end{Bmatrix}=\begin{Bmatrix}0\\0\\q\\0\\0\end{Bmatrix} \tag{29}$$

where:

$$C=\begin{bmatrix}c_{11}&c_{12}&c_{13}&c_{14}&c_{15}\\c_{21}&c_{22}&c_{23}&c_{24}&c_{25}\\c_{31}&c_{32}&c_{33}&c_{34}&c_{35}\\c_{41}&c_{42}&c_{43}&c_{44}&c_{45}\\c_{51}&c_{52}&c_{53}&c_{54}&c_{55}\end{bmatrix},\ M=\begin{bmatrix}m_{11}&m_{12}&m_{13}&m_{14}&m_{15}\\m_{21}&m_{22}&m_{23}&m_{24}&m_{25}\\m_{31}&m_{32}&m_{33}&m_{34}&m_{35}\\m_{41}&m_{42}&m_{43}&m_{44}&m_{45}\\m_{51}&m_{52}&m_{53}&m_{54}&m_{55}\end{bmatrix} \tag{30}$$

The components of the matrix C are defined by:

$$c_{11}=A_{11}\alpha^2+A_{66}\beta^2$$
$$c_{12}=(A_{12}+A_{66})\alpha\beta$$
$$c_{13}=-C_1\left[E_{11}\alpha^2+(E_{12}+2E_{66})\beta^2\right]\alpha$$
$$c_{14}=(B_{11}-C_1E_{11})\alpha^2+(B_{66}-C_1E_{66})\beta^2$$
$$c_{15}=\left[(B_{12}-C_1E_{12})+(B_{66}-C_1E_{66})\right]\alpha\beta$$
$$c_{22}=A_{66}\alpha^2+A_{22}\beta^2$$
$$c_{23}=-C_1\left[E_{22}\beta^2+(E_{12}+2E_{66})\alpha^2\right]\beta$$
$$c_{24}=c_{15}$$

$$c_{25}=(B_{66}-C_1E_{66})\alpha^2+(B_{22}-C_1E_{22})\beta^2 \tag{31}$$
$$c_{33}=(A_{55}-2C_1D_{55}+C_1^2F_{55})\alpha^2+(A_{44}-2C_1D_{44}+C_1^2F_{44})\beta^2+C_1^2\left[H_{11}\alpha^4+2(H_{12}+2H_{66})\alpha^2\beta^2+H_{22}\beta^4\right]$$
$$c_{34}=(A_{55}-2C_1D_{55}+C_1{}^2F_{55})\alpha-C_1\left[(F_{11}-C_1H_{11})\alpha^3+((F_{12}-C_1H_{12})+2(F_{66}-C_1H_{66}))\alpha\beta^2\right]$$
$$c_{35}=(A_{44}-2C_1D_{44}+C_1^2F_{44})\beta-C_1\left[(F_{22}-C_1H_{22})\beta^3+((F_{12}-C_1H_{12})+2(F_{66}-C_1H_{66}))\alpha^2\beta\right]$$
$$c_{44}=(A_{55}-2C_1D_{55}+C_1^2F_{55})+(D_{11}-2C_1F_{11}+C_1^2H_{11})\alpha^2+(D_{66}-2C_1F_{66}+C_1^2H_{66})\beta^2$$
$$c_{45}=\left[(D_{12}-2C_1F_{12}+C_1^2H_{12})+(D_{66}-2C_1F_{66}+C_1^2H_{66})\right]\alpha\beta$$
$$c_{55}=(A_{44}-6C_1D_{44}+9C_1^2F_{44})+(D_{66}-2C_1F_{66}+C_1^2H_{66})\alpha^2+(D_{22}-2C_1F_{22}+C_1^2H_{22})\beta^2$$

And the components of the matrix M are defined by:

$$\begin{aligned}
m_{11}&=I_0\\
m_{22}&=I_0\\
m_{33}&=I_0+C_1^2I_6(\alpha^2+\beta^2)\\
m_{34}&=-C_1(I_4-C_1I_6)\alpha\\
m_{35}&=-C_1(I_4-C_1I_6)\beta\\
m_{44}&=I_2-2C_1I_4+C_1^2I_6\\
m_{55}&=I_2-2C_1I_4+C_1^2I_6
\end{aligned} \tag{32}$$

The equations (28) can be specialized to static response, buckling, and vibrations. To achieve the natural frequency, we set $q_{mn}=0$, therefore equations (29) become as following:

$$[C]\begin{Bmatrix}U_1\\U_2\\U_3\\\phi_1\\\phi_2\end{Bmatrix}-\varpi^2[M]\begin{Bmatrix}U_1\\U_2\\U_3\\\phi_1\\\phi_2\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\\0\end{Bmatrix}$$

or

$$\left[c_{ij}-m_{ij}\varpi^2\right]\begin{Bmatrix}U_1\\U_2\\U_3\\\phi_1\\\phi_2\end{Bmatrix}=0 \tag{33}$$

Setting the determinant of matrix $\lfloor c_{ij} - m_{ij}\bar{\omega}^2 \rfloor$ equal to zero and solving the achieved eqation for $\bar{\omega}$ (frequency), values of natural frequency for simply supported rectangular FGM plate will be derived.

3. First-order theory of shear deformation FSDT

Using the achieved equations for FGM rectangular plate in TSDT , we will drive the above equations for a rectangular plate in FSDT. To do this, by setting $C_1 = 0$ in any equation of the last chapter, new equations will be drived with FSDT. It is important to know by setting $C_1 = 0$, the value of shear strain become independent of thickness.

3.1. Displacement field

Setting $C_1 = 0$ in equation (2), we obtain displacement field for FSDT as following:

$$\begin{cases} u(x,y,z) = u_0(x,y) + z \cdot \phi_1(x,y) \\ v(x,y,z) = v_2(x,y) + z \cdot \phi_2(x,y) \\ w(x,y,z) = w_0(x,y) \end{cases} \tag{34}$$

3.2. Strain

In the same way, by setting $C_1 = 0$ in equation (3)-(6), The linear strain relations are obtained:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} = \begin{Bmatrix} k_{11}^{(\circ)} \\ k_{22}^{(\circ)} \\ k_{12}^{(\circ)} \end{Bmatrix} + z \begin{Bmatrix} k^{(1)}{}_{11} \\ k^{(1)}{}_{22} \\ k^{(1)}{}_{12} \end{Bmatrix} \tag{35}$$

$$\begin{Bmatrix} \varepsilon_{13} \\ \varepsilon_{23} \end{Bmatrix} = \begin{Bmatrix} \gamma_{13}^{(\circ)} \\ \gamma_{23}^{(\circ)} \end{Bmatrix} \tag{36}$$

where:

$$\begin{Bmatrix} k_{11}^{(0)} \\ k_{22}^{(0)} \\ k_{12}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix}, \qquad \begin{Bmatrix} k_{11}^{(1)} \\ k_{22}^{(1)} \\ k_{12}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_1}{\partial x} \\ \frac{\partial \phi_2}{\partial y} \\ \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \end{Bmatrix} \tag{37}$$

$$\begin{Bmatrix} \gamma_{13}^{(\circ)} \\ \gamma_{23}^{(\circ)} \end{Bmatrix} = \begin{Bmatrix} (\phi_1 + \frac{\partial w_0}{\partial x}) \\ (\phi_2 + \frac{\partial w_0}{\partial y}) \end{Bmatrix} \tag{38}$$

3.3. Equations of motion

The equation of motion of FSDT by using hamilton’s principle are:

$$\frac{\partial N_{11}}{\partial x} + \frac{\partial N_{12}}{\partial y} = -q_1 + I_0 \ddot{u}_1 + I_1 \ddot{\phi}_1 \tag{39}$$

$$\frac{\partial N_{22}}{\partial y} + \frac{\partial N_{12}}{\partial x} = -q_2 + I_0 \ddot{u}_2 + I_1 \ddot{\phi}_2 \tag{40}$$

$$\frac{\partial Q_{13}}{\partial x} + \frac{\partial Q_{23}}{\partial y} = -q_3 + I_0 \ddot{u}_3 \tag{41}$$

$$\frac{\partial M_{11}}{\partial x} + \frac{\partial M_{12}}{\partial y} - Q_{13} = I_1 \ddot{u}_1 + I_2 \ddot{\phi}_1 \tag{42}$$

$$\frac{\partial M_{22}}{\partial y} + \frac{\partial M_{12}}{\partial x} - Q_{23} = I_1 \ddot{u}_2 + I_2 \ddot{\phi}_2 \tag{43}$$

where:

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \cdot \begin{Bmatrix} \{k^{(0)}\} \\ \{k^{(1)}\} \end{Bmatrix} \qquad i, j = 1,2,6 \tag{44}$$

$$\{Q\} = [A]\{\gamma^{(0)}\} \qquad i, j = 4,5 \tag{45}$$

and

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1, z, z^2)\,\mathrm{dz} \tag{46}$$

$$I_i = \int_{-h/2}^{h/2} \rho(z)^i\, dz \tag{47}$$

3.4. Natural frequency equation of simply supported FGM rectangular plate (FSDT)

Substitution of equations (25a)-(25e) into equations (39)-(43) and simplifying the resultants relations,then we obtain the following equation:

$$[C]\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \phi_1 \\ \phi_2 \end{Bmatrix} - \omega^2 [M]\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q \\ 0 \\ 0 \end{Bmatrix} \quad (48)$$

where:

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} \end{bmatrix}, M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} \end{bmatrix} \quad (49)$$

The components of the matrix C are defined by:

$$\begin{aligned}
c_{11} &= A_{11}\alpha^2 + A_{66}\beta^2 \\
c_{12} &= (A_{12} + A_{66})\alpha\beta \\
c_{13} &= 0 \\
c_{14} &= B_{11}\alpha^2 + B_{66}\beta^2 \\
c_{15} &= [B_{12} + B_{66}]\alpha\beta \\
c_{22} &= A_{66}\alpha^2 + A_{22}\beta^2 \\
c_{23} &= 0 \\
c_{24} &= c_{15} \\
c_{25} &= B_{66}\alpha^2 + B_{22}\beta^2 \\
c_{33} &= A_{55}\alpha^2 + A_{44}\beta^2 \\
c_{34} &= A_{55}\alpha \\
c_{35} &= A_{44}\beta \\
c_{44} &= A_{55} + D_{11}\alpha^2 + D_{66}\beta^2 \\
c_{45} &= [D_{12} + D_{66}]\alpha\beta \\
c_{55} &= A_{44} + D_{66}\alpha^2 + D_{22}\beta^2
\end{aligned} \quad (50)$$

And the components of the matrix M are defined by:

$$\begin{aligned}
m_{11} &= I_0 \\
m_{22} &= I_0 \\
m_{33} &= I_0 \\
m_{34} &= 0 \\
m_{35} &= 0 \\
m_{44} &= I_2 \\
m_{55} &= I_2
\end{aligned} \quad (51)$$

The other components zero. To derive the natural frequency we set $q = 0$, so (38) becomes as following:

$$[C]\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \phi_1 \\ \phi_2 \end{Bmatrix} - \varpi^2 [M] \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

or

$$\left[c_{ij} - m_{ij}\varpi^2\right]\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \phi_1 \\ \phi_2 \end{Bmatrix} = 0 \tag{52}$$

Similar to TSDT, setting the determinant of matrix $\left|c_{ij} - m_{ij}\varpi^2\right|$ equal to zero and solving the achieved eqation for ϖ (frequency), values of natural frequency for simply supported rectangular FGM plate will be derived.

4. Result and discussion

The numerical results for an isotropic plate with $\upsilon = 0.3$ and FGM plate with $\upsilon = 0.3$, $\rho_1 / \rho_2 = 300/2707$ and $G_1 / G_2 = 151/70$ are given which ρ_1 and ρ_2 are ceramic and metal density. Also G_1 and G_2 are shear modulus of ceramic and metal, respectively. Also m, n are parameters mentioned in relations (25a)-(25e) and p relates to (8). The above results are the same for any FGMs according to (8).

From figures 1 to 10, we understand :

1. The nondimansionalized values of natural frequencies of both first and third theory decrease by increase of a/h, so that the diference between the nondimansionalized frequency in $a/h = 10$ and $a/h = 20$ is almost %80 but this decreasement get shower and the figure becomes horizontal and the nondimansionalized natural frequencies with increase of a/h, leave constant.

2. By increasement of a/h the results of FSDT and HSDT are completely synchronize, in other words, by decrease of h (thickness), the results of FSDT and HSDT are completely synchronize. This matter is in derived relations, because h in HSDT effects on transvers shear stresses as a coefficient, but in FSDT the transverse shear stresses are constant along thickness and are independent of h. In other word with increase of h the difference between FSDT and HSDT increases, also the figures show that FSDT represent higher values.

3. figure 11 shows that, with increase of p, the amount of natural frequency decrease, but at first the range of decrease is high and then get gradually and smoothly after. While $p = 0$ shows pure ceramic, so by increase of metal percentage the natural frequency decreases. As the elastisity modulus of ceramic is higher than elastisity modulus of metal, this fact is correct.

5. Conclusion

FSDT and HSDT theories can be used replace by another one for thin plates with high accuracy, but HSDT has higher accuracy for thick plates , so is better to use this theory for thick plates.

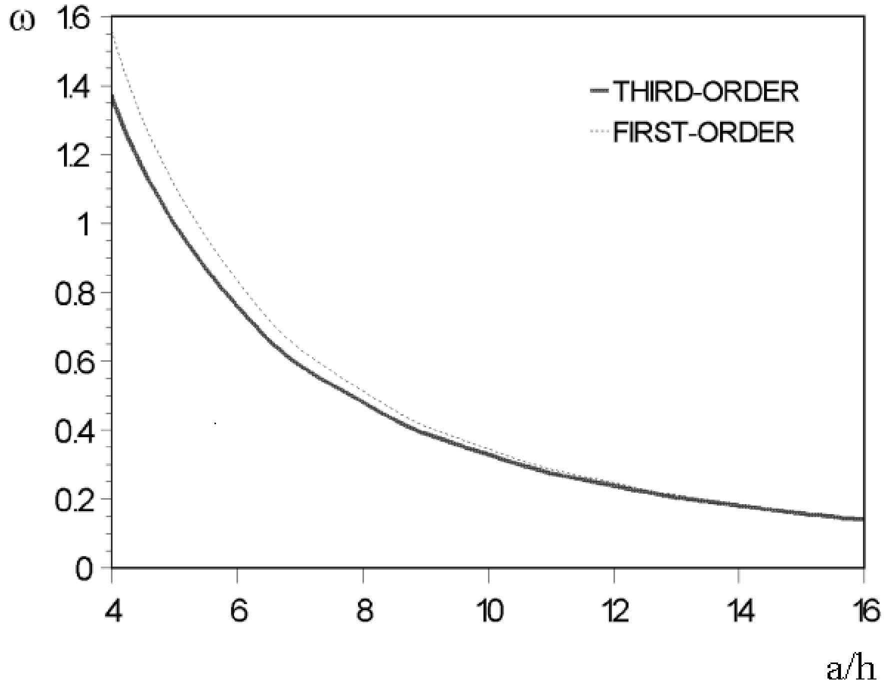

Figure 1-Values of nondimansionalized natural frequency $\varpi \cdot h\sqrt{\frac{\rho_1}{G_1}}$ for rectangular FGM plate in case $m = n = 2$, $\upsilon = 0.3$, $\rho_1 / \rho_2 = 300/2707$, $G_1 / G_2 = 151/70$ and $p = 0$ (Isotropic-Material)

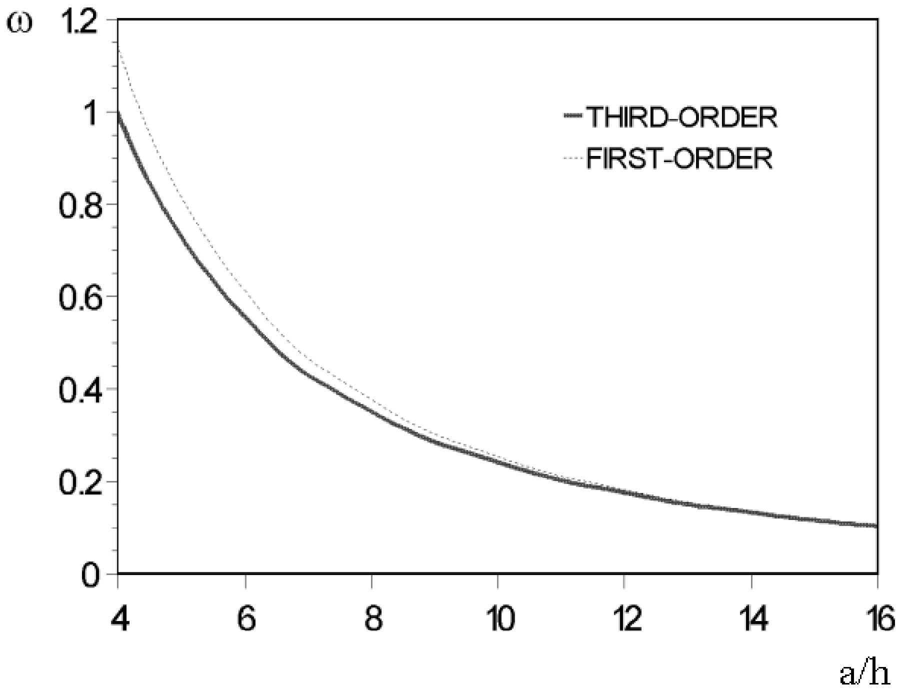

Figure 2-Values of nondimansionalized natural frequency $\varpi \cdot h\sqrt{\frac{\rho_1}{G_1}}$ for rectangular FGM plate in case $m = n = 2$, $\upsilon = 0.3$, $\rho_1 / \rho_2 = 300/2707$, $G_1 / G_2 = 151/70$ and $p = 0.1$.

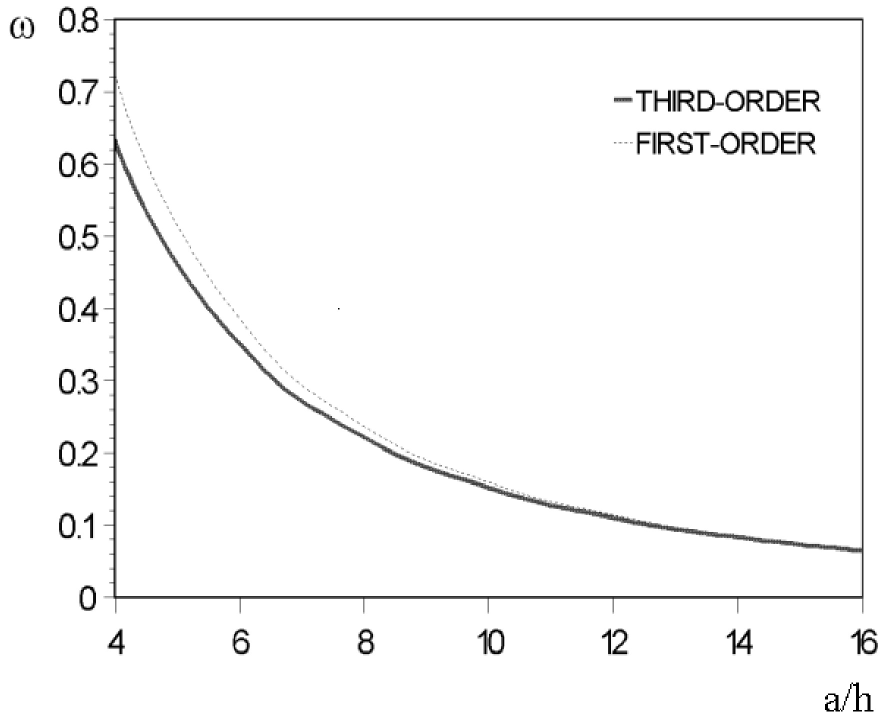

Figure 3-Values of nondimansionalized natural frequency $\varpi \cdot h\sqrt{\frac{\rho_1}{G_1}}$ for rectangular FGM plate in case $m = n = 2$, $\upsilon = 0.3$, $\rho_1 / \rho_2 = 300/2707$, $G_1 / G_2 = 151/70$ and $p = 0.5$.

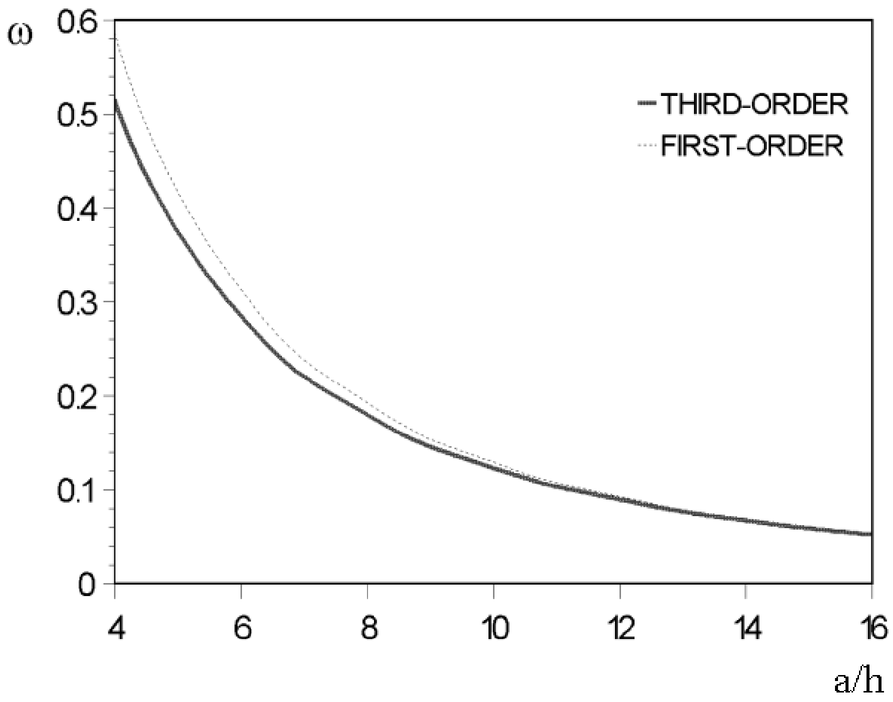

Figure 4-Values of nondimansionalized natural frequency $\varpi \cdot h\sqrt{\frac{\rho_1}{G_1}}$ for rectangular FGM plate in case $m = n = 2$, $\upsilon = 0.3$, $\rho_1 / \rho_2 = 300/2707$, $G_1 / G_2 = 151/70$ and $p = 1$.

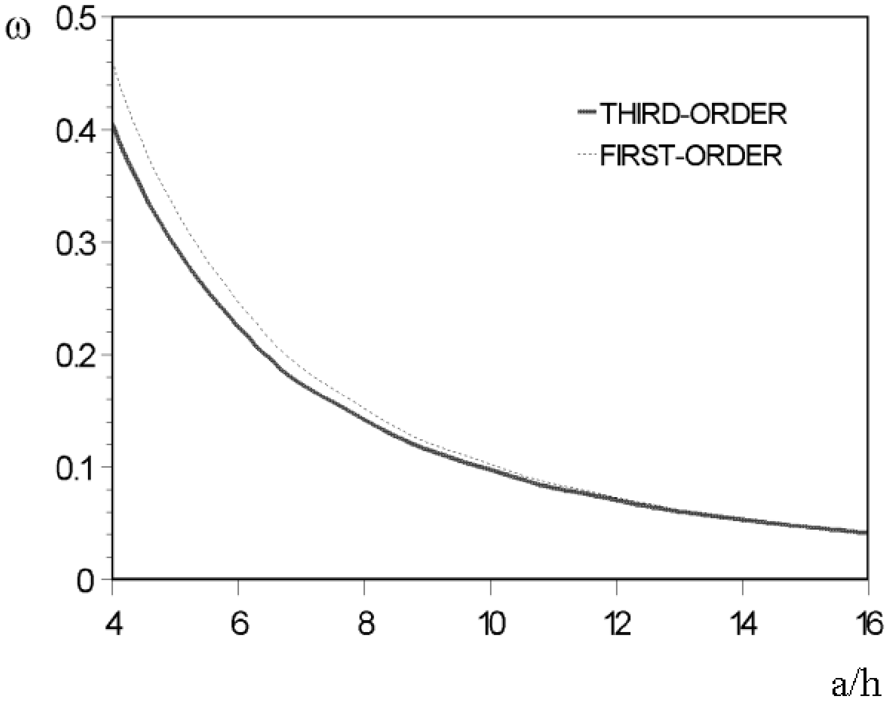

Figure 5-Values of nondimansionalized natural frequency $\varpi \cdot h\sqrt{\frac{\rho_1}{G_1}}$ for rectangular FGM plate in case $m = n = 2$, $\upsilon = 0.3$, $\rho_1/\rho_2 = 300/2707$, $G_1/G_2 = 151/70$ and $p = 3$.

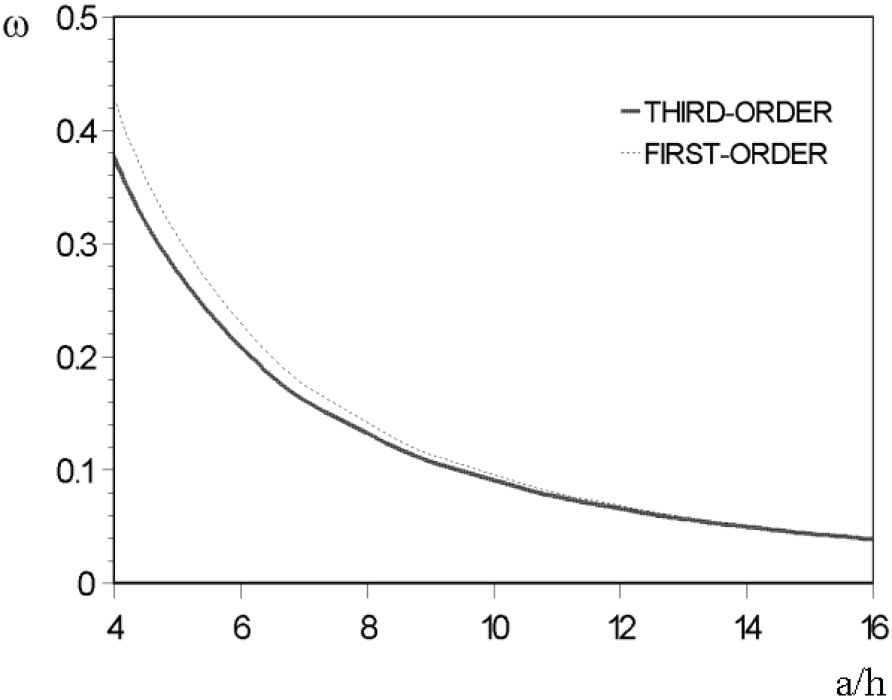

Figure 6-Values of nondimansionalized natural frequency $\varpi \cdot h\sqrt{\frac{\rho_1}{G_1}}$ for rectangular FGM plate in case $m = n = 2$, $\upsilon = 0.3$, $\rho_1/\rho_2 = 300/2707$, $G_1/G_2 = 151/70$ and $p = 5$.

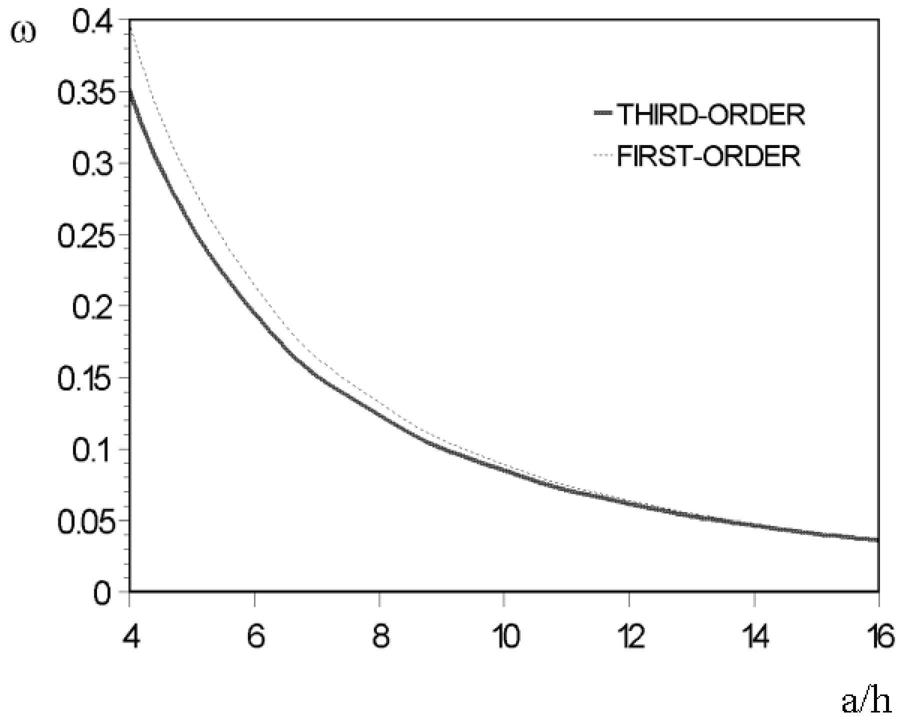

Figure 7-Values of nondimansionalized natural frequency $\varpi \cdot h\sqrt{\frac{\rho_1}{G_1}}$ for rectangular FGM plate in case $m = n = 2$, $\upsilon = 0.3$, $\rho_1 / \rho_2 = 300/2707$, $G_1 / G_2 = 151/70$ and $p = 10$.

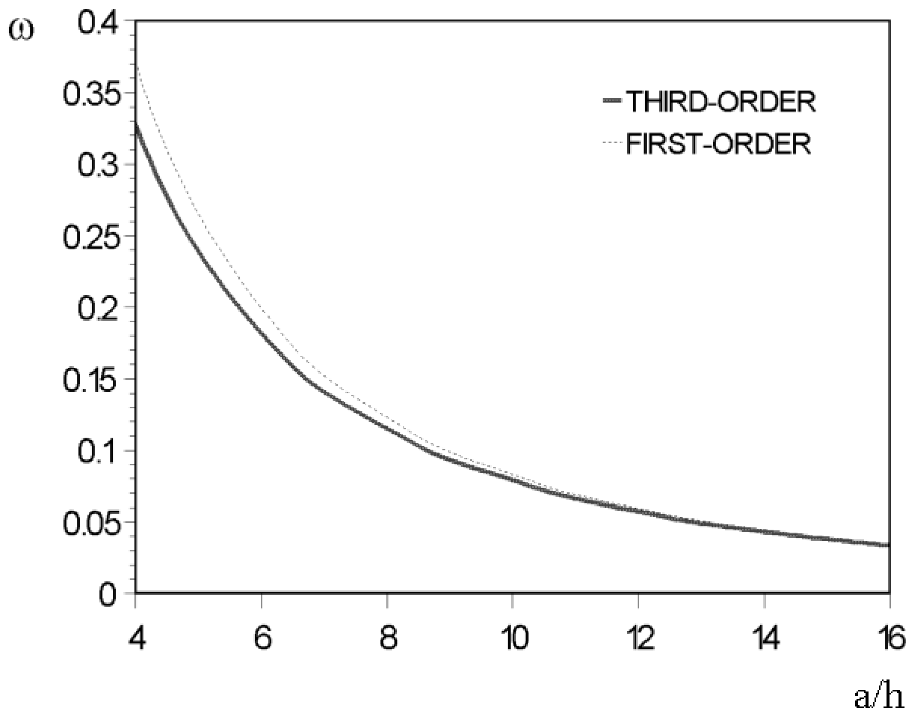

Figure 8-Values of nondimansionalized natural frequency $\varpi \cdot h\sqrt{\frac{\rho_1}{G_1}}$ for rectangular FGM plate in case $m = n = 2$, $\upsilon = 0.3$, $\rho_1 / \rho_2 = 300/2707$, $G_1 / G_2 = 151/70$ and $p = 30$.

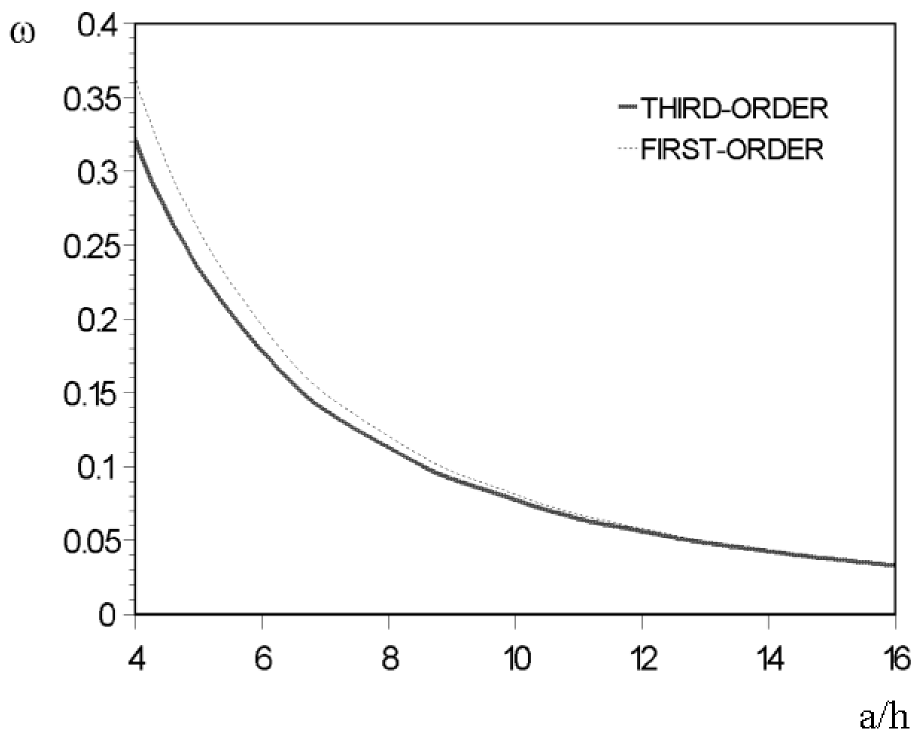

Figure 9- Values of nondimansionalized natural frequency $\overline{\omega} \cdot h\sqrt{\frac{\rho_1}{G_1}}$ for rectangular FGM plate in case $m = n = 2$, $\upsilon = 0.3$, $\rho_1 / \rho_2 = 300/2707$, $G_1 / G_2 = 151/70$ and $p = 50$.

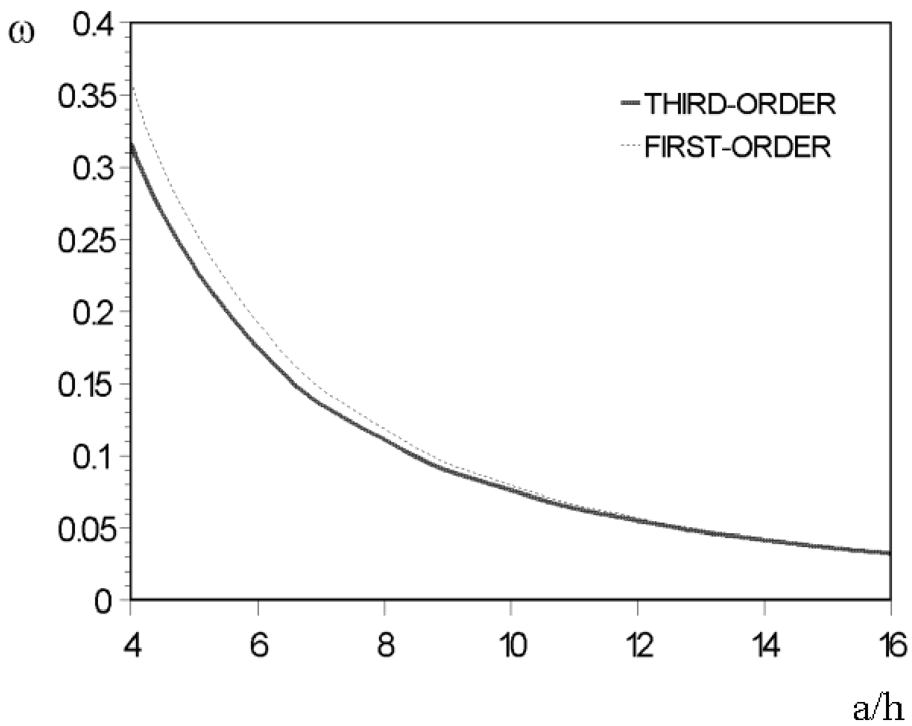

Figure 10- Values of nondimansionalized natural frequency $\overline{\omega} \cdot h\sqrt{\frac{\rho_1}{G_1}}$ for rectangular FGM plate in case $m = n = 2$, $\upsilon = 0.3$, $\rho_1 / \rho_2 = 300/2707$, $G_1 / G_2 = 151/70$ and $p = 100$.

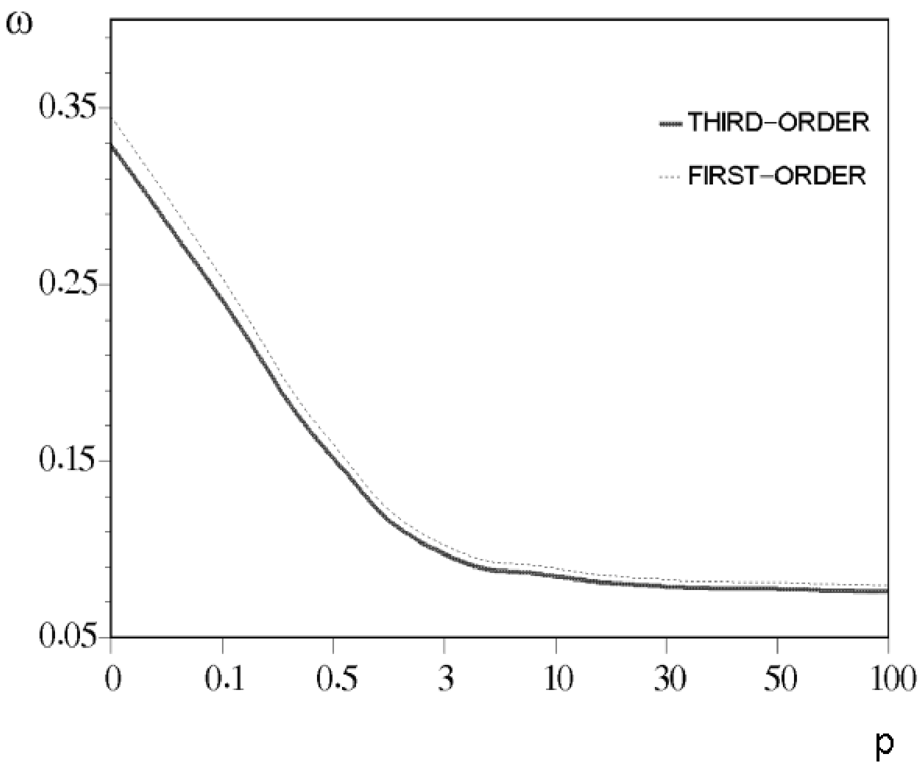

Figure 11- Values of nondimansionalized natural frequency $\overline{\omega} \cdot h\sqrt{\frac{\rho_1}{G_1}}$ for rectangular FGM plate in case $m = n = 2$, $\upsilon = 0.3$, $\rho_1 / \rho_2 = 300/2707$, $G_1 / G_2 = 151/70$.

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