Effect of Material Gradient on Stresses of Thick FGM Spherical Pressure Vessels with Exponentially-Varying Properties

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ARTICLE INFO

Article history:

Received 15 Apr. 2014
Accepted 23 May 2014
Available online 31 Aug. 2014

Keywords:

Thick sphere
Pressure vessel
Exponential
Frobenius series method (FSM)
Finite element method (FEM)

ABSTRACT

Using the Frobenius series method (FSM), an analytical solution is developed to obtain mechanical stresses of thick spherical pressure vessels made of functionally graded materials (FGMs). The cylinder pressure vessel is subjected to uniform internal pressure. The modulus of elasticity is graded along the radial direction according to power functions of the radial direction. It is assumed that Poisson's ratio is constant across the cylinder thickness. Primarily, displacements and stresses have been obtained as closed-form solution. Next, the profiles are plotted for different values of inhomogeneity constant along the radial direction. Finally, the problem was solved by using the finite element method (FEM). The obtained results of finite element method were compared with those of the analytical method. The analytical solutions and the solutions carried out through the FEM showed good agreement. The values used in this study are arbitrarily chosen to demonstrate the effect of inhomogeneity on displacements, and stresses distributions.

1. Introduction

In materials science, functionally graded materials (FGMs) are advanced composite materials whose mechanical properties vary continuously from one surface to another at macro level. From the perspective of solid mechanics, FGMs are non-homogeneous elastic mediums. Using this property in the FGMs, an engineer can design composite materials such that any portion of the materials reaches the same safety level [1].

Tutuncu and Ozturk [2], by using the infinitesimal theory of elasticity, obtained closed-form solutions for stresses and displacements in functionally graded cylindrical and spherical vessels subjected to internal pressure. Assuming that the material properties except Poisson's ratio are variable as power law of radius, Nayak et al. [3] investigated displacements, strains, and stresses of the FGM spherical vessel. Using an accurate method, elastic analysis of internally pressurized thick-walled spherical pressure vessels of functionally graded materials was studied by You et al. [4]. Tutuncu [5] derived the solutions of FGM thick cylindrical shells with exponentially-varying properties. Assuming that the material properties are

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variable in any arbitrary direction, and Poisson's ratio is constant, by using tensor analysis, Zamani Nejad et al. [6] obtained a complete and consistent 3-D set of field equations of FGM thick shells of revolution with arbitrary curvature and variable thickness. Chen and Lin [7] obtained the approximate solution for thick cylinders and spherical pressure vessels. Tutuncu and Temel [8] gained axisymmetric displacements and stresses in functionally graded hollow cylinders, disks and spheres subjected to uniform internal pressure using plane elasticity theory and complementary functions method. Assuming that the displacement function is unknown in the governing equation, Chen and Lin [9] obtained displacement and stresses in thick-walled cylinders and spheres made of functionally graded materials. Zamani Nejad et al. [10] presented an analytical solution and a numerical solution for stresses and radial displacement of parabolic FGM solid spheres with parabolic varying properties subjected uniform external pressure. In another study, using an analytical method, Zamani Nejad et al. [11] obtained an exact solution for stresses and displacements of pressurized thick spheres made of functionally graded material with exponentially-varying properties.

In this paper, by using the Frobenius series method, a closed-form analytical solution for displacements and stresses of FGM thick spherical pressure vessels with exponential varying properties has been obtained. For the numerical solution, a commercial finite element program ANSYS 12 has been used.

2. Experimental

Consider a functionally graded hollow sphere with inner radius a and outer radius b. The sphere is subjected to internal uniform pressure P. (Fig. 1)

The material is isotropic and functionally graded while the Poisson's ratio remains constant throughout the entire sphere. Modulus of elasticity E is assumed to vary exponential form as follows:

$$E = E_i e^{\beta\left(\frac{r-a}{b-a}\right)} \quad [1]$$

Where E_i and β are modulus of elasticity in inner surface and inhomogeneity constant, respectively. In the symmetrical deformation case, the equilibrium equation for the radial and circumferential stress components (σ_{rr} and $\sigma_{\theta\theta}$ respectively) disregarding the body forces and inertia terms takes the following form:

$$\frac{d\sigma_{rr}}{dr} + 2\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \quad [2]$$

Two radial and circumferential strain components (ε_{rr} and $\varepsilon_{\theta\theta}$ respectively) can be expressed as

$$\varepsilon_{rr} = \frac{du_r}{dr} \quad [3]$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r} \quad [4]$$

Here, the displacement in the *r*-direction is denoted by "u_r".

The stress and strain relations for non-homogeneous and isotropic spherical shell are

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} = \sigma_{\phi\phi} \end{bmatrix} = E \begin{bmatrix} A & 2B \\ B & A+B \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} = \varepsilon_{\phi\phi} \end{bmatrix} \quad [5]$$

Where *A* and *B* are related to Poisson's ratio ν as

$$A = \frac{(1-\nu)}{(1+\nu)(1-2\nu)} \quad [6]$$

$$B = \frac{\nu}{(1+\nu)(1-2\nu)} \quad [7]$$

Using Eqs. [1] to [5], the Navier equation in terms of the radial displacement is

$$r^2 u_r'' + r(2+hr)u_r' - 2(1-h\nu_1 r)u_r = 0 \quad [8]$$

Where

$$\begin{cases} \nu_1 = \dfrac{\nu}{1-\nu} \\ h = \dfrac{\beta}{b-a} \end{cases} \quad [9]$$

Eq. [9] can be solved by using power series method with the solution in the form of

$$u_r = \sum_{n=0}^{\infty} a_n r^{n+s} \quad [10]$$

Substituting Eq. [10] into Eq. [8]

$$a_0\left[(s-1)(s+2)\right]r^s + \sum_{n=1}^{\infty}\left[(n+s-1)(n+s+2)a_n + h(n+s-1+2\nu_1)a_{n-1}\right]r^{n+s} = 0 \quad [11]$$

Eq. [11] has an indicial equation and a recurrence relation, respectively. Since $a_0 \neq 0$

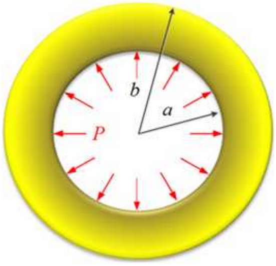

Fig. 1. Cross section of the FGM thick-walled spherical pressure vessel

$$(s-1)(s+2)=0 \quad \rightarrow \quad \begin{cases} s_1=+1 \\ s_2=-2 \end{cases} \qquad [12]$$

$$a_n=-\frac{h\left(n+s-1+2\nu_1\right)}{\left(n+s-1\right)\left(n+s+2\right)}a_{n-1} \qquad [13]$$

The indicial equation has roots that differ by an integer, thus only one of the solutions is in the form of Eq. [10]. Expansion of the above recurrence relation, the coefficient a_n in terms of a_0 and Gamma functions are obtained.

$$a_n=\frac{(-1)^n\,\Gamma\left(n+s+2\nu_1\right)\Gamma\left(s+3\right)^2 a_0 h^n}{s\left(s+1\right)\left(s+2\right)\Gamma\left(s+2\nu_1\right)\Gamma\left(n+s\right)\Gamma\left(n+s+3\right)} \qquad [14]$$

For the first root $(s_1=+1\,,\,a_0=1)$, the first solution of Eq. [8] obtained in the form of

$$u_1=r+\sum_{n=1}^{\infty}a_n\left(s_1\right)r^{n+1} \qquad [15]$$

Where

$$a_n\left(s_1\right)=\frac{6(-1)^n\,\Gamma\left(n+1+2\nu_1\right)}{\Gamma\left(2\nu_1+1\right)\Gamma\left(n+1\right)\Gamma\left(n+4\right)}h^n \qquad [16]$$

The second solution for $s_2=-2$ will be of the form

$$u_2=Qu_1\ln\left(r\right)+r^{s_2}\left[1+\sum_{n=1}^{\infty}C_n\left(s_2\right)r^n\right] \qquad [17]$$

Where

$$\begin{cases} Q=\lim_{s\to s_2}\left(s-s_2\right)a_N\left(s\right)=-\dfrac{h^3\nu_1\left(1-\nu_1\right)\left(1-2\nu_1\right)}{3} \\ C_n\left(s_2\right)=\dfrac{d}{ds}\left[\left(s-s_2\right)a_n\left(s\right)\right] \end{cases} \qquad [18]$$

Assuming that

$$L=\frac{(-1)^n\,\Gamma\left(n+s+2\nu_1\right)\Gamma\left(s+3\right)^2}{s\left(s+1\right)\Gamma\left(s+2\nu_1\right)\Gamma\left(n+s\right)\Gamma\left(n+s+3\right)}h^n \qquad [19]$$

hence,

$$L_{s_2=-2}=\frac{(-1)^n\,\Gamma\left(n+2\nu_1-2\right)}{2\left(n\right)!\left(n-3\right)!\,\Gamma\left(2\nu_1-2\right)}h^n \qquad [20]$$

and

$$\frac{d}{ds}\left[\ln\left(L\right)\right]\Big|_{s=s_2}=1.5+2\frac{\Gamma'(1)}{\Gamma(1)}+\frac{\Gamma'\left(n+2\nu_1-2\right)}{\Gamma\left(n+2\nu_1-2\right)} -\frac{\Gamma'\left(2\nu_1-2\right)}{\Gamma\left(2\nu_1-2\right)}-\frac{\Gamma'\left(n-2\right)}{\Gamma\left(n-2\right)}-\frac{\Gamma'\left(n+1\right)}{\Gamma\left(n+1\right)} \qquad [21]$$

It is obvious that

$$\Psi\left(z\right)=\frac{\Gamma'\left(z\right)}{\Gamma\left(z\right)} \qquad [22]$$

Where $\Psi(z)$ is called Psi (Digamma) function. With substituting Eq. [22] into Eq. [21]

$$\frac{d}{ds}\left[\ln\left(L\right)\right]\Big|_{s=s_2}=1.5+2\Psi\left(1\right)+\Psi\left(n+2\nu_1-2\right) -\Psi\left(2\nu_1-2\right)-\Psi\left(n-2\right)-\Psi\left(n+1\right) \qquad [23]$$

Hence, the final form of $C_n\left(s_2\right)$ will be as follows

$$C_n\left(s_2\right)=\frac{(-1)^n\,\Gamma\left(n+2\nu_1-2\right)h^n}{2n!\left(n-3\right)!\,\Gamma\left(2\nu_1-2\right)} \left[1.5+2\Psi\left(1\right)+\Psi\left(n+2\nu_1-2\right) -\Psi\left(2\nu_1-2\right)-\Psi\left(n-2\right)-\Psi\left(n+1\right)\right] \qquad [24]$$

Thus, the complete solution for u_r is expressed as

$$u_r=c_1u_1+c_2u_2=c_1\left[r+\sum_{n=1}^{\infty}a_n\left(s_1\right)r^{n+1}\right] +c_2\left[Q\left[r+\sum_{n=1}^{\infty}a_n\left(s_1\right)r^{n+1}\right]\ln\left(r\right) +\left[r^{-2}+\sum_{n=1}^{\infty}C_n\left(s_2\right)r^{n-2}\right]\right] \qquad [25]$$

Substituting Eq. [25] into Eq. [5]

$$\sigma_{rr}=AEc_1\left[1+2\nu_1+\sum_{n=1}^{\infty}Fa_n\left(s_1\right)r^n\right] +AEc_2\left[Q\left[\left(1+2\nu_1\right)\ln\left(r\right)+1\right]-2\frac{1-\nu_1}{r^3} +Q\sum_{n=1}^{\infty}\left[F\ln\left(r\right)+1\right]a_n\left(s_1\right)r^n+\sum_{n=1}^{\infty}GC_n\left(s_2\right)r^{n-3}\right] \qquad [26]$$

Where

$$\begin{cases} F = n+1+2\nu_1 \\ G = n-2+2\nu_1 \end{cases} \quad [27]$$

The constant coefficients c_1 and c_2 are found using the boundary conditions $\sigma_{rr}(r=a)=-P$ and $\sigma_{rr}(r=b)=0$

$$\begin{cases} c_1 = \dfrac{-PD_4}{AE_i\left[D_1D_4 - D_2D_3\right]} \\ c_2 = \dfrac{PD_3}{AE_i\left[D_1D_4 - D_2D_3\right]} \end{cases} \quad [28]$$

Where

$$\begin{cases} D_1 = 1+2\nu_1 + \sum_{n=1}^{\infty} Fa_n(s_1)a^n \\ D_2 = Q\left[(1+2\nu_1)\ln(a)+1\right] - 2\dfrac{1-\nu_1}{a^3} \\ +Q\sum_{n=1}^{\infty}\left[F\ln(a)+1\right]a_n(s_1)a^n + \sum_{n=1}^{\infty} GC_n(s_2)a^{n-3} \\ D_3 = 1+2\nu_1 + \sum_{n=1}^{\infty} Fa_n(s_1)b^n \\ D_4 = Q\left[(1+2\nu_1)\ln(b)+1\right] - 2\dfrac{1-\nu_1}{b^3} \\ +Q\sum_{n=1}^{\infty}\left[F\ln(b)+1\right]a_n(s_1)b^n + \sum_{n=1}^{\infty} GC_n(s_2)b^{n-3} \end{cases} \quad [29]$$

We have the displacement and stresses equations in the following form:

$$u_r = \frac{-P}{AE_i\left[D_1 - kD_2\right]}\left[r\left(1-kQ\ln(r)\right) - \frac{k}{r^2} + \sum_{n=1}^{\infty}\left[1-kQ\ln(r)\right]a_n(s_1)r^{n+1} - k\sum_{n=1}^{\infty} C_n(s_2)r^{n-2}\right] \quad [30]$$

$$\sigma_{rr} = \frac{-Pe^{\beta\left(\frac{r-a}{b-a}\right)}}{D_1 - kD_2}\left[1+2\nu_1 - k\left[Q\left[(1+2\nu_1)\ln(r) + 1\right] - 2\frac{1-\nu_1}{r^3}\right] + \sum_{n=1}^{\infty}\left[F - kQ\left[F\ln(r) + 1\right]\right]a_n(s_1)r^n - k\sum_{n=1}^{\infty} GC_n(s_2)r^{n-3}\right] \quad [31]$$

$$\sigma_{\phi\phi} = \frac{-Pe^{\beta\left(\frac{r-a}{b-a}\right)}}{D_1 - kD_2}\left[1+2\nu_1 - k\left[Q\left[(1+2\nu_1)\ln(r) + \nu_1\right] + \frac{1-\nu_1}{r^3}\right] + \sum_{n=1}^{\infty}\left[F^* - kQ\left[F^*\ln(r) + \nu_1\right]\right]a_n(s_1)r^n - k\sum_{n=1}^{\infty} G^*C_n(s_2)r^{n-3}\right] \quad [32]$$

Where

$$\begin{cases} k = \dfrac{D_3}{D_4} \\ F^* = \nu_1(n+2)+1 \\ G^* = \nu_1(n-1)+1 \end{cases} \quad [33]$$

3. Results and Discussion

Consider a thick spherical pressure vessel under the internal pressure of 40 MPa. The pressure vessel has the inner and outer radii 30 cm and 60 cm, respectively. In this study, it is supposed that the modulus of elasticity at the internal radius has the value of 200 GPa and the Poisson's ratio, ν, has a constant value of 0.3. For a comparative study on numerical analysis of this problem, a geometric specimen is modeled using commercial finite element code, ANSYS 12. Due to the geometrical symmetry in the sphere, only a quarter of the specimen geometry in the finite element model was considered. In order to represent the non-homogeneous specimen, an 8-node axisymmetric quadrilateral element was used. The variation in material properties was implemented by 20 layers, with each layer having a constant value of material properties, for modeling of the FGM spherical pressure vessel.

Fig. 2 shows the distribution of elastic modulus in the radial direction. The elastic modulus increases as β increases.

Fig. 3 shows the distribution of tensile radial displacement along the normalized radial direction. It is obvious in this curve that the radial displacement decreases as β increases at the same position.

In Fig. 4 distribution of compressive radial stress along normalized radial direction is shown. It is perceived that the radial stress

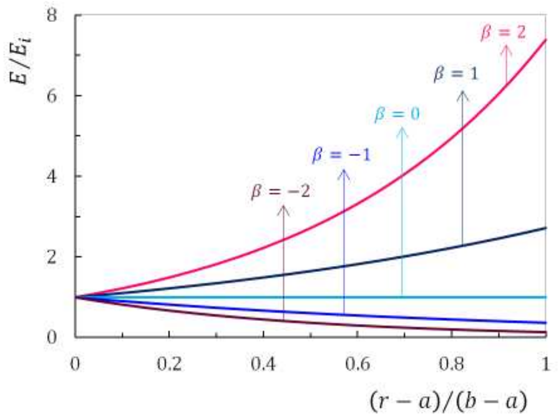

Fig. 2. Radial distribution of modulus of elasticity

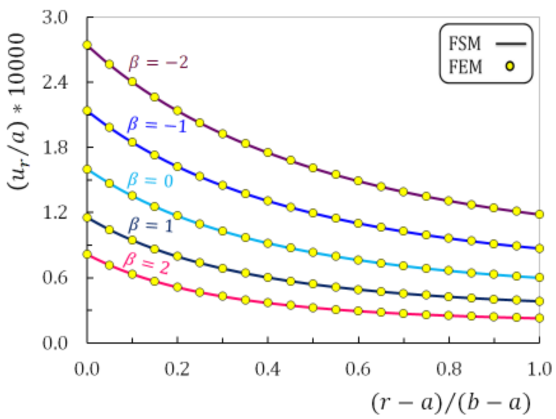

Fig. 3. Radial distribution of radial displacement

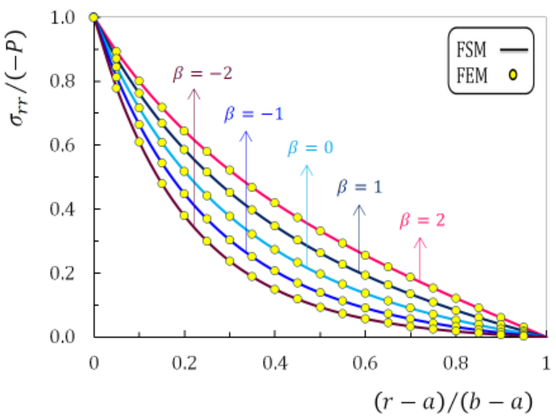

Fig. 4. Radial distribution of radial stress

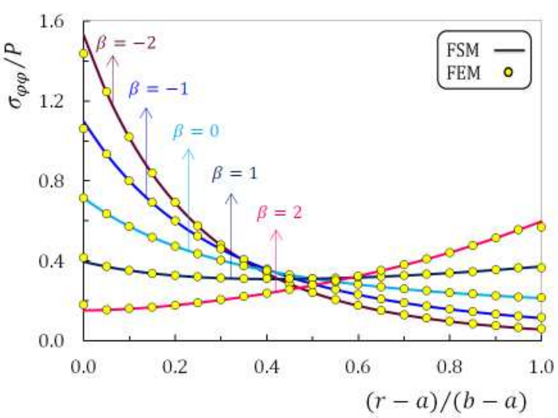

Fig. 5. Radial distribution of circumferential stress

increases for higher values of β.

In Fig. 5 the tensile circumferential stress along normalized radial direction for different values of β is plotted. Here, it should be noted that in the same situation, approximately, for $(r-a/b-a)<0.5$, the value of the circumferential stress decreases as β increases, whereas for $(r-a/b-a)>0.5$ this situation was reversed.

Further, along the radial direction, approximately, for $\beta<1$ the circumferential stress decreases, while almost for $\beta>1$, the circumferential stress increases.

It should also be noted that for all the considered conditions, for $\beta=1$ the circumferential stress remains almost uniform along the radius of the sphere. The issue can be a valuable factor for controlling the stress.

For the purpose of studying the stress distribution along the spherical pressure vessel radius, in Fig. 6 the von Mises equivalent stress ($\sigma_{eq}=\sigma_r-\sigma_\phi$) is plotted in the radial direction.

The von Mises equivalent stress decreases as the radius increases for all β values.

In Figs. 7 to 10, the radial displacements, radial, circumferential and von Mises equivalent stresses values obtained from ANSYS commercial finite elements analysis program are represented.

For verification, the Frobenius series method results are compared with those of the numerical solution of Chen and Lin method [9] (Figs 11 and 12 ($a=30$ cm, $b=60$ cm, $\nu=0.3$)).

4. Conclusions

The Frobenius series method is a powerful technique for obtaining solutions of certain differential equations which occur in applications. Based on basic equations of elasticity and using FSM, closed-form solutions have been derived for stresses and the

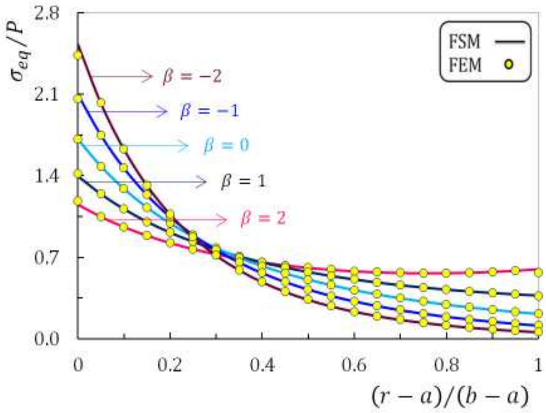

Fig. 6. Radial distribution of von Mises equivalent stress

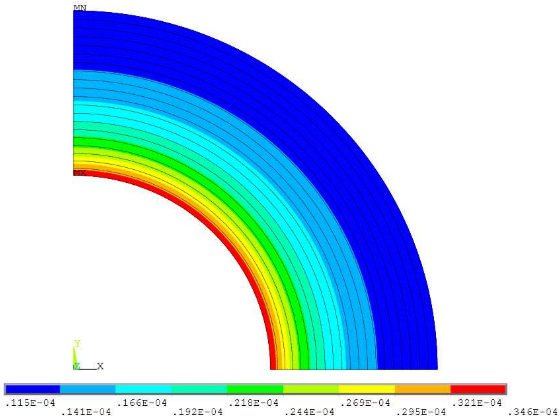

Fig. 7. Radial displacement obtained from ANSYS code in an FGM spherical pressure vessel ($\beta = 1$)

Fig. 8. Radial stress obtained from ANSYS code in an FGM spherical pressure vessel ($\beta = 1$)

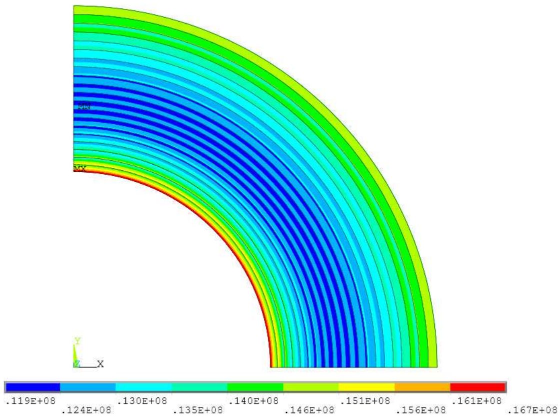

Fig. 9. Circumferential stress obtained from ANSYS code in an FGM spherical pressure vessel ($\beta = 1$)

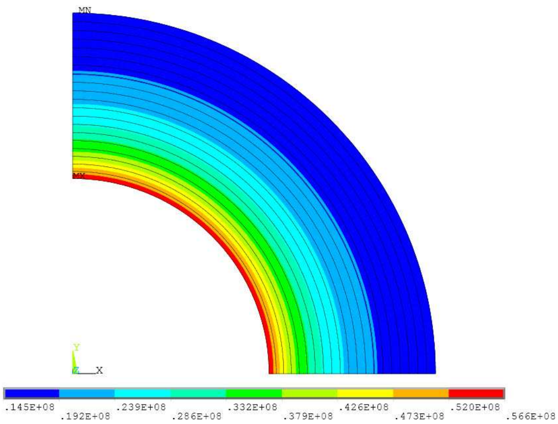

Fig. 10. Von Mises equivalent stress obtained from ANSYS code in an FGM spherical pressure vessel ($\beta = 1$)

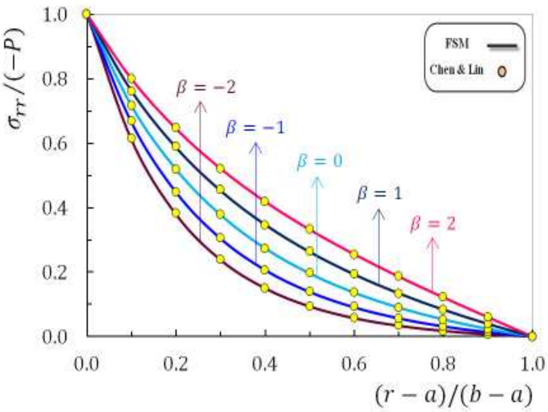

Fig. 11. Non dimensional radial stress in spherical pressure vessel with an inner pressure P

displacements of thick spherical pressure vessels made of functionally graded materials with exponential varying properties. Following this, profiles are plotted for different values of inhomogeneity constant for the radial displacement, radial stress, and circumferential

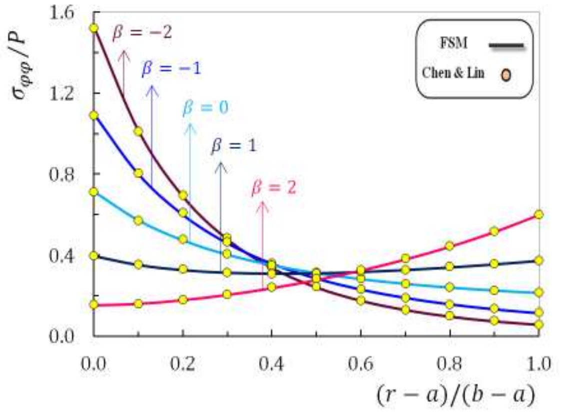

Fig. 12. Non dimensional tangential stress in spherical pressure vessel with an inner pressure P

stress, as a function of radial direction. In this study, a numerical solution using a commercial finite elements code, ANSYS 12, is also presented. Good agreement was found between the analytical solutions and the solutions carried out through finite element code. The presented results show that the inhomogeneity constant has a significant influence on mechanical behaviors of the thick spherical pressure vessels made of functionally graded material with exponential varying properties.

References

1. M. Ghannad, M. Z. Nejad, “Elastic solution of pressurized clamped-clamped thick cylindrical shells made of functionally graded materials”, J. Theor. Appl. Mech., Vol. 51, 2013, pp. 1067-1079.
2. N. Tutuncu, M. Ozturk, “Exact solution for stresses in functionally graded pressure vessels”, Compos. Part B-Eng., Vol. 32, 2001, pp. 683-686.
3. P. Nayak, S. C. Mondal, A. Nandi, “Stress, strain and displacement of a functionally graded thick spherical vessel”, Int. J. Eng. Sci., Vol. 3, 2001, pp. 2659-2671.
4. L. H. You, J. J. Zhang, J. J. You, “Elastic analysis of internally pressurized thick-walled spherical pressure vessels of functionally graded materials”, Int. J. Pres. Ves. Pip., Vol. 82, 2005, pp. 347-354.
5. N. Tutuncu, “Stresses in thick-walled FGM cylinders with exponentially varying properties”, Eng. Struct., Vol. 29, 2007, pp. 2032-2035.
6. M. Z. Nejad, G. H. Rahimi, M. Ghannad, “Set of field equations for thick shell of revolution made of functionally graded materials in curvilinear coordinate system”, Mechanika, Vol. 77, 2009, pp. 18-26.
7. Y. Z. Chen, X. Y. Lin, “Elastic analysis for thick cylinders and spherical pressure vessels made of functionally graded materials”, Comp. Mater. Sci., Vol. 44, 2008, pp. 581-587.
8. N. Tutuncu, B. Temel, “A novel approach to stress analysis of pressurized FGM cylinders, disks and spheres”, Compos. Struct., Vol. 91, 2009, pp. 385-390.
9. Y. Z. Chen, X. Y. Lin, “An alternative numerical solution of thick-walled cylinders and spheres made of functionally graded materials”, Comp. Mater. Sci., Vol. 48, 2010, pp. 640-647.
10. M. Z. Nejad, M. Abedi, M. H. Lotfian, M. Ghannad, “Exact and numerical solutions for stresses in pressurized FGM solid sphere with parabolic varying properties”, Am. J. Sci. Res., Vol. 32, 2011, pp. 82-89.
11. M. Z. Nejad, M. Abedi, M. H. Lotfian, M. Ghannad, “An exact solution for stresses and displacements of pressurized FGM thick-walled spherical shells with exponential-varying properties”, J. Mech. Sci. Technol., Vol. 26, 2012, pp. 4081-4087.