

## Some fixed point results of $T$ -contractions on partially ordered cone metric spaces under $c$ -distances

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**Abstract.** The main purpose of this article is to state some fixed point results of  $T$ -contractions on partially ordered cone metric spaces under  $c$ -distances using two ways; directed and indirected ways. Some notes and corollaries are also added to demonstrate the applicability of main results.

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**Keywords:**  $T$ -contraction, partially ordered cone metric space,  $c$ -distance, fixed point.

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### 1. Introduction and preliminaries

In 1996, Kada et al. [10] proposed the following concept of  $w$ -distances on metric spaces to improve Eklund's variational principle and Takahashi's non-convex minimization theorem.

**Definition 1.1** Assume  $(X, d)$  is a metric space and  $p : X \times X \rightarrow [0, +\infty)$  is a function satisfying the following conditions for all  $x, y, z \in X$ :

- (w<sub>1</sub>)  $p(x, z) \leq p(x, y) + p(y, z)$ ;
- (w<sub>2</sub>)  $p$  is lower semi-continuous in its second variable;
- (w<sub>3</sub>) for each  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $p(z, x) \leq \delta$  and  $p(z, y) \leq \delta$  imply that  $d(x, y) \leq \varepsilon$ .

Then  $p$  is named a  $w$ -distance on  $X$ .

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Note that each metric is a  $w$ -distance, but the converse is not correct. Also, it should be mentioned that a  $w$ -distance has two important differences from a usual metric.  $w(x, y) = 0$  is not equivalent to  $x = y$  and a  $w$ -distance is not necessarily symmetric. For these, it is enough to take  $w(x, y) = y$  for any  $x, y \in [0, +\infty)$ . For other examples, convergent properties of  $w$ -distances and various fixed point results regarding this distance, see [10, 23, 27] and their references. In 2011, Cho et al. [6] defined a cone version of the  $w$ -distance which is called a  $c$ -distance and proved several fixed point theorems in ordered cone metric spaces in which cone metric spaces are introduced by Huang and Zhang [9] and partially ordered metric spaces are stated by Ran and Reurings [24].

**Definition 1.2** Assume  $E$  is a real Banach space and  $\theta$  denote the zero element in  $E$ . A subset  $P$  of  $E$  is a cone if the followings are held:

- (a)  $P$  is closed, non-empty and  $P \neq \{\theta\}$ ;
- (b)  $a, b \in [0, +\infty)$  and  $x, y \in P$  imply that  $ax + by \in P$ ;
- (c)  $x \in P$  and  $-x \in P$  imply  $x = \theta$ .

Given a  $P \subset E$ , we define a partial order  $\preceq$  with respect to  $P$  by  $x \preceq y$  iff  $y - x \in P$ . We write  $x \prec y$  if  $x \preceq y$  and  $x \neq y$ . Also, we take  $x \ll y$  iff  $y - x \in \text{int}P$ ,  $\text{int}P$  is interior of  $P$ . If  $\text{int}P \neq \emptyset$ , the cone  $P$  is called solid.

**Definition 1.3** [9, Huang and Zhang, 2007] Assume  $X \neq \emptyset$  and  $E$  is a real Banach space equipped with the partial ordering  $\preceq$  with respect to the cone  $P \subset E$ . A mapping  $d : X \times X \rightarrow P$  is called a cone metric on  $X$  if, for all  $x, y, z \in X$ , the following conditions are held:

- (d<sub>1</sub>)  $d(x, y) = \theta$  iff  $x = y$ ;
- (d<sub>2</sub>)  $d(x, y) = d(y, x)$ ;
- (d<sub>3</sub>)  $d(x, z) \preceq d(x, y) + d(y, z)$ .

In this manner,  $(X, d)$  is called a cone metric space.

**Definition 1.4** [6, Cho et al., 2011] Assume  $(X, d)$  is a cone metric space. A function  $q : X \times X \rightarrow P$  is called a  $c$ -distance on  $X$  if, for all  $x, y, z \in X$ , the following properties hold:

- (q<sub>1</sub>)  $q(x, z) \preceq q(x, y) + q(y, z)$ ;
- (q<sub>2</sub>) if  $q(x, y_n) \preceq u$  for some  $u = u_x$  and all  $n \geq 1$ , then  $q(x, y) \preceq u$  when  $\{y_n\}$  is a sequence in  $X$  converging to a  $y \in X$ ;
- (q<sub>3</sub>) for all  $c \in \text{int}P$ , there is  $e \in E$  with  $\theta \ll e$  so that  $q(z, x) \ll e$  and  $q(z, y) \ll e$  induce  $d(x, y) \ll c$ .

Note that each cone metric is a  $c$ -distance, but the converse is not correct. Also, it should be mentioned that a  $c$ -distance has two important differences from a cone metric.  $q(x, y) = \theta$  is not equivalent to  $x = y$  and a  $c$ -distance is not necessarily symmetric.

**Lemma 1.5** [6] Assume  $(X, d)$  is a cone metric space,  $q$  is a  $c$ -distance on  $X$ ,  $\{x_n\}$  and  $\{y_n\}$  are sequences in  $X$  and  $x, y, z \in X$ . Also, suppose  $\{u_n\}$  and  $\{v_n\}$  are two sequences in  $P$  converging to  $\theta$ . Then

- (qp<sub>1</sub>) If  $q(x_n, y) \preceq u_n$  and  $q(x_n, z) \preceq v_n$  for  $n \in \mathbb{N}$ , then  $y = z$ . Specially, if  $q(x, y) = \theta$  and  $q(x, z) = \theta$ , then  $y = z$ .
- (qp<sub>2</sub>) If  $q(x_n, y_n) \preceq u_n$  and  $q(x_n, z) \preceq v_n$  for  $n \in \mathbb{N}$ ,  $\{y_n\}$  converges to  $z$ .
- (qp<sub>3</sub>) If  $q(x_n, x_m) \preceq u_n$  for  $m > n$ ,  $\{x_n\}$  is a Cauchy sequence in  $X$ .
- (qp<sub>4</sub>) If  $q(y, x_n) \preceq u_n$  for  $n \in \mathbb{N}$ ,  $\{x_n\}$  is a Cauchy sequence in  $X$ .

For other examples, convergent properties of  $c$ -distances and some fixed point results regarding this distance, see [2, 20–22] and their references. Moreover, there have been defined several types of weak distances in various metric spaces by many researchers that some of them can be found in [3, 7, 8, 12, 13, 16, 26] and references therein.

On the other hand, Chi [5, 2009] defined the concept of a  $T$ -contraction. After that, many authors applied this concept to prove some well-known fixed point theorems in [14, 15, 17–19, 22] and their reference.

**Definition 1.6** Assume  $(X, d)$  is a metric space and  $f, T : X \rightarrow X$  are two mappings. Then  $f$  is called a  $T$ -contraction if there is  $\alpha \in [0, 1)$  so that

$$d(Tfx, Tfy) \leq \alpha d(Tx, Ty) \quad (1)$$

for all  $x, y \in X$ .

It is clear that if  $T$  is an identity mapping, then  $T$ -contraction and Banach contraction will be equal. To prove the existence of a fixed point for  $T$ -contraction mapping, we need two conditions for  $T$ , which are defined below.

**Definition 1.7** [5, 15] Assume  $(X, d)$  is a (cone) metric space and  $T : X \rightarrow X$  is a mapping. Then  $T$  is called

- (i) sequentially convergent if  $\{Tx_n\}$  is convergent for every sequence  $\{x_n\}$ , then  $\{x_n\}$  is convergent;
- (ii) continuous if  $\lim_{n \rightarrow \infty} x_n = x$  implies that  $\lim_{n \rightarrow \infty} Tx_n = Tx$  for all  $\{x_n\}$  in  $X$ .

But, in 2012, Aydi et al. [1] proved that the fixed point results of  $T$ -contractions are equivalent to former fixed point results (also, see [4]). The same is done for a  $w$ -distance ( $c$ -distance) in [11, 27] and references therein.

**Proposition 1.8** [1, 4, 11, 27] Assume  $(X, d)$  is a complete metric space and  $p$  is a  $w$ -distance on  $X$ . Also, suppose  $T$  is a continuous, injective and sequentially convergent mapping on  $X$ . Presume  $d^* : X \times X \rightarrow \mathbb{R}$  and  $p^* : X \times X \rightarrow [0, +\infty)$  are defined by

$$d^*(x, y) = d(Tx, Ty) \quad \text{and} \quad p^*(x, y) = p(Tx, Ty) \quad (2)$$

for all  $x, y \in X$ , respectively. Then  $d^*$  is a complete metric and  $p^*$  is a  $w$ -distance.

Note that Proposition 1.8 is held for both cone metric spaces and partially metric spaces and combination of both spaces, named a partially ordered cone metric space.

## 2. Main results

The following result is the first main theorem of this paper that shows the existence of fixed point for a  $T$ -contraction mapping of Chatterjea type on a cone metric space under a  $c$ -distance.

**Theorem 2.1** Assume  $(X, \sqsubseteq, d)$  is a complete partially ordered cone metric space and  $q$  is a  $c$ -distance on  $X$ . Also, presume  $f, T : X \rightarrow X$  are two mapping so that  $T$  is injective, continuous and sequentially convergent and  $f$  is continuous and nondecreasing respect to  $\sqsubseteq$ . Moreover, suppose that there are  $\alpha, \beta, \gamma : X \rightarrow [0, 1)$  so that the following conditions are held:

- ( $t_1$ )  $\alpha(fx) \leq \alpha(x)$ ,  $\beta(fx) \leq \beta(x)$  and  $\gamma(fx) \leq \gamma(x)$  for all  $x \in X$ ;  
 ( $t_2$ )  $(\alpha + 2\beta + 2\gamma)(x) < 1$  for all  $x \in X$ ;  
 ( $t_3$ ) for all comparable  $x, y \in X$ ,

$$q(Tfx, Tfy) \preceq \alpha(x)q(Tx, Ty) + \beta(x)q(Tx, Tfy) + \gamma(x)q(Ty, Tfx), \quad (3)$$

$$q(Tfy, Tfx) \preceq \alpha(x)q(Ty, Tx) + \beta(x)q(Tfy, Tx) + \gamma(x)q(Tfx, Ty). \quad (4)$$

If there exists  $x_0 \in X$  so that  $x_0 \sqsubseteq fx_0$ , then  $f$  has a fixed point. Moreover, if  $fz = z$ , then  $q(Tz, Tz) = \theta$ .

**Proof.** We will prove this theorem via two ways; directed and indirected methods.

**Directed method.** If  $fx_0 = x_0$ , then  $x_0$  is a fixed point of  $f$  and the proof ends. Now, presume  $fx_0 \neq x_0$ . As  $f$  is nondecreasing with respect to  $\sqsubseteq$  and  $x_0 \sqsubseteq fx_0$ , we have by induction that

$$x_0 \sqsubseteq x_1 = fx_0 \sqsubseteq \cdots \sqsubseteq x^n = f^n x_0 \sqsubseteq \cdots$$

in which  $x_n = fx_{n-1} = f^n x_0$  for all  $n \in \mathbb{N}$ . Setting  $x = x_n$  and  $y = x_{n-1}$  in (3), we obtain

$$\begin{aligned} q(Tx_{n+1}, Tx_n) &= q(Tfx_n, Tfx_{n-1}) \\ &\preceq \alpha(x_n)q(Tx_n, Tx_{n-1}) + \beta(x_n)q(Tx_n, Tx_n) \\ &\quad + \gamma(x_n)q(Tx_{n-1}, Tx_{n+1}) \\ &\preceq \alpha(fx_{n-1})q(Tx_n, Tx_{n-1}) \\ &\quad + \beta(fx_{n-1})[q(Tx_n, Tx_{n+1}) + q(Tx_{n+1}, Tx_n)] \\ &\quad + \gamma(fx_{n-1})[q(Tx_{n-1}, Tx_n) + q(Tx_n, Tx_{n+1})] \\ &\preceq \alpha(x_{n-1})q(Tx_n, Tx_{n-1}) + (\beta + \gamma)(x_{n-1})q(Tx_n, Tx_{n+1}) \\ &\quad + \beta(x_{n-1})q(Tx_{n+1}, Tx_n) + \gamma(x_{n-1})q(Tx_{n-1}, Tx_n) \\ &\quad \vdots \\ &\preceq \alpha(x_0)q(Tx_n, Tx_{n-1}) + (\beta + \gamma)(x_0)q(Tx_n, Tx_{n+1}) \\ &\quad + \beta(x_0)q(Tx_{n+1}, Tx_n) + \gamma(x_0)q(Tx_{n-1}, Tx_n). \end{aligned} \quad (5)$$

Similarly, setting  $x = x_n$  and  $y = x_{n-1}$  in (4), we have

$$\begin{aligned} q(Tx_n, Tx_{n+1}) &\preceq \alpha(x_0)q(Tx_{n-1}, Tx_n) + \beta(x_0)q(Tx_n, Tx_{n+1}) \\ &\quad + (\beta + \gamma)(x_0)q(Tx_{n+1}, Tx_n) + \gamma(x_0)q(Tx_n, Tx_{n-1}). \end{aligned} \quad (6)$$

Adding up (5) and (6), we obtain

$$\begin{aligned} q(Tx_{n+1}, Tx_n) + q(Tx_n, Tx_{n+1}) &\preceq (\alpha + \gamma)(x_0)[q(Tx_n, Tx_{n-1}) + q(Tx_{n-1}, Tx_n)] \\ &\quad + (2\beta + \gamma)(x_0)[q(Tx_{n+1}, Tx_n) + q(Tx_n, Tx_{n+1})]. \end{aligned}$$

Setting

$$u_n = q(Tx_{n+1}, Tx_n) + q(Tx_n, Tx_{n+1}),$$

we have

$$u_n \preceq (\alpha + \gamma)(x_0)u_{n-1} + (2\beta + \gamma)(x_0)u_n.$$

Thus, we have  $u_n \preceq \lambda u_{n-1}$  in which, by (t2),

$$\lambda = \frac{(\alpha + \gamma)(x_0)}{1 - (2\beta + \gamma)(x_0)} < 1.$$

Following this process, we have  $u_n \preceq \lambda^n u_0$  for all  $n \in \mathbb{N}$ . Thus,

$$q(Tx_n, Tx_{n+1}) \preceq u_n \preceq \lambda^n [q(Tx_1, Tx_0) + q(Tx_0, Tx_1)]. \quad (7)$$

Assume  $m > n$  for  $m, n \in \mathbb{N}$ . It follows from (7) and  $\lambda \in [0, 1)$  that

$$q(Tx_n, Tx_m) \preceq \frac{\lambda^n}{1 - \lambda} [q(Tx_1, Tx_0) + q(Tx_0, Tx_1)].$$

Using Lemma 1.5,  $\{Tx_n\}$  is a Cauchy sequence on  $X$ . As  $X$  is complete,  $\{Tx_n\}$  is a convergent sequence. Since  $T$  is injective, continuous and sequentially convergent, we conclude that there exists a  $x' \in X$  so that  $x_n \rightarrow x'$  as  $n \rightarrow \infty$ . Since  $f$  is continuous,  $fx_n \rightarrow fx'$  as  $n \rightarrow \infty$  and  $Tfx_n \rightarrow Tfx'$ . Because of the uniqueness of limit, we have  $Tfx' = Tx'$ . It follows from the injectivity of  $T$  that  $fx' = x'$ ; that is,  $x'$  is a fixed point of  $f$ . Now, suppose that  $fz = z$ . Then, (3) implies that

$$\begin{aligned} q(Tz, Tz) &= q(Tfz, Tfz) \\ &\preceq \alpha(z)q(Tz, Tz) + \beta(z)q(Tz, Tfz) + \gamma(z)q(Tz, Tfz) \\ &= (\alpha + \beta + \gamma)(z)q(Tz, Tz). \end{aligned}$$

Since

$$(\alpha + \beta + \gamma)(z) < (\alpha + 2\beta + 2\gamma)(z) < 1,$$

we have  $q(Tz, Tz) = \theta$ . This completes the proof.

**Indirected method.** Applying Proposition 3.1 of Karimizad's work [11] and Proposition 1.8, we conclude that  $d^* : X \times X \rightarrow \mathbb{R}$  and  $q^* : X \times X \rightarrow [0, +\infty)$  defined by

$$\begin{cases} d^*(x, y) = d(Tx, Ty), \\ q^*(x, y) = q(Tx, Ty), \end{cases}$$

for all  $x, y \in X$  are a complete metric and a  $c$ -distance, respectively. In this case, we reach Theorem 3.1 of [21] with notations  $d^*$  and  $q^*$ , and thus, the resident of the proof follows the proof of Theorem 3.1 of [21]. ■

Note that if  $T$  is an identity mapping, we have the same Theorem 3.1 of [21].

**Corollary 2.2** Assume  $(X, \sqsubseteq, d)$  is a complete partially ordered cone metric space and  $q$  is a  $c$ -distance on  $X$ . Also, suppose  $f, T : X \rightarrow X$  are two mapping so that  $T$  is injective, continuous and sequentially convergent and  $f$  is continuous and nondecreasing respect to  $\sqsubseteq$ . Moreover, presume there are  $\alpha, \beta, \gamma > 0$  so that the following conditions hold:

- ( $t_1$ )  $\alpha + 2\beta + 2\gamma < 1$ ;
- ( $t_2$ ) for all comparable  $x, y \in X$ ,

$$\begin{aligned} q(Tfx, Tfy) &\preceq \alpha q(Tx, Ty) + \beta q(Tx, Tfy) + \gamma q(Ty, Tfx), \\ q(Tfy, Tfx) &\preceq \alpha q(Ty, Tx) + \beta q(Tfy, Tx) + \gamma q(Tfx, Ty). \end{aligned}$$

If there exists  $x_0 \in X$  so that  $x_0 \sqsubseteq fx_0$ , then  $f$  has a fixed point. Moreover, if  $fz = z$ , then  $q(Tz, Tz) = \theta$ .

**Proof.** It is enough to take  $\alpha(x) = \alpha$ ,  $\beta(x) = \beta$  and  $\gamma(x) = \gamma$  in Theorem 2.1. ■

Again if  $T$  is an identity mapping, we have the same Corollary 3.1 of [21].

Next result is the second main theorem of this paper that shows the existence of fixed point for a  $T$ -contraction mapping of Kannan type on a cone metric space under a  $c$ -distance.

**Theorem 2.3** Assume  $(X, \sqsubseteq, d)$  is a complete partially ordered cone metric space and  $q$  is a  $c$ -distance on  $X$ . Also, presume  $f, T : X \rightarrow X$  are two mapping so that  $T$  is injective, continuous and sequentially convergent and  $f$  is continuous and nondecreasing respect to  $\sqsubseteq$ . Moreover, suppose that there are  $\alpha, \beta, \gamma : X \rightarrow [0, 1)$  so that the following conditions are held:

- ( $t_1$ )  $\alpha(fx) \leq \alpha(x)$ ,  $\beta(fx) \leq \beta(x)$  and  $\gamma(fx) \leq \gamma(x)$  for all  $x \in X$ ;
- ( $t_2$ )  $(\alpha + \beta + \gamma)(x) < 1$  for all  $x \in X$ ;
- ( $t_3$ ) for all  $x, y \in X$  with  $x \sqsubseteq y$ ,

$$q(Tfx, Tfy) \preceq \alpha(x)q(Tx, Ty) + \beta(x)q(Tx, Tfx) + \gamma(x)q(Ty, Tfy). \quad (8)$$

If there exists  $x_0 \in X$  so that  $x_0 \sqsubseteq fx_0$ , then  $f$  has a fixed point. Moreover, if  $fz = z$ , then  $q(Tz, Tz) = \theta$ .

**Proof.** The proof of Theorem 2.1 shows that it is not wise to prove such theorems in a directed way while we can obtain them from an indirected way with a short proposition. Hence, Applying Proposition 3.1 of Karimizad's work [11] and Proposition 1.8, we conclude that  $d^* : X \times X \rightarrow \mathbb{R}$  and  $q^* : X \times X \rightarrow [0, +\infty)$  defined by

$$\begin{cases} d^*(x, y) = d(Tx, Ty), \\ q^*(x, y) = q(Tx, Ty), \end{cases}$$

for all  $x, y \in X$  are a complete metric and a  $c$ -distance, respectively. In this case, we reach Theorem 3.1 of [25] with notations  $d^*$  and  $q^*$ , and so, the resident of the proof follows the proof of Theorem 3.1 of [25]. ■

**Corollary 2.4** Assume  $(X, \sqsubseteq, d)$  is a complete partially ordered cone metric space and  $q$  is a  $c$ -distance on  $X$ . Also, suppose  $f, T : X \rightarrow X$  are two mapping so that  $T$  is injective, continuous and sequentially convergent and  $f$  is continuous and nondecreasing respect to  $\sqsubseteq$ . Moreover, presume there are  $\alpha, \beta, \gamma > 0$  so that the following conditions hold:

- ( $t_1$ )  $\alpha + \beta + \gamma < 1$ ;  
 ( $t_2$ ) for all  $x, y \in X$  with  $x \sqsubseteq y$ ,

$$q(Tfx, Tfy) \preceq \alpha q(Tx, Ty) + \beta q(Tx, Tfx) + \gamma q(Ty, Tfy) \quad (9)$$

If there exists  $x_0 \in X$  so that  $x_0 \sqsubseteq fx_0$ , then  $f$  has a fixed point. Moreover, if  $fz = z$ , then  $q(Tz, Tz) = \theta$ .

**Proof.** It is enough to take  $\alpha(x) = \alpha$ ,  $\beta(x) = \beta$  and  $\gamma(x) = \gamma$  in Theorem 2.3. ■

Note that if  $T$  is an identity mapping in Theorem 2.3 and Corollary 2.4, we have the same Theorem 3.1 of [25] and Theorem 3.1 of [6], respectively. Moreover, if we take  $\beta(x) = \gamma(x) = 0$  and  $\beta = \gamma = 0$ , we can state well-known contraction, named Banach type.

**Theorem 2.5** Assume  $(X, \sqsubseteq, d)$  is a complete partially ordered cone metric space and  $q$  is a  $c$ -distance on  $X$ . Also, presume  $f, T : X \rightarrow X$  are two mapping so that  $T$  is injective, continuous and sequentially convergent and  $f$  is continuous and nondecreasing respect to  $\sqsubseteq$ . Moreover, suppose that there is  $\alpha : X \rightarrow [0, 1)$  so that

$$\alpha(fx) \leq \alpha(x)$$

for all  $x \in X$  and

$$q(Tfx, Tfy) \preceq \alpha(x)q(Tx, Ty) \quad (10)$$

for all  $x, y \in X$  with  $x \sqsubseteq y$ . If there exists  $x_0 \in X$  so that  $x_0 \sqsubseteq fx_0$ , then  $f$  has a fixed point. Moreover, if  $fz = z$ , then  $q(Tz, Tz) = \theta$ .

**Corollary 2.6** Assume  $(X, \sqsubseteq, d)$  is a complete partially ordered cone metric space and  $q$  is a  $c$ -distance on  $X$ . Also, suppose  $f, T : X \rightarrow X$  are two mapping so that  $T$  is injective, continuous and sequentially convergent and  $f$  is continuous and nondecreasing respect to  $\sqsubseteq$ . Moreover, presume there is  $\alpha \in [0, 1)$  so that

$$q(Tfx, Tfy) \preceq \alpha q(Tx, Ty) \quad (11)$$

for all  $x, y \in X$  with  $x \sqsubseteq y$ . If there exists  $x_0 \in X$  so that  $x_0 \sqsubseteq fx_0$ , then  $f$  has a fixed point. Moreover, if  $fz = z$ , then  $q(Tz, Tz) = \theta$ .

Note that if  $T$  is an identity mapping in Theorem 2.5 and Corollary 2.6, we have the same Theorem 3.2 of [11], respectively. It should be noted that if we take  $E = \mathbb{R}$  and  $P = [0, +\infty)$ , we can obtain Theorems 2.1-2.3-2.5 and Corollaries 2.2-2.4-2.6 in metric spaces under a  $w$ -distance.

Note that all examples in references of this paper can be arranged by the main theorems and their corresponding corollaries to show the existence of fixed points of  $T$ -contractions on a partially ordered cone metric space under a  $c$ -distance.

### 3. Conclusion

In this paper, we proved some famous fixed point theorems of  $T$ -contraction on partially ordered cone metric spaces under  $c$ -distances using two methods; directed and indirected

ways. Thus, our theorems and corollaries unify, extend and generalize well-known comparable results of fixed point theory in cone metric spaces under  $c$ -distances. Moreover, we gave some examples and remarks to show the importance of obtained results.

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