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Various versions of distances on different metric spaces and fixed point theory; a discussion of Rahimi et al.'s works

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Abstract. The main purpose of this article is to investigate the relation between different kinds of distances on divers metric spaces. Also, we consider some fixed point theorems with respect to a specified distance and show the validity of them regarding other distances. This helps researchers avoid constructing new theorems and additional proofs, which are indeed an illogical task. In particular, we discussed the fixed point theorems of Rahimi et al. with respect to w-distances (c-distances) and wt-distances (ct-distances).

Keywords: w-distance, c-distance, wt-distance, ct-distance, Banach algebra.

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1. Introduction

Since 1906, various definitions and theorems of metric spaces, introduced by Fréchet [19], have been suggested by many researchers, leading to the creation of a major branch of mathematics called mathematical analysis. On the other hand, in 1922, Banach [9] introduced his famous principle, named Banach contraction principle, which helped him to show the existence and uniqueness of a fixed point for a mapping in complete metric spaces, leading to the creation of an important subfield in mathematical analysis called fixed point theory. Also, motivated by these works, several new metric spaces have been proposed by many mathematicians, some of which are cone metric spaces, tvs-cone metric spaces, b-metric spaces, cone b-metric spaces, tvs-cone b-metric spaces, cone metric spaces over a Banach algebra, cone b-metric spaces over a Banach algebra. The combination

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of these concepts and convergence theorems, along with some other topics, led to the creation of a key branch of analysis that is now called nonlinear functional analysis. To learn more about these spaces, their history and topology, convergence theorems, comparing them with each other, and having examples of them, one can refer to [2, 13, 21, 22, 28, 29, 33, 37, 39, 47] and references therein.

Considering the importance of metric spaces and specifying the completeness of such spaces, Kada et al. [27] defined the concept of w-distance in a metric space X. They also discussed non-convex minimization theorems and fixed point theorems in complete metric spaces with respect to such distances. Since then, numerous distances have been introduced with respect to their metric spaces, some of which are w-cone distances [12, 2012], c-distances [11, 14, 43, 2011], tvs version of c-distances [14, 2011], wt-distances [25, 2014], generalised c-distances [10, 2015], c-distances over a Banach algebra [23, 2015], tvs version of generalized c-distances [44, 2018], w_0-distances [35, 2019], w_b-cone distances [6, 2020], generalized c-distances over Banach algebra [1, 2020], wt_0-distances [34, 2023], pre-symmetric w-distances [42, 2024], pre-symmetric w-cone distances [32, 2024], pre-symmetric c-distances [31, 2025]. In addition to the definition of such distances, other properties and (common) fixed point theorems regarding them are proved in [1, 3–8, 10–12, 14, 17, 18, 20, 23, 25, 27, 30–32, 34–36, 38–40, 42–46] and references therein.

In the present article, we first review the relation between some distances on divers metric spaces, especially those are introduced and applied by Rahimi and Soleimani Rad, and then consider some fixed point theorems regarding a specified distance to show the validity of them with respect to other distances. This helps the reader avoid constructing new theorems and additional proofs. It should be noted that there are many other generalized metric spaces, and it is quite natural to have distances on them, and their relations can be monitored as this article. Note that the purpose of this paper is not to show a direct proof of the theorem, but rather to provide a way to derive several theorems for each other to avoid constructing new theorems that cannot be presented as a new research paper. An illustrative example is drawn from previous results to verify our claim.

2. Various distances and their relationship

We start by the definition of a w-distance on a metric space (X, d).

Definition 2.1 [27, 1996] A function $p : X \times X \to [0, +\infty)$ is called a w-distance on X if the following are satisfied:

(w_1) $p(x, z) \leqslant p(x, y) + p(y, z)$ for all $x, y, z \in X$;
(w_2) p is lower semi-continuous in its second variable;
(w_3) for any $\varepsilon > 0$, there is $\delta > 0$ such that $p(z, x) \leqslant \delta$ and $p(z, y) \leqslant \delta$ imply $d(x, y) \leqslant \varepsilon$.

Note that a metric d is a w-distance, but the converse doesn't necessarily hold. For this, it is enough to take $d, p : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ by $d(x, y) = |x - y|$ and $p(x, y) = y$ for all $x, y \in [0, +\infty)$. The following is a key lemma that can help the researchers understand the Cauchy condition and convergence of a sequence with respect to a w-distance.

Lemma 2.2 [27] Let p be a w-distance on a metric space (X, d). Also, presume $\{x_n\}$ and $\{y_n\}$ are two sequences in X, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, +\infty)$ converging to zero and $x, y, z \in X$. Then the following properties are held:

(i) if $p(x_n, y) \leqslant \alpha_n$ and $p(x_n, z) \leqslant \beta_n$ for each $n \in \mathbb{N}$, then $y = z$. Moreover, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$;

(ii) if $p(x_n, y_n) \leqslant \alpha_n$ and $p(x_n, z) \leqslant \beta_n$ for $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;

(iii) if $p(x_n, x_m) \leqslant \alpha_n$ for all $m, n \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence in X;

(iv) if $p(y, x_n) \leqslant \alpha_n$ for each $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.

From Definition 2.1, some researchers could derive at least five new distances as follows:

Definition 2.3 [35, 2019] A function $p_0 : X \times X \to [0, \infty)$ is called a w_0-distance on X if the following are satisfied:

(w_{01}) $p_0(x, z) \leqslant p_0(x, y) + p_0(y, z)$ for any $x, y, z \in X$;

(w_{02}) for any $x \in X$, two functions $p_0(x, \cdot), p_0(\cdot, x) : X \to [0, \infty)$ are lower semicontinuous;

(w_{03}) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p_0(z, x) \leqslant \delta$ and $p_0(z, y) \leqslant \delta$ imply $d(x, y) \leqslant \varepsilon$.

Note that the notion of w_0-distance is more general than the standard notion of metric, but less general than the w-distance. It should be noted that a w-distance is not necessarily symmetric and $p(x, y) = 0$ is not absolutely equivalent to $x = y$. On the other hand, note that if $w(x, y) = \max\{p(x, y), p(y, x)\}$ for each $x, y \in X$, then the condition (w_2) is not generally valid for w. But if p is symmetric, then $w = p$ and w satisfies (w_2). Also, it is clear that if we substitute p by p_0 in $w(x, y)$, then w satisfies (w_2). With this interpretation, Romaguera and Tirado defined the concept of pre-symmetric w-distances as follows:

Definition 2.4 [42, 2024] A w-distance p is said to be pre-symmetric if

(1) when $(x_n) \subset X$ provided that $x_n \to x$ and $p(x_n, x) \to 0$ for $x \in X$, there is a subsequence $(x_{k(n)})$ of (x_n) so that $p(x, x_{k(n)+1}) \leqslant p(x_{k(n)}, x)$ for any positive integer n.

Note that a symmetric w-distance is pre-symmetric, but the converse doesn't necessarily hold. In fact, there exist some pre-symmetric w-distances that are not symmetric [42].

Assume that E is a Banach space and P is a cone therein with the same notations and concepts such as partial order, $\preceq$, $\ll$ and $int\, P$ introduced in [24, 38]. Returning to a cone metric space (X, d) defined by Hunag and Zhang [24], the third definition can be obtained from Definition 2.1 as follows:

Definition 2.5 [11, 2011] A function $q : X \times X \to P$ is named a c-distance if the following are satisfied:

(c_1) $q(x, z) \preceq q(x, y) + q(y, z)$ for every $x, y, z \in X$;

(c_2) for any $x \in X$, when $q(x, y_n) \preceq u$ for a $u = u_x$ and any $n \geqslant 1$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a convergent sequence in X to a y;

(c_3) for any $c \in int\, P$ and $x, y, z \in X$, there is $e \in int\, P$ provided that $q(z, x) \ll e$ and $q(z, y) \ll e$ entail $d(x, y) \ll c$.

Note that a cone metric d is a c-distance but the converse doesn't necessarily hold. For example, it is enough to take $d, q : [0, +\infty) \times [0, +\infty) \to P$ by $d(x, y)(t) = |x - y|\psi(t)$ and $q(x, y)(t) = y\psi(t)$ for all $x, y \in [0, +\infty)$, in which $E = C^1_{\mathbb{R}}[0, 1]$ is a Banach space with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and the cone $P = \{x \in E : x(t) \geqslant 0 \text{ on } [0, 1]\}$, and $\psi : [0, 1] \to \mathbb{R}$ with $\psi(t) = e^t$ for each $t \in [0, 1]$. Also, note that a w-distance is a c-distance with $E = \mathbb{R}$ and $P = [0, +\infty)$, so a c-distance is a generalization of a w-distance. The following is a key lemma that can help the researchers understand the Cauchy condition and convergence of a sequence with respect to a c-distance.

Lemma 2.6 [11] Let (X, d) be a cone metric space and q be a c-distance on X. Also, presume $\{x_n\}$ and $\{y_n\}$ are two sequences in X, $x, y, z \in X$, and $\{u_n\}$ and $\{v_n\}$ are two sequences in P converging to θ. Then the following conditions are held:

- (i) if $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \in \mathbb{N}$, then $y = z$. Moreover, if $q(x, y) = \theta$ and $q(x, z) = \theta$, then $y = z$;
- (ii) if $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;
- (iii) if $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X;
- (iv) if $q(y, x_n) \preceq u_n$ for $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.

Comparing Lemmas 2.2 and 2.6 shows that if all the cases are true in one, they are also valid for the other, and this only changes the part of the proof that goes back to the definition and notation of distances and metric spaces.

Again, assume that E is a Banach space and P is a cone therein with the same notations and concepts such as partial order, $\preceq$, $\ll$ and $int\, P$ introduced in [15]. Returning to a *tvs*-cone metric space (X, d) defined by Du [15], next definition can be considered as a generalization of Definition 2.1.

Definition 2.7 [12, 2012] A function $q_c : X \times X \to P$ is named w-cone distance on X when for each $x, y, z \in X$,

- (qc_1) $q_c(x, z) \preceq q_c(x, y) + q_c(y, z)$;
- (qc_2) $q_c(x, \cdot) : X \to P$ is lower semicontinuous;
- (qc_3) for each $c \in int\, P$, there is $e \in int\, P$ provided that $q_c(z, x) \ll e$ and $q_c(z, y) \ll e$ imply $d(x, y) \ll c$.

Note that a function $f : X \to P$ is lower semicontinuous at $x \in X$ if there is $n_0 \in \mathbb{N}$ provided that $f(x) \preceq f(x_n) + c$ for any $c \in int\, P$ and $n \geqslant n_0$ in which $x_n \to x$ in X. The same properties of c-distance is also true for w-cone distance, but they aren't equal. In fact, the conditions (c_2) and (qc_2) aren't equal and there is a different in their definitions. Moreover, it is clear that a c-distance (w-cone distance) is not necessarily symmetric and $q(x, y) = \theta$ ($q_c(x, y) = \theta$) is not absolutely equivalent to $x = y$ for all $x, y \in X$.

Similar to the definition of a pre-symmetric w-distance (Definition 2.4), we are able to define a pre-symmetric c-distance (w-cone distance) which is an extension of pre-symmetric w-distance.

Definition 2.8 [31, 32, 2024-2025] A c-distance (w-cone distance) q on a X is said to be pre-symmetric if $x_n \to x$ in X and $q(x_n, x) \to \theta$ for a $x \in X$, then there is a subsequent $(x_{k(n)})_{n \in \mathbb{N}}$ of (x_n) so that $q(x, x_{k(n)+1}) \preceq q(x_{k(n)}, x)$ for each $n \in \mathbb{N}$.

Now, if we consider a b-metric space (X, D), we are able to define last generalized distance as an extension of a w-distance, calling wt-distance.

Definition 2.9 [25] Let $b \geqslant 1$. A function $\rho : X \times X \to [0, +\infty)$ is called a wt-distance on X if the following are satisfied:

- (wt_1) $\rho(x, z) \leqslant b[\rho(x, y) + \rho(y, z)]$ for all $x, y, z \in X$;
- (wt_2) ρ is b-lower semi-continuous in its second variable, that is, if $x \in X$ and $y_n \to y$ in X, then $\rho(x, y) \leqslant b \liminf_n \rho(x, y_n)$;
- (wt_3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(z, x) \leqslant \delta$ and $\rho(z, y) \leqslant \delta$ imply $D(x, y) \leqslant \varepsilon$.

Note that each w-distance is a wt-distance with $b = 1$, but the converse doesn't hold. Also, note that a b-metric D is a wt-distance, but the converse doesn't necessarily hold. It should be noted that a wt-distance is not necessarily symmetric and $\rho(x, y) = 0$ is not absolutely equivalent to $x = y$. For these properties, it is enough to take $D, \rho : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ by $D(x, y) = |x - y|^2$ and $\rho(x, y) = y^2$ for all $x, y \in [0, +\infty)$. The combination of Definitions 2.3 and 2.9 can results in a new distance on X as follows:

Definition 2.10 [34, 2023] Let $b \geqslant 1$. A function $\rho_0 : X \times X \to [0, \infty)$ is called a wt_0-distance on X if the following are satisfied:

(wt_{01}) $\rho_0(x, z) \leqslant b[\rho_0(x, y) + \rho_0(y, z)]$ for any $x, y, z \in X$;
(wt_{02}) for any $x \in X$, functions $\rho_0(x, \cdot), \rho_0(\cdot, x) : X \to [0, \infty)$ are lower b-semicontinuous;
(wt_{03}) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho_0(z, x) \leqslant \delta$ and $\rho_0(z, y) \leqslant \delta$ imply $D(x, y) \leqslant \varepsilon$.

Note that the notion of wt_0-distance is more general than the standard notion of b-metric, but less general than the wt-distance. Also, w_0-distance is a wt_0 with $b = 1$.

Now, the combination of Definitions 2.5 and 2.9 can lead to the following definition of a distance on a cone b-metric space (X, D), which is named a generalized c-distance (or, a ct-distance). For this, assume that E is a Banach space and P is a cone therein with the same notations and concepts such as partial order, $\preceq$, $\ll$ and $int\, P$ introduced in [24, 38].

Definition 2.11 [10, 2015] Let $b \geqslant 1$. A function $Q : X \times X \to P$ is called a ct-distance on X if the following are satisfied:

(qt_1) $Q(x, z) \preceq s[Q(x, y) + Q(y, z)]$ for all $x, y, z \in X$;
(qt_2) for $x \in X$, if $Q(x, y_n) \preceq u$ for some $u = u_x$ and all $n \geqslant 1$, then $Q(x, y) \preceq su$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;
(qt_3) for all $c \in int\, P$, there is $e \in int\, P$ so that $Q(z, x) \ll e$ and $Q(z, y) \ll e$ imply $D(x, y) \ll c$.

Note that a cone b-metric D is a ct-distance but the converse doesn't necessarily hold. For example, it is enough to take $D, Q : [0, +\infty) \times [0, +\infty) \to P$ by $D(x, y)(t) = |x - y|^2\psi(t)$ and $Q(x, y)(t) = y^2\psi(t)$ for all $x, y \in [0, +\infty)$, in which $E = C^1_{\mathbb{R}}[0, 1]$ is a Banach space with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and the cone $P = \{x \in E : x(t) \geqslant 0$ on $[0, 1]\}$, and $\psi : [0, 1] \to \mathbb{R}$ with $\psi(t) = e^t$ for each $t \in [0, 1]$. Also, a wt-distance is a ct-distance with $E = \mathbb{R}$ and $P = [0, +\infty)$, so a ct-distance is an extension of a wt-distance. Finally, note that each c-distance is a ct-distance with $b = 1$, but the converse doesn't hold.

Again, assume that E is a Banach space and P is a cone therein with the same notations and concepts such as partial order, $\preceq$, $\ll$ and $int\, P$ introduced in [16, 28]. Returning to a tvs-cone b-metric space (X, D) defined by Kadelburg et al. [28], next definition can be considered as a generalization of Definition 2.9.

Definition 2.12 [6, 2020] Let $b \geqslant 1$. A function $Q_c : X \times X \to P$ is named w_b-cone distance on X when for each $x, y, z \in X$,

(Qc_1) $Q_c(x, z) \preceq b[Q_c(x, y) + Q_c(y, z)]$;
(Qc_2) $Q_c(x, \cdot) : X \to P$ is b-lower semicontinuous;
(Qc_3) for each $c \in int\, P$, there is $e \in int\, P$ provided that $Q_c(z, x) \ll e$ and $Q_c(z, y) \ll e$ imply $D(x, y) \ll c$.

Note that a function $f : X \to P$ is b-lower semicontinuous at $x \in X$ if there is $n_0 \in \mathbb{N}$

provided that $f(x) \preceq bf(x_n) + c$ for any $c \in int\, P$ and $n \geqslant n_0$ in which $x_n \to x$ in X. The same properties of ct-distance is also true for w_b-cone distance, but they aren't equal. In fact, the conditions (qt_2) and (Qc_2) aren't equal and there is a different in their definitions. Moreover, it is clear from previous example that a ct-distance (w_b-cone distance) is not necessarily symmetric and $Q(x, y) = \theta$ ($Q_c(x, y) = \theta$) is not absolutely equivalent to $x = y$ for all $x, y \in X$.

Now, we can recall the generalization of Definitions 2.5 and 2.11 by setting a Banach algebra $\mathcal{E}$ instead of a Banach space E. Presume $\mathcal{E}$ is a Banach algebra. A non-empty and proper closed subset $\mathcal{P}$ of $\mathcal{E}$ is said to be a cone if $\mathcal{P} \cap (-\mathcal{P}) = \{\theta\}$, $\mathcal{P} + \mathcal{P} \subset \mathcal{P}$ and $\lambda\mathcal{P} \subset \mathcal{P}$ for $\lambda \geqslant 0$. Now, with respect to an optional cone $\mathcal{P}$ in $\mathcal{E}$, we define a partial ordering $\preceq$ by $x \preceq y$ iff $y - x \in \mathcal{P}$. If $x \preceq y$ and $x \neq y$, then we apply $x \prec y$. Also, $x \ll y$ iff $y - x \in int\, \mathcal{P}$, where $int\, \mathcal{P}$ is the interior of $\mathcal{P}$. Also, $\mathcal{P}$ is named a solid cone if $int\, \mathcal{P} \neq \emptyset$. Moreover, $\mathcal{P}$ is named a solid cone if there is a number K such that $\theta \preceq x \preceq y$ imply $\|x\| \leqslant K\|y\|$ for all $x, y \in \mathcal{E}$.

Returning to a cone metric space $(\mathcal{X}, d_{\mathcal{E}})$ (resp. a cone b-metric space $(\mathcal{X}, D_{\mathcal{E}})$) over a Banach algebra $\mathcal{E}$, next definitions can be considered as a generalization of Definitions 2.5 (resp. 2.11).

Definition 2.13 [23, 2015] A function $q_{\mathcal{E}} : \mathcal{X} \times \mathcal{X} \to \mathcal{P}$ is called a $c_{\mathcal{E}}$-distance on $\mathcal{X}$ if it satisfies the following conditions:

- (q_1) $q_{\mathcal{E}}(x, y) \preceq q_{\mathcal{E}}(x, z) + q_{\mathcal{E}}(z, y)$ for all $x, y, z \in \mathcal{X}$;
- (q_2) for $x \in \mathcal{X}$ and a sequence $\{y_n\}$ in $\mathcal{X}$ converging to $y \in \mathcal{X}$, if $q_{\mathcal{E}}(x, y_n) \preceq u$ for some $u = u_x \in \mathcal{P}$ and all $n \geqslant 1$, then $q_{\mathcal{E}}(x, y) \preceq u$;
- (q_3) for all $c \in int\, \mathcal{P}$, there is $e \in int\, \mathcal{P}$ so that $q_{\mathcal{E}}(z, x) \ll e$ and $q_{\mathcal{E}}(z, y) \ll e$ imply $d_{\mathcal{E}}(x, y) \ll c$.

Definition 2.14 [1, 3, 2020-2021] Let $b \geqslant 1$. A function $Q_{\mathcal{E}} : \mathcal{X} \times \mathcal{X} \to \mathcal{P}$ is called a $ct_{\mathcal{E}}$-distance on $\mathcal{X}$ if it satisfies the following conditions:

- (qt_1) $Q_{\mathcal{E}}(x, y) \preceq b[Q_{\mathcal{E}}(x, z) + Q_{\mathcal{E}}(z, y)]$ for all $x, y, z \in \mathcal{X}$;
- (qt_2) for $x \in \mathcal{X}$ and a sequence $\{y_n\}$ in $\mathcal{X}$ converging to $y \in \mathcal{X}$, if $Q_{\mathcal{E}}(x, y_n) \preceq u$ for some $u = u_x \in \mathcal{P}$ and all $n \geqslant 1$, then $Q_{\mathcal{E}}(x, y) \preceq bu$;
- (qt_3) for all $c \in int\, \mathcal{P}$, there is $e \in int\, \mathcal{P}$ so that $Q_{\mathcal{E}}(z, x) \ll e$ and $Q_{\mathcal{E}}(z, y) \ll e$ imply $D_{\mathcal{E}}(x, y) \ll c$.

Remark 1 *Regarding Definitions 2.1-2.14, the following important points will help researchers avoid proposing repetitive theorems and imitatively proving them.*

1. *It should be mentioned that Lemmas 2.2 and 2.6 are held for other defined distances with the same hypothesis or slight changes, which are recalled here with their own notations and concepts, but they may change the proof slightly.*
2. *Note that the following diagram show the relations between various versions of distances. Thus, considering item (1) and the properties of each distance, it is clear that we can prove many fixed point theorems with respect to all of them, but in fact they imitate an equal method. Also, we can see equal results for a constant example that will show in next section.*
3. *New theorems are only interesting when we can show the existence and uniqueness of a fixed point with respect to a distance that does not work with respect to another distance. Otherwise, the theorem would be meaningless, its proof redundant, and its application repetitive.*
4. *If our spaces X and $\mathcal{X}$ are endowed by a partial order introduced by Ran and Reurings [41] or a graph stated by Jachymski [26] in all definitions, the former*

items of this remark are held. Hence, they cannot also add new effective results to fixed point theory.

$$\begin{tikzcd}
 & & \boxed{w\text{-distance}} \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
\boxed{w\text{-cone distance}} \arrow[d] & \boxed{c_{\mathcal{E}}\text{-distance}} \arrow[d] & & \boxed{c\text{-distance}} \arrow[dl] & \boxed{wt\text{-distance}} \arrow[dll] \\
\boxed{w_b\text{-cone distance}} & \boxed{ct_{\mathcal{E}}\text{-distance}} & \boxed{ct\text{-distance}} & &
\end{tikzcd}$$

Also, the following questions can be considered as a supplementary research paper on distances.

Question 2.15 Can one extend (define) the notions of a c_0-distance and a ct_0-distance from the w_0-distance and the wt_0-distance? If yes, how can the results of [34, 35] be converted?

Question 2.16 Can one extend (define) the notion of a pre-symmetric $c_{\mathcal{E}}$-distance over a Banach algebra from a pre-symmetric c-distance? If yes, how can the characterization of [31] be converted?

Question 2.17 Is it meaningful to define a pre-symmetric wt-distance or generalize it to a ct-distance or $ct_{\mathcal{E}}$-distance over a Banach algebra? If yes, how can the characterization of [31, 32, 42] be converted?

In fact, our goal in Questions 2.15-2.17 is to explore a way to complete the following diagrams, along with examples and practical applications.

$$\begin{tikzcd}
 & \boxed{w_0\text{-distance}} \arrow[dl] \arrow[dr] & \\
\boxed{wt_0\text{-distance}} \arrow[dr] & & \boxed{c_0\text{-distance?}} \arrow[dl] \\
 & \boxed{ct_0\text{-distance?}} &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \boxed{\text{pre-symmetric } wt\text{-distance?}} \arrow[dl] \arrow[dr] & \\
\boxed{\text{pre-symmetric } ct\text{-distance?}} & & \boxed{\text{pre-symmetric } w_b\text{-cone distance?}}
\end{tikzcd}$$

3. Fixed point theory; some comparisons and examples

In this section, we consider some theorems and some of their corresponding examples.

Theorem 3.1 Let $(X, \sqsubseteq)$ be a partially ordered set and there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.

(1) [45, 2019, Theorem 2.8] Presume (X, D) is a complete cone b-metric space with given real number $b \geqslant 1$, Q is a ct-distance on X and $f : X \to X$ is a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Also, assume that there are mappings $\alpha, \beta, \gamma : X \to [0, 1)$ provided that the following conditions hold:
(t_1) $\alpha(fx) \leqslant \alpha(x)$, $\beta(fx) \leqslant \beta(x)$ and $\gamma(fx) \leqslant \gamma(x)$ for all $x \in X$;
(t_2) $\big(b(\alpha + 2\beta) + (b^2 + b)\gamma\big)(x) < 1$ for all $x \in X$;
(t_3) for all $x, y \in X$ with $y \sqsubseteq x$,

$$Q(fx, fy) \preceq \alpha(x)Q(x, y) + \beta(x)Q(x, fy) + \gamma(x)Q(y, fx);$$

(t_4) for all $x, y \in X$ with $y \sqsubseteq x$,

$$Q(fy, fx) \preceq \alpha(x)Q(y, x) + \beta(x)Q(fy, x) + \gamma(x)Q(fx, y).$$

Then f has a fixed point. Moreover, if $fv = v$ for $v \in X$, then $Q(v, v) = \theta$.

(2) [7, 2022, Corollary 2.3] Presume (X, D) is a complete b-metric space with given real number $b \geqslant 1$, ρ is a wt-distance on X and $f : X \to X$ is a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Also, assume that there are mappings $\alpha, \beta, \gamma : X \to [0, 1)$ provided that the following conditions hold:
(t_1) $\alpha(fx) \leqslant \alpha(x)$, $\beta(fx) \leqslant \beta(x)$ and $\gamma(fx) \leqslant \gamma(x)$ for all $x \in X$;
(t_2) $\big(b(\alpha + 2\beta) + (b^2 + b)\gamma\big)(x) < 1$ for all $x \in X$;
(t_3) for all $x, y \in X$ with $y \sqsubseteq x$,

$$\rho(fx, fy) \leqslant \alpha(x)\rho(x, y) + \beta(x)\rho(x, fy) + \gamma(x)\rho(y, fx);$$

(t_4) for all $x, y \in X$ with $y \sqsubseteq x$,

$$\rho(fy, fx) \leqslant \alpha(x)\rho(y, x) + \beta(x)\rho(fy, x) + \gamma(x)\rho(fx, y).$$

Then f has a fixed point. Moreover, if $fv = v$ for $v \in X$, then $\rho(v, v) = 0$.

Proof. With the proof of (1) in [45], the proof of (2) is not important and can be simply an imitation of (1) and conversely. ■

Further, we can see that when we have either (1) or (2) of Theorem 3.1, the other has no new impact on our example.

***Example* 3.2** Take $E = C^1([0, 1], \mathbb{R})$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and the non-normal cone $P = \{x \in E : x(t) \geqslant 0 \text{ for all } t \in [0, 1]\}$. Also, assume $X = [0, 1]$ and $D, Q : X \times X \to P$ are defined by $D(x, y)(t) = (x - y)^2\psi(t)$ and $Q(x, y)(t) = D(x, y)(t)$ for all $x, y \in X$, where $\psi : [0, 1] \to \mathbb{R}$ is defined by $\psi(t) = e^t$ for all $t \in [0, 1]$. Then (X, D) is a complete cone b-metric space with $b = 2$ and Q is a ct-distance. Moreover, presume $\sqsubseteq$ is defined by $x \sqsubseteq y$ iff $x \preceq y$ and a mapping $f : X \to X$ is considered by $fx = \frac{x^2}{4}$ for all $x \in X$. Define $\alpha(x) = \frac{(x+1)^2}{16}$ and $\beta(x) = \gamma(x) = 0$ for all $x \in X$. Note that

(t_1) $\alpha(fx) = \frac{1}{16}\left(\frac{x^2}{4}+1\right)^2 \leqslant \frac{1}{16}\left(x^2+1\right)^2 \leqslant \frac{(x+1)^2}{16} = \alpha(x)$ for all $x \in X$. Also, $\beta(fx) = 0 \leqslant 0 = \beta(x)$ and $\gamma(fx) = 0 \leqslant 0 = \gamma(x)$ for all $x \in X$;

(t_2) $\left(2(\alpha+2\beta)+(2^2+2)\gamma\right)(x) = 2\frac{(x+1)^2}{16} = \frac{(x+1)^2}{8} < 1$ for all $x \in X$;

(t_3) for all $x, y \in X$ with $y \sqsubseteq x$, we get

$$\begin{aligned} Q(fx, fy)(t) &= \left(\frac{x^2}{4} - \frac{y^2}{4}\right)^2 \psi(t) \\ &\preceq \alpha(x)Q(x,y)(t) + \beta(x)Q(x,fy)(t) + \gamma(x)Q(y,fx)(t); \end{aligned}$$

(t_4) for all $x, y \in X$ with $y \sqsubseteq x$, we get

$$Q(fy, fx)(t) \preceq \alpha(x)Q(y,x)(t) + \beta(x)Q(fy,x)(t) + \gamma(x)Q(fx,y)(t).$$

Moreover, f is a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Therefore, all conditions of Theorem 3.1(1) are satisfied and f has a fixed point $x = 0$ with $Q(0,0)(t) = 0$ [45]. Now, we show the existence of fixed point with Theorem 3.1(2) that is easier than of using Theorem 3.1(1). For this, take the same X, $\alpha(x)$, $\beta(x)$ and $\gamma(x)$. Also, presume $D, \rho : X \times X \to [0, +\infty)$ are defined by $D(x,y) = (x-y)^2$ and $\rho(x,y) = D(x,y)$ for all $x, y \in X$ and $\sqsubseteq$ is defined by $x \sqsubseteq y$ iff $x \leqslant y$. Also, (t_1) and (t_2) hold and for (t_3) and (t_4), we have

$$\begin{aligned} \rho(fx, fy) = \left(\frac{x^2}{4} - \frac{y^2}{4}\right)^2 &\leqslant \alpha(x)\rho(x,y) + \beta(x)\rho(x,fy) + \gamma(x)\rho(y,fx), \\ \rho(fy, fx) &\leqslant \alpha(x)\rho(y,x) + \beta(x)\rho(fy,x) + \gamma(x)\rho(fx,y). \end{aligned}$$

Moreover, f is a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Therefore, all conditions of Theorem 3.1(2) are satisfied and f has a fixed point $x = 0$ with $\rho(0,0) = 0$.

This example can clearly show that when we have fewer conditions, it is better to use them instead of solving a problem with more conditions. Anyhow, in 2019, the results of Soleimani Rad et al. [45] were valuable as their results didn't considered with respect to a wt-distance. But, if they showed their results in 2022, they couldn't be more attractive as Babaei et al. [7, Theorem 2.1] proved the same for a big class of contractions with respect to wt-distances that their corollaries contained the same cone version of contractions in Soleimani Rad et al. [45, Theorem 2.8].

Corollary 3.3 Let $(X, \sqsubseteq)$ be a partially ordered set, $b = 1$ and there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.

(1) [11, 43, 2011] Change ct-distance Q into c-distance q, cone b-metric D into cone metric d and (t_2) to $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$ in item (1) of Theorem 3.1. Then f has a fixed point. Moreover, if $fv = v$ for $v \in X$, then $q(v,v) = \theta$.

(2) Change wt-distance ρ into w-distance p, b-metric D into metric d and (t_2) to $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$ in item (2) of Theorem 3.1. Then f has a fixed point. Moreover, if $fv = v$ for $v \in X$, then $p(v,v) = 0$.

Proof. With the proof of (1) in [11, 43], the proof of (2) is not important and can be simply an imitation of (1) and conversely. ■

To end this note, we consider Corollary 2 of Lakzian et al. [36, 2019], which is introduced in the framework of w-distance, and make it in the framework c-distance.

Theorem 3.4 Presume q is a c-distance on a complete cone metric space (X, d) and $f : X \to X$ is a q-Kannan contraction, i.e., there is $\lambda \in [0, 1)$ such that

$$q(fx, fy) \preceq \frac{\lambda}{2}[q(fx, x) + q(fy, y)]$$

or

$$q(fx, fy) \preceq \frac{\lambda}{2}[q(fx, x) + q(y, fy)]$$

for all $x, y \in X$. Then f has a unique point $z \in X$. Moreover, $q(z, z) = \theta$.

Proof. To prove of this theorem, it is enough to follow Theorem 2.1 and Corollary 2 of [36, 2019] and using the notion of c-distance along with Lemma 2.6. ■

Note that all results of Lakzian et al. [36, 2019] can be changed into the framework of c-distance by this technique. But they cannot add new techniques to fixed point theory.

4. Conclusion

In this article, we collected all distances introduced on (abstract) metric spaces and reviewed the relations among those distances. Then, as a results, we compare two fixed point theorems with respect to both wt-distances and ct-distances. Also, we provided a fixed point theorem with respect to w-distance by a c-distance. From this note, we conclude that

(1) Fixed point results with respect to a w-distance in [27, 35, 36, 40, 42, 46] can be considered in the framework of a c-distance (w-cone distance), and conversely, fixed point results with respect to a c-distance (w-cone distance) in [4, 11, 12, 14, 17, 18, 31, 32, 38, 39, 43] can be considered in the framework of a w-distance.
(2) Fixed point results with respect to a wt-distance in [5, 7, 25, 30, 34] can be considered in the framework of a ct-distance (w_b-cone distance), and conversely, fixed point results with respect to a ct-distance (w_b-cone distance) in [4, 6, 8, 10, 44, 45] can be considered in the framework of a wt-distance.
(3) For all articles on w-distances (wt-distances), c-distances (ct-distances) and w-cone distances (w_b-cone distances), items (1) and (2) are true.
(4) All fixed point results with respect to a $ct_{\mathcal{E}}$-distance in [1, 3] can be valid for fixed point results with respect to a $c_{\mathcal{E}}$ with $b = 1$, and conversely, all fixed point results with respect to a $c_{\mathcal{E}}$-distance in [20, 23] can be extended to fixed point results with respect to a $ct_{\mathcal{E}}$-distance. This item is valid for all other fixed point articles with respect to both $c_{\mathcal{E}}$-distances and $ct_{\mathcal{E}}$-distances.

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