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Fixed point results for self-mappings in 2-normed spaces

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Abstract. This paper presents an extension and generalization of key results in fixed point theory by formulating common fixed point theorems for various forms of self-mappings within closed subsets of linear 2-Banach spaces. The analysis addresses the existence and uniqueness of common fixed points in several settings, including individual mappings, pairs of mappings, their positive powers, and sequences of self-mappings.

Keywords: 2-normed spaces, fixed point, 2-Banach spaces, contraction mappings, nonexpansive mappings.

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1. Introduction

In recent years, linear 2-normed spaces have attracted considerable interest because of their deep impact on functional analysis, fixed point theory, and nonlinear mathematics. The pioneering work of Freese and Cho [4] established the foundational framework for understanding the geometric and algebraic structures of these spaces, offering crucial insights into their mathematical properties. Since then, researchers have expanded upon these ideas, investigating iterative methods, accretive operators, and generalizations of classical fixed-point theorems within this framework.

Within this context, fixed-point theory has been a major focus of research, with Berinde [2] making substantial contributions to iterative approximation methods. His work has enhanced our understanding of fixed-point existence and convergence in various normed and metric spaces. Expanding on these advancements, Harikrishnan and Ravindran [5]

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explored accretive operators in linear 2-normed spaces, thereby extending classical operator theory and nonlinear functional analysis. Their research aligns with Banachs [1] seminal work, which laid the theoretical foundation for contraction mappings and fixed-point results.

Further developments in fixed-point theory arose from the contributions of Kannan [6] and Chatterjea [3], who introduced distinct contractive conditions that refined and generalized Banachs contraction principle. More recent studies by Kr and Acikgoz [8] have investigated the completion of quasi-2-normed spaces, while Modi and Gupta [10], along with Seshagiri and Kalyani [11], have extended fixed-point results to Hilbert spaces, thus broadening their applicability.

This work offers an in-depth extension and broadening of core results related to self-mappings in a closed subset of a linear 2-Banach space. Specifically, it examines the existence and uniqueness of common fixed points across various configurations, including single mappings, pairs of mappings, positive powers of two mappings, and sequences of self-mappings. By incorporating rational inequalities, this study broadens the applicability of these theorems and provides deeper analytical insights into the structure and behavior of fixed points in 2-Banach spaces. These findings contribute significantly to the advancement of linear functional analysis.

2. Preliminaries

This section outlines key definitions and results necessary for the progression of our study.

Definition 2.1 [4] Let F be a real linear space with $\dim F > 1$, and consider a function $\|.,.\| : F \times F \to [0,\infty)$. The structure $(F, \|.,.\|)$ is referred to as a linear 2-normed space if the following conditions are satisfied for all $\zeta, \eta, \kappa \in F$ and $\lambda \in \mathbb{R}$:

(1) $\|\zeta, \eta\| = 0$ if and only if ζ and η are linearly dependent;
(2) $\|\zeta, \eta\| = \|\zeta, \eta\|$;
(3) $\|\lambda\zeta, \eta\| = |\lambda| \|\zeta, \eta\|$;
(4) $\|\zeta + \eta, \kappa\| \leqslant \|\zeta, \kappa\| + \|\eta, \kappa\|$.

Definition 2.2 [4] A sequence $\{\zeta_n\}$ in a 2-normed space $(F, \|.,.\|)$ is said to be convergent if there exists a point $\zeta \in F$ such that $\lim_{n\to\infty} \|\zeta_n - \zeta, \kappa\| = 0$ for every $\kappa \in F$. If $\{\zeta_n\}$ converges to ζ, we denote this as $\zeta_n \to \zeta$ as $n \to \infty$.

Definition 2.3 [7] A sequence $\{\zeta_n\}$ in a 2-normed space $(F, \|.,.\|)$ is said to be a Cauchy sequence if $\lim_{m,n\to\infty} \|\zeta_n - \zeta_m, \kappa\| = 0$ for all $\kappa \in F$.

Definition 2.4 [5] A linear 2-normed space is termed complete if every Cauchy sequence within it converges to a point in F. When a 2-normed space possesses this completeness property, it is referred to as a 2-Banach space.

Definition 2.5 [5] Let $(F, \|.,.\|)$ be a linear 2-normed space, and let $C \subset F$. The *closure* of the set C is defined by

$$\bar{C} = \Big\{\zeta \in E : \text{there exists a sequence } \{\zeta_n\} \subset C \text{ such that } \zeta_n \to \zeta\Big\}.$$

The set C is called sequentially closed if $\bar{C} = C$.

Definition 2.6 [9] Consider a linear 2-normed space $(F, \|.,.\|)$ and let B be a nonempty subset of F. For a fixed element $e \in B$, the set B is said to be F-bounded if there exists a constant $M > 0$ such that $\|x, e\| \leqslant M$ for all $x \in B$. If this condition holds for every $e \in B$, then B is referred to as a bounded set.

3. Main results

Theorem 3.1 Let $(K, \|.,.\|)$ denote a linear 2-Banach space, and F be a nonempty closed subset of K. Consider a mapping $T : F \to F$ such that

$$\begin{aligned}\|T\zeta - T\eta, \kappa\| \leqslant \lambda &\frac{\|\zeta - T\zeta, \kappa\|^2 + \|\eta - T\eta, \kappa\|^2}{\|\zeta - T\zeta, \kappa\| + \|\eta - T\eta, \kappa\|} \\ &+ \mu \frac{\|\zeta - T\eta, \kappa\|^2 + \|\eta - T\zeta, \kappa\|^2}{\|\zeta - T\eta, \kappa\| + \|\eta - T\zeta, \kappa\|} + \nu \|\zeta - \eta, \kappa\|\end{aligned}$$

for all $\zeta, \eta, \kappa \in F$ with $\zeta \neq \eta$, where the constants λ, μ satisfying $0 \leqslant \lambda, \mu < \frac{1}{2}$, $\nu \geqslant 0$, and $2\lambda + 2\mu + \nu < 1$. Then T has a unique fixed point in F.

Proof. Let F be a non-empty closed subset of a linear 2-Banach space, and consider a mapping $T : F \to F$ that maps F into itself. Let $\zeta_0 \in F$ be arbitrary. We construct a sequence $\{\zeta_n\}$ by defining $\zeta_{n+1} = T\zeta_n$ for all $n = 0, 1, 2, \ldots$. For some n, if we have $\zeta_n = \zeta_{n+1}$, then $\zeta_{n+1} = \zeta_n = T\zeta_n$. Thus, ζ_n is a fixed point of T, proving the result immediately. Now, consider the case where $\zeta_{n+1} \neq \zeta_n$ for all $n = 0, 1, 2, \ldots$. We have

$$\begin{aligned}\|\zeta_{n+1} - \zeta_n, \kappa\| &= \|T\zeta_n - T\zeta_{n-1}, \kappa\| \\ &\leqslant \lambda \frac{\|\zeta_n - T\zeta_n, \kappa\|^2 + \|\zeta_{n-1} - T\zeta_{n-1}, \kappa\|^2}{\|\zeta_n - T\zeta_n, \kappa\| + \|\zeta_{n-1} - T\zeta_{n-1}, \kappa\|} \\ &\quad + \mu \frac{\|\zeta_n - T\zeta_{n-1}, \kappa\|^2 + \|\zeta_{n-1} - T\zeta_n, \kappa\|^2}{\|\zeta_n - T\zeta_{n-1}, \kappa\| + \|\zeta_{n-1} - T\zeta_n, \kappa\|} + \nu \|\zeta_n - \zeta_{n-1}, \kappa\|.\end{aligned}$$

Rewriting the terms,

$$\begin{aligned}\|\zeta_{n+1} - \zeta_n, \kappa\| &\leqslant \lambda \frac{\|\zeta_n - \zeta_{n+1}, \kappa\|^2 + \|\zeta_{n-1} - \zeta_n, \kappa\|^2}{\|\zeta_n - \zeta_{n+1}, \kappa\| + \|\zeta_{n-1} - \zeta_n, \kappa\|} \\ &\quad + \mu \frac{\|\zeta_n - \zeta_n, \kappa\|^2 + \|\zeta_{n-1} - \zeta_{n+1}, \kappa\|^2}{\|\zeta_n - \zeta_n, \kappa\| + \|\zeta_{n-1} - \zeta_{n+1}, \kappa\|} + \nu \|\zeta_n - \zeta_{n-1}, \kappa\| \\ &\leqslant \lambda \Big(\|\zeta_n - \zeta_{n+1}, \kappa\| + \|\zeta_{n-1} - \zeta_n, \kappa\| \Big) \\ &\quad + \mu \Big(\|\zeta_{n-1} - \zeta_{n+1}, \kappa\| \Big) + \nu \|\zeta_n - \zeta_{n-1}, \kappa\|.\end{aligned}$$

This implies that

$$(1 - \lambda - \mu)\|\zeta_{n+1} - \zeta_n, \kappa\| \leqslant (\lambda + \mu + \nu)\|\zeta_n - \zeta_{n-1}, \kappa\|,$$
$$\|\zeta_{n+1} - \zeta_n, \kappa\| \leqslant \rho \|\zeta_n - \zeta_{n-1}, \kappa\|,$$

where $\rho = \frac{\lambda+\mu+\nu}{1-\lambda-\mu} < 1$, since $2\lambda + 2\mu + \nu < 1$. Iterating this inequality,

$$\|\zeta_{n+1} - \zeta_n, \kappa\| \leqslant \rho^n \|\zeta_1 - \zeta_0, \kappa\|, \quad \forall n \geqslant 1.$$

We proceed to demonstrate that the sequence $\{\zeta_n\}$ is a Cauchy sequence. For any $p > 0$, we obtain

$$\begin{aligned}\|\zeta_n - \zeta_{n+p}, \kappa\| &\leqslant \sum_{i=n}^{n+p-1} \|\zeta_i - \zeta_{i+1}, \kappa\| \\ &\leqslant (\rho^n + \rho^{n+1} + \cdots + \rho^{n+p-1})\|\zeta_1 - \zeta_0, \kappa\| \\ &\leqslant \rho^n(1 + \rho + \rho^2 + \cdots + \rho^{p-1})\|\zeta_1 - \zeta_0, \kappa\| \\ &\leqslant \frac{\rho^n}{1-\rho}\|\zeta_1 - \zeta_0, \kappa\|.\end{aligned}$$

Since $\rho < 1$, it follows that $\|\zeta_n - \zeta_{n+p}, \kappa\| \to 0$ as $n \to \infty$, proving that $\{\zeta_n\}$ is a Cauchy sequence. As F is a closed subset of a 2-Banach space, there exists an element $\zeta \in F$ such that $\zeta_n \to \zeta$. Next we demonstrate that ζ is a fixed point of the mapping T.

$$\begin{aligned}\|\zeta - T\zeta, \kappa\| &= \|\zeta - \zeta_n + \zeta_n - T\zeta, \kappa\| \\ &\leqslant \|\zeta - \zeta_n, \kappa\| + \|\zeta_n - T\zeta, \kappa\| \\ &\leqslant \|\zeta - \zeta_n, \kappa\| + \|T\zeta_{n-1} - T\zeta, \kappa\|.\end{aligned}$$

Using the given condition,

$$(1 - \lambda - \mu)\|\zeta - T\zeta, \kappa\| \leqslant \lambda\|\zeta_{n-1} - \zeta_n, \kappa\| + \mu\|\zeta - \zeta_n, \kappa\| + \nu\|\zeta_{n-1} - \zeta, \kappa\|.$$

Taking limits, we obtain $\|\zeta - T\zeta, \kappa\| = 0$, implying $\zeta = T\zeta$. Hence, ζ is a fixed point of T. To establish uniqueness, suppose there exists another fixed point $\eta \in F$ with $\eta \neq \zeta$ such that $T(\eta) = \eta$. Then

$$\begin{aligned}\|\zeta - \eta, \kappa\| &= \|T\zeta - T\eta, \kappa\| \\ &\leqslant \lambda\frac{||\zeta - T\zeta, \kappa||^2 + ||\eta - T\eta, \kappa||^2}{||\zeta - T\zeta, \kappa|| + ||\eta - T\eta, \kappa||} + \mu\frac{||\zeta - T\eta, \kappa||^2 + ||\eta - T\zeta, \kappa||^2}{||\zeta - T\eta, \kappa|| + ||\eta - T\zeta, \kappa||} + \nu||\zeta - \eta, \kappa|| \\ &\leqslant \lambda\Big(||\zeta - T\zeta, \kappa|| + ||\eta - T\eta, \kappa||\Big) + \mu\Big(||\zeta - T\eta, \kappa|| + ||\eta - T\zeta, \kappa||\Big) + \nu||\zeta - \eta, \kappa|| \\ &\leqslant (2\mu + \nu)\|\zeta - \eta, \kappa\|.\end{aligned}$$

Given that $1 - 2\mu - \nu > 0$, it follows that $\|\zeta - \eta, \kappa\| = 0$, which implies $\zeta = \eta$. Hence, ζ is the unique fixed point of the mapping T. ■

***Example* 3.2** Let us consider the space $K = \mathbb{R}^2$ equipped with the standard 2-norm defined by $\|\zeta, \eta\| = |\zeta_1\eta_2 - \zeta_2\eta_1|$, which represents the area of the parallelogram spanned by the vectors ζ and η. Take $F = \mathbb{R}^2$, a closed and nonempty subset of K. Define a self-map $T : F \to F$ by $T(\zeta_1, \zeta_2) = \left(\frac{\zeta_1}{4}, \frac{\zeta_2}{4}\right)$. Choose the elements $\zeta = (1, 0)$, $\eta = (0, 1)$, and $\kappa = (1, 1)$. Let the constants be $\lambda = \mu = \nu = \frac{1}{6}$. Then

$$T\zeta = \left(\frac{1}{4}, 0\right), \quad T\eta = \left(0, \frac{1}{4}\right), \quad T\zeta - T\eta = \left(\frac{1}{4}, -\frac{1}{4}\right),$$
$$\|T\zeta - T\eta, \kappa\| = \left|\frac{1}{4} \cdot 1 - (-\frac{1}{4} \cdot 1)\right| = \frac{1}{2}.$$

Now,

$$\zeta - T\zeta = \left(\frac{3}{4}, 0\right) \Rightarrow \|\zeta - T\zeta, \kappa\| = \left|\frac{3}{4} \cdot 1 - 0 \cdot 1\right| = \frac{3}{4},$$
$$\eta - T\eta = \left(0, \frac{3}{4}\right) \Rightarrow \|\eta - T\eta, \kappa\| = \left|0 \cdot 1 - \frac{3}{4} \cdot 1\right| = \frac{3}{4},$$
$$\zeta - T\eta = \left(1, -\frac{1}{4}\right) \Rightarrow \|\zeta - T\eta, \kappa\| = \left|1 \cdot 1 - (-\frac{1}{4} \cdot 1)\right| = \frac{5}{4},$$
$$\eta - T\zeta = \left(-\frac{1}{4}, 1\right) \Rightarrow \|\eta - T\zeta, \kappa\| = \left|-\frac{1}{4} \cdot 1 - 1 \cdot 1\right| = \frac{5}{4},$$
$$\|\zeta - \eta, \kappa\| = |1 \cdot 1 - 0 \cdot 1 - (0 \cdot 1 - 1 \cdot 1)| = 2.$$

Substituting into the inequality:

$$\|T\zeta - T\eta, \kappa\| = \frac{1}{2},$$
$$\lambda \frac{\|\zeta - T\zeta, \kappa\|^2 + \|\eta - T\eta, \kappa\|^2}{\|\zeta - T\zeta, \kappa\| + \|\eta - T\eta, \kappa\|} = \frac{1}{6} \cdot \frac{(3/4)^2 + (3/4)^2}{3/4 + 3/4} = \frac{1}{6} \cdot \frac{9/8}{3/2} = \frac{1}{6} \cdot \frac{3}{4} = \frac{1}{8},$$
$$\mu \frac{\|\zeta - T\eta, \kappa\|^2 + \|\eta - T\zeta, \kappa\|^2}{\|\zeta - T\eta, \kappa\| + \|\eta - T\zeta, \kappa\|} = \frac{1}{6} \cdot \frac{(5/4)^2 + (5/4)^2}{5/4 + 5/4} = \frac{1}{6} \cdot \frac{50/16}{5/2} = \frac{1}{6} \cdot \frac{25}{20} = \frac{5}{24},$$
$$\nu\|\zeta - \eta, \kappa\| = \frac{1}{6} \cdot 2 = \frac{1}{3}.$$

Adding up the right-hand side, $\frac{1}{8} + \frac{5}{24} + \frac{1}{3} = \frac{3}{24} + \frac{5}{24} + \frac{8}{24} = \frac{16}{24} = \frac{2}{3}$. Since $\|T\zeta - T\eta, \kappa\| = \frac{1}{2} \leqslant \frac{2}{3}$, the inequality is satisfied. Hence, all the conditions of the Theorem 3.1 are verified. Therefore, the mapping T has a unique fixed point in F, which is $(0, 0)$.

Theorem 3.3 Let $(K, ||., .||)$ denote a linear 2-Banach space and F be a non-empty closed subset of K. Suppose that the self mappings $T_1, T_2 : F \to F$ satisfy

$$||T_1\zeta - T_2\eta, \kappa|| \leqslant \lambda \frac{||\zeta - T_1\zeta, \kappa||^2 + ||\eta - T_2\eta, \kappa||^2}{||\zeta - T_1\zeta, \kappa|| + ||\eta - T_2\eta, \kappa||} + \mu \frac{||\zeta - T_2\eta, \kappa||^2 + ||\eta - T_1\zeta, \kappa||^2}{||\zeta - T_2\eta, \kappa|| + ||\eta - T_1\zeta, \kappa||} + \nu||\zeta - \eta, \kappa||$$

for all $\zeta, \eta, \kappa \in F$, $\zeta \neq \eta$, where $0 \leqslant \lambda, \mu < \frac{1}{2}$, $\nu \geqslant 0$, and $2\lambda + 2\mu + \nu < 1$. Then T_1 and T_2 have a unique common fixed point in F.

Proof. Let ζ_0 be an arbitrary element in F. We construct a sequence $\{\zeta_n\}$ in F defined by

$$\zeta_{2n+1} = T_1\zeta_{2n},$$
$$\zeta_{2n+2} = T_2\zeta_{2n+1}, \quad \text{for all } n = 0, 1, 2, \dots$$

If $\zeta_n = \zeta_{n+1} = \zeta_{n+2}$ for some index n, then ζ_n is evidently a common fixed point of both T_1 and T_2. Otherwise, we proceed under the assumption that no two consecutive terms in the sequence are identical. Now, we analyze

$$\begin{aligned}
||\zeta_{2n+1} - \zeta_{2n}, \kappa|| &= ||T_1\zeta_{2n} - T_2\zeta_{2n-1}, \kappa|| \\
&\leqslant \lambda \frac{||\zeta_{2n} - T_1\zeta_{2n}, \kappa||^2 + ||\zeta_{2n-1} - T_2\zeta_{2n-1}, \kappa||^2}{||\zeta_{2n} - T_1\zeta_{2n}, \kappa|| + ||\zeta_{2n-1} - T_2\zeta_{2n-1}, \kappa||} \\
&\quad + \mu \frac{||\zeta_{2n} - T_2\zeta_{2n-1}, \kappa||^2 + ||\zeta_{2n-1} - T_1\zeta_{2n}, \kappa||^2}{||\zeta_{2n} - T_2\zeta_{2n-1}, \kappa|| + ||\zeta_{2n-1} - T_1\zeta_{2n}, \kappa||} + \nu||\zeta_{2n} - \zeta_{2n-1}, \kappa|| \\
&\leqslant \lambda\Big(\|\zeta_{2n} - \zeta_{2n+1}, \kappa\| + \|\zeta_{2n-1} - \zeta_{2n}, \kappa\|\Big) + \mu\Big(\|\zeta_{2n-1} - \zeta_{2n+1}, \kappa\|\Big) \\
&\quad + \nu\|\zeta_{2n} - \zeta_{2n-1}, \kappa\|.
\end{aligned}$$

We obtain $||\zeta_{2n+1} - \zeta_{2n}, \kappa|| \leqslant \rho||\zeta_{2n} - \zeta_{2n-1}, \kappa||$, where $\rho = \frac{\lambda+\mu+\nu}{1-\lambda-\mu} < 1$ since $2\lambda+2\mu+\nu < 1$. Continuing this process, we obtain $||\zeta_{n+1} - \zeta_n, \kappa|| \leqslant \rho^n||\zeta_1 - \zeta_0, \kappa||$ for all $n \geqslant 1$. Thus, for $p > 0$,

$$||\zeta_n - \zeta_{n+p}, \kappa|| \leqslant \sum_{j=0}^{p-1} ||\zeta_{n+j} - \zeta_{n+j+1}, \kappa|| \leqslant \sum_{j=0}^{p-1} \rho^{n+j}||\zeta_1 - \zeta_0, \kappa||.$$

Given that $\rho < 1$, it follows that the sequence $\{\zeta_n\}$ is Cauchy. As F is a closed subset of a 2-Banach space, there exists $\zeta_1 \in F$ such that $\lim_{n\to\infty} \zeta_n = \zeta_1$. We now proceed to demonstrate that ζ_1 is a fixed point of both T_1 and T_2. Suppose, for the sake of contradiction, that $T_1\zeta_1 \neq \zeta_1$. Then

$$\begin{aligned}
\|\zeta_1 - T_1\zeta_1, \kappa\| &= \|\zeta_1 - \zeta_{2n+2} + \zeta_{2n+2} - T_1\zeta_1, \kappa\| \\
&\leqslant \|\zeta_1 - \zeta_{2n+2}, \kappa\| + \|\zeta_{2n+2} - T_1\zeta_1, \kappa\| \\
&= \|\zeta_1 - \zeta_{2n+2}, \kappa\| + \|T_1\zeta_1 - T_2\zeta_{2n+1}, \kappa\| \\
&\leqslant \|\zeta_1 - \zeta_{2n+2}, \kappa\| + \lambda \frac{\|\zeta_1 - T_1\zeta_1, \kappa\|^2 + \|\zeta_{2n+1} - T_2\zeta_{2n+1}, \kappa\|^2}{\|\zeta_1 - T_1\zeta_1, \kappa\| + \|\zeta_{2n+1} - T_2\zeta_{2n+1}, \kappa\|} \\
&\quad + \mu \frac{\|\zeta_1 - T_2\zeta_{2n+1}, \kappa\|^2 + \|\zeta_{2n+1} - T_1\zeta_1, \kappa\|^2}{\|\zeta_1 - T_2\zeta_{2n+1}, \kappa\| + \|\zeta_{2n+1} - T_1\zeta_1, \kappa\|} + \nu\|\zeta_1 - \zeta_{2n+1}, \kappa\| \\
&\leqslant \|\zeta_1 - \zeta_{2n+2}, \kappa\| + \lambda\Big(\|\zeta_1 - T_1\zeta_1, \kappa\| + \|\zeta_{2n+1} - T_2\zeta_{2n+1}, \kappa\|\Big) \\
&\quad + \mu\Big(\|\zeta_1 - T_2\zeta_{2n+1}, \kappa\| + \|\zeta_{2n+1} - T_1\zeta_1, \kappa\|\Big) + \nu||\zeta_1 - \zeta_{2n+1}, \kappa|| \\
\Rightarrow (1 - \lambda - \mu)||\zeta_1 - T_1\zeta_1, \kappa|| &\leqslant \lambda||\zeta_{2n+1} - \zeta_{2n+2}, \kappa|| + (\mu + 1)||\zeta_1 - \zeta_{2n+2}, \kappa|| \\
&\quad + \nu||\zeta_1 - \zeta_{2n+1}, \kappa||
\end{aligned}$$

Taking limits as $n \to \infty$, we get $||\zeta_1 - T_1\zeta_1, \kappa|| = 0$, implying $T_1\zeta_1 = \zeta_1$. Similarly, we can show $T_2\zeta_1 = \zeta_1$, so ζ_1 is a common fixed point of T_1 and T_2. Finally, assume that there exists another common fixed point η_1 with $\eta_1 \neq \zeta_1$.

$$\begin{aligned}
||\zeta_1 - \eta_1, \kappa|| &= ||T_1\zeta_1 - T_2\eta_1, \kappa|| \\
&\leqslant \lambda \frac{||\zeta_1 - T_1\zeta_1, \kappa||^2 + ||\eta_1 - T_2\eta_1, \kappa||^2}{||\zeta_1 - T_1\zeta_1, \kappa|| + ||\eta_1 - T_2\eta_1, \kappa||} \\
&\quad + \mu \frac{||\zeta_1 - T_2\eta_1, \kappa||^2 + ||\eta_1 - T_1\zeta_1, \kappa||^2}{||\zeta_1 - T_2\eta_1, \kappa|| + ||\eta_1 - T_1\zeta_1, \kappa||} + \nu||\zeta_1 - \eta_1, \kappa|| \\
&\leqslant \lambda\Big(||\zeta_1 - T_1\zeta_1, \kappa|| + ||\eta_1 - T_2\eta_1, \kappa||\Big) \\
&\quad + \mu\Big(||\zeta_1 - T_2\eta_1, \kappa|| + ||\eta_1 - T_1\zeta_1, \kappa||\Big) + \nu||\zeta_1 - \eta_1, \kappa||.
\end{aligned}$$

Since $T_1\zeta_1 = \zeta_1$ and $T_2\eta_1 = \eta_1$, this simplifies to

$$(1 - 2\mu - \nu)||\zeta_1 - \eta_1, \kappa|| \leqslant 0 \Rightarrow \zeta_1 = \eta_1.$$

Thus, ζ_1 is the unique common fixed point of T_1 and T_2. ■

***Example* 3.4** Consider the linear 2-Banach space $K = \mathbb{R}^2$ with the standard 2-norm defined by $\|\zeta, \eta\| = |\zeta_1\eta_2 - \zeta_2\eta_1|$, which represents the area of the parallelogram spanned by ζ and η. Let $F = \mathbb{R}^2$ be a non-empty closed subset of K. Define two self-mappings $T_1, T_2 : F \to F$ as follows:

$$T_1(\zeta_1, \zeta_2) = \left(\frac{\zeta_1}{4}, \frac{\zeta_2}{4}\right), \quad T_2(\zeta_1, \zeta_2) = \left(\frac{\zeta_1}{5}, \frac{\zeta_2}{5}\right).$$

Now, take $\zeta = (1, 0)$, $\eta = (0, 1)$ and $\kappa = (1, 1)$. Let $\lambda = \mu = \nu = \frac{1}{6}$. Clearly, these satisfy $0 \leqslant \lambda, \mu < \frac{1}{2}, \quad \nu \geqslant 0, \quad 2\lambda + 2\mu + \nu = \frac{5}{6} < 1$. Now, compute the necessary quantities:

$$\begin{aligned}
T_1\zeta &= \left(\frac{1}{4}, 0\right), \quad T_2\eta = \left(0, \frac{1}{5}\right), \\
T_1\zeta - T_2\eta &= \left(\frac{1}{4}, -\frac{1}{5}\right), \quad \|T_1\zeta - T_2\eta, \kappa\| = \left|\frac{1}{4}\cdot 1 - (-\frac{1}{5}\cdot 1)\right| = \frac{9}{20}. \\
\zeta - T_1\zeta &= \left(\frac{3}{4}, 0\right), \quad \|\zeta - T_1\zeta, \kappa\| = \left|\frac{3}{4}\cdot 1 - 0\cdot 1\right| = \frac{3}{4}, \\
\eta - T_2\eta &= \left(0, \frac{4}{5}\right), \quad \|\eta - T_2\eta, \kappa\| = \left|0\cdot 1 - \frac{4}{5}\cdot 1\right| = \frac{4}{5}. \\
\zeta - T_2\eta &= \left(1, -\frac{1}{5}\right), \quad \|\zeta - T_2\eta, \kappa\| = \left|1\cdot 1 - (-\frac{1}{5}\cdot 1)\right| = \frac{6}{5}, \\
\eta - T_1\zeta &= \left(-\frac{1}{4}, 1\right), \quad \|\eta - T_1\zeta, \kappa\| = \left|-\frac{1}{4}\cdot 1 - 1\cdot 1\right| = \frac{5}{4}, \\
\|\zeta - \eta, \kappa\| &= |1\cdot 1 - 0\cdot 1 - (0\cdot 1 - 1\cdot 1)| = 2.
\end{aligned}$$

Now, verify Left-hand side: $\|T_1\zeta - T_2\eta, \kappa\| = \frac{9}{20}$. Right-hand side:

$$\lambda \cdot \frac{\left(\frac{3}{4}\right)^2 + \left(\frac{4}{5}\right)^2}{\frac{3}{4} + \frac{4}{5}} = \frac{1}{6} \cdot \frac{\frac{9}{16} + \frac{16}{25}}{\frac{15}{20} + \frac{16}{20}} = \frac{1}{6} \cdot \frac{\frac{649}{400}}{\frac{31}{20}} = \frac{1}{6} \cdot \frac{649}{620} \approx 0.1745,$$

$$\mu \cdot \frac{\left(\frac{6}{5}\right)^2 + \left(\frac{5}{4}\right)^2}{\frac{6}{5} + \frac{5}{4}} = \frac{1}{6} \cdot \frac{\frac{36}{25} + \frac{25}{16}}{\frac{59}{20}} = \frac{1}{6} \cdot \frac{\frac{1521}{400}}{\frac{59}{20}} = \frac{1}{6} \cdot \frac{1521}{1180} \approx 0.2148,$$

$$\nu \cdot \|\zeta - \eta, \kappa\| = \frac{1}{6} \cdot 2 = \frac{1}{3} \approx 0.3333.$$

Summing up RHS $\approx 0.1745 + 0.2148 + 0.3333 = 0.7226 > \frac{9}{20} = 0.45$. Since the left-hand side is less than the right-hand side, the inequality is satisfied. Therefore, the mappings T_1 and T_2 satisfy the conditions of Theorem 3.3 and have a unique common fixed point, which is easily verified to be $(0, 0)$.

Theorem 3.5 Let $(K, ||\cdot, \cdot||)$ be a linear 2-Banach space and F be a non-empty closed subset of K. Suppose that the self-mappings $T_1^p, T_2^q : F \to F$ satisfy

$$||T_1^p\zeta - T_2^q\eta, \kappa|| \leqslant \lambda \frac{||\zeta - T_1^p\zeta, \kappa||^2 + ||\eta - T_2^q\eta, \kappa||^2}{||\zeta - T_1^p\zeta, \kappa|| + ||\eta - T_2^q\eta, \kappa||} + \mu \frac{||\zeta - T_2^q\eta, \kappa||^2 + ||\eta - T_1^p\zeta, \kappa||^2}{||\zeta - T_2^q\eta, \kappa|| + ||\eta - T_1^p\zeta, \kappa||} + \nu||\zeta - \eta, \kappa||$$

for all $\zeta, \eta, \kappa \in F$, where $\zeta \neq \eta$, $0 \leqslant \lambda, \mu < \frac{1}{2}$, $\nu \geqslant 0$, and $2\lambda + 2\mu + \nu < 1$. Then T_1 and T_2 have a unique common fixed point in F.

Proof. By applying Theorem 3.2, we conclude that T_1^p and T_2^q possess a unique common fixed point in F, denoted by ζ^*. That is, $T_1^p\zeta^* = \zeta^*$ and $T_2^q\zeta^* = \zeta^*$. Now, we consider $T_1\zeta^* = T_1(T_1^p\zeta^*) = T_1^p(T_1\zeta^*)$. This indicates that $T_1\zeta^*$ is also a fixed point of T_1^p. Since T_1^p has a unique fixed point, we conclude $T_1\zeta^* = \zeta^*$. Similarly, we obtain $T_2\zeta^* = \zeta^*$. Hence, ζ^* is a common fixed point of T_1 and T_2. To establish uniqueness, assume that there exists another common fixed point η^*. So η^* is also a common fixed point of T_1^p and T_2^q, and by the uniqueness property, we conclude that $\zeta^* = \eta^*$. Hence, the common fixed point is unique. ■

Theorem 3.6 Let F be a non-empty closed subset of a linear 2-Banach space and $\{T_j\}$ be a sequence of mappings on F converges pointwise to a mapping T. Suppose that T_j satisfies

$$||T_j\zeta - T_j\eta, \kappa|| \leqslant \lambda \frac{||\zeta - T_j\zeta, \kappa||^2 + ||\eta - T_j\eta, \kappa||^2}{||\zeta - T_j\zeta, \kappa|| + ||\eta - T_j\eta, \kappa||} + \mu \frac{||\zeta - T_j\eta, \kappa||^2 + ||\eta - T_j\zeta, \kappa||^2}{||\zeta - T_j\eta, \kappa|| + ||\eta - T_j\zeta, \kappa||} + \nu||\zeta - \eta, \kappa||$$

for all $\zeta, \eta \in F$, where $\zeta \neq \eta$, $0 \leqslant \lambda, \mu < \frac{1}{2}$, $\nu \geqslant 0$, and $2\lambda + 2\mu + \nu < 1$. If ξ_j and ξ are the fixed points of T_j and T respectively, then the sequence $\{\xi_j\}$ converges to ξ.

Proof. Since ξ_j is a fixed point of T_j, we have

$$||\xi - \xi_j, \kappa|| = ||T\xi - T_j\xi + T_j\xi - T_j\xi_j, \kappa|| \leqslant ||T\xi - T_j\xi, \kappa|| + ||T_j\xi - T_j\xi_j, \kappa||.$$

Applying the given condition,

$$\begin{aligned} ||T_j\xi - T_j\xi_j, \kappa|| \leqslant \lambda & \frac{||\xi - T_j\xi, \kappa||^2 + ||\xi_j - T_j\xi_j, \kappa||^2}{||\xi - T_j\xi, \kappa|| + ||\xi_j - T_j\xi_j, \kappa||} \\ & + \mu \frac{||\xi - T_j\xi_j, \kappa||^2 + ||\xi_j - T_j\xi, \kappa||^2}{||\xi - T_j\xi_j, \kappa|| + ||\xi_j - T_j\xi, \kappa||} + \nu||\xi - \xi_j, \kappa||. \end{aligned}$$

Using $\frac{a^2+b^2}{a+b} \leqslant a + b$, we obtain

$$\begin{aligned} ||T_j\xi - T_j\xi_j, \kappa|| \leqslant & \lambda(||\xi - T_j\xi, \kappa|| + ||\xi_j - T_j\xi_j, \kappa||) \\ & + \mu(||\xi - T_j\xi_j, \kappa|| + ||\xi_j - T_j\xi, \kappa||) + \nu||\xi - \xi_j, \kappa||. \end{aligned}$$

Taking the limit as $j \to \infty$ and noting that $T_j\xi \to T\xi$, we get $||T_j\xi - T\xi, \kappa|| \to 0$. Thus,

$$\lim_{j\to\infty} ||\xi - \xi_j, \kappa|| \leqslant \lambda||\xi - T_j\xi, \kappa|| + \mu(||\xi - T_j\xi_j, \kappa|| + ||\xi_j - T_j\xi, \kappa||) + \nu||\xi - \xi_j, \kappa||.$$

Since $2\lambda + 2\mu + \nu < 1$, we conclude that $\{\xi_j\}$ converges to ξ. ■

4. Conclusion

In this study, we have extended and generalized key foundational results by establishing common fixed-point theorems for various arrangements of self-mappings within a closed subset of a linear 2-Banach space. Our investigation focused on the existence and uniqueness of common fixed points for individual mappings, pairs of mappings, their respective positive powers, as well as sequences of self-mappings. The findings incorporate a range of inequalities involving rational expressions, thereby enriching the theoretical structure of fixed-point theory in the context of 2-Banach spaces. These developments provide a solid platform for further exploration in functional analysis and its potential applications.

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