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Parabolic transformation and solution of 3D Ricci flow equations using killing vector fields

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Abstract. Ricci flow equations are among the most fundamental equations in Riemannian geometry and classical field theory, playing a crucial role in modeling physical phenomena such as relativistic gravity and quantum field theory. In this paper, we transform the Ricci flow equations for three-dimensional manifolds into a parabolic form by applying appropriate coordinate changes and solve them using invariant geometric structures, particularly the Killing vector field. Additionally, we propose a method for diagonalizing metrics on three-dimensional manifolds, which simplifies the dynamical analysis of these equations. This approach extends known results on two-dimensional Ricci flow equations and, by leveraging algebraic structures related to Toda equations, provides a more precise examination of possible solutions.

Keywords: Riemann solitons, killing vector field, general relativity.

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1. Introduction

The Ricci flow has been a fundamental topic in differential geometry and mathematical physics due to its deep connection with the uniformization theorem and the classification of manifolds. In the case of two-dimensional manifolds, the Ricci flow plays a crucial role in proving the uniformization theorem, which asserts that every compact, orientable surface with genus g admits a unique constant curvature metric, positive for $g = 0$, zero for $g = 1$, and negative for $g \geqslant 2$ [11]. Bakas demonstrated that the two-dimensional conformal Ricci flow is equivalent to the continuous limit of the Toda field equations [9, 10]. This correspondence facilitates the use of algebraic techniques to explicitly solve the Ricci flow equations [4, 8].

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The three-dimensional uniformization conjecture, which plays a crucial role in quantum gravity models, provides a geometric framework analogous to the two-dimensional case [2]. Moreover, Perelman's groundbreaking work applied the Ricci flow to prove the Poincar conjecture, resolving one of the most fundamental problems in topology [18–20]. His proof built upon Hamilton's Ricci flow approach, introducing techniques to control singularities and ensure convergence [15, 16]. However, unlike the two-dimensional case, where the uniformization theorem provides a classification of Riemann surfaces in terms of constant curvature geometries, the three-dimensional uniformization conjecture remains an open problem, first proposed by Thurston [21]. While Thurstons geometrization conjecture, a generalization of uniformization, was proven by Perelman [20], the full understanding of three-dimensional uniformization, particularly in the context of quantum gravity, is still an area of active research [1, 17].

Since then, Ricci flow has become an important tool in geometric analysis and has been applied to a variety of problems in topology and theoretical physics. The ability to transform Ricci flow equations into parabolic form, as done in this paper, is a critical advancement in the study of three-dimensional manifolds, as it allows for a more systematic approach to solving these equations. Riemannian solitons emerged in the study of the geometric evolution of Riemannian manifolds and are closely related to Ricci solitons. Ricci solitons were introduced as a generalization of Einstein spaces and play a crucial role in the evolution of the Ricci flow, which was introduced by Richard Hamilton in the 1980s. A Ricci soliton is a triplet (M, g, X) that satisfies the equation

$$\mathcal{L}_X g + 2\,\mathrm{Ric} = \lambda g, \tag{1}$$

where $\mathcal{L}_X g$ is the Lie derivative of the metric along the vector field X, Ric is the Ricci tensor, and λ is a constant scalar. If $X = \nabla f$ for a smooth function f, the soliton is called a gradient Ricci soliton.

Riemannian solitons have been introduced as a weaker and more flexible version of Ricci solitons, where the solitonic equation allows for additional modifications, particularly in the context of Finsler geometry and more complex structures. The relationship between these two concepts is significant in the study of the evolutionary behavior of metrics under geometric flows, as both Ricci solitons and Riemannian solitons appear as self-similar models in manifold evolution. They play a fundamental role in understanding the dynamics of the Ricci flow and the classification of geometric structures. For more details on Riemannian solitons, see [5–7].

The relationship between Ricci solitons and Riemannian solitons plays a crucial role in understanding the underlying structures governing geometric evolution equations. Ricci solitons serve as self-similar solutions to the Ricci flow and often describe singularity models, whereas Riemannian solitons provide a broader framework for studying equilibrium configurations in geometric flows. The results of this work contribute to this understanding by demonstrating how transformed Ricci flow equations, when expressed in terms of Killing vector fields, preserve or modify solitonic properties. This insight helps in classifying self-similar solutions and provides a deeper geometric interpretation of stability and rigidity phenomena in evolving Riemannian manifolds. Furthermore, by examining the interplay between Ricci and Riemannian solitons in this transformed setting, the study sheds light on new potential invariant structures that emerge under parabolic evolution.

In this paper, we extend Bakas's results on the 2D Ricci flow to 3-manifolds using a Killing vector field. This approach offers a concise and efficient method for analyzing the three-dimensional Ricci flow equations. A key observation is that, due to the interplay between coordinate scaling and invariance under subspace transformations, the equations do not admit a unique global solution. Consequently, we analyze two distinct exact analytical solutions and demonstrate their differing behaviors.

The paper is structured as follows: Section 2 reviews Bakas's results on two-dimensional Ricci flow. Section 3 discusses the transformation of the Ricci flow equations into a parabolic form and explores coordinate changes to simplify the metric. Section 4 presents explicit solutions to the equations, followed by the conclusions.

2. Preliminaries

The two-dimensional Ricci flow equation for a Riemannian metric g_{ij} is given by $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$, where R_{ij} is the Ricci tensor. The last two terms (commonly referred to as the De Turck terms) account for all diffeomorphism degrees of freedom [12]. To optimize the evolution equation, De Turck introduced the vector field $V^k = g^{ij}(\Gamma^k_{ij} - \tilde{\Gamma}^k_{ij})$, where Γ^k_{ij} are the Christoffel symbols of the metric g_{ij}. The goal is to transform the Ricci flow equations into a strictly parabolic form. Bakas analyzed the conformal case:

$$\frac{\partial \phi}{\partial t} = \Delta\phi + e^{\phi}, \tag{2}$$

which corresponds to the nonlinear heat equation. The Toda field equations, which describe the relation between two-dimensional fields and Cartan matrices, take the form:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = e^{\phi}. \tag{3}$$

Bakas demonstrated that equation (2) is the continuous limit of equation (3), allowing general solutions in terms of power series expansions around a potential field.

A Killing vector field was first introduced by Isenberg and Jackson in 1993 to study Ricci flow on minisuperspace models [13]. We assume that the metric admits at least one Killing vector field ξ^μ in coordinates (x^1, x^2, x^3). The metric is then written as:

$$ds^2 = e^{2\lambda}(dx^1)^2 + e^{2\mu}(dx^2)^2 + e^{2\nu}(dx^3)^2,$$

where λ, μ, and ν depend only on the coordinates x^i (for $i = 1, 2, 3$). Using the De Turck modification:

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \nabla_i V_j + \nabla_j V_i,$$

we obtain a fully parabolic system.

We demonstrated that the three-dimensional Ricci flow equations, when restricted to manifolds admitting a Killing vector, can be solved using techniques similar to the two-dimensional case. By appropriately choosing the De Turck term, the system remains parabolic. The key question remains whether this method can be extended to fully generic three-dimensional metrics. Future work will explore the diagonalization of arbitrary three-metrics. We consider a 3-manifold (M, g) admitting a Killing vector field ξ^μ. In an adapted coordinate system (x, y, z), where $\xi = \partial_z$, the metric takes a diagonal form:

$$ds^2 = e^{2\lambda(x,y,t)}dx^2 + e^{2\mu(x,y,t)}dy^2 + e^{2\nu(x,y,t)}dz^2.$$

Using the proposed transformation $\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \nabla_i \nu \nabla_j \nu$, we ensure diagonalization, simplifying the evolution equations significantly. This equation deforms the metric in a manner analogous to heat diffusion, leading to the homogenization of geometric structures. In lower dimensions, particularly for two-dimensional surfaces, Ricci flow has profound connections with the uniformization theorem and integrable systems [9]. An interesting case arises when the manifold admits a Killing vector field. The presence of such a symmetry reduces the system to an effective two-dimensional problem, where methods from conformal geometry and soliton theory can be applied. This reduction offers a powerful framework for understanding the long-term behavior of the flow, including convergence to homogeneous geometries. Since the metric components are independent of z, the Ricci tensor components simplify as $R_{ij} = \tilde{R}_{ij} - \frac{1}{2}e^{2\nu}\nabla_i \nu \nabla_j \nu$, where $\tilde{R}ij$ represents the Ricci tensor of the induced 2D metric $g_{ab} = e^{2\lambda}dx^2 + e^{2\mu}dy^2$. The Ricci flow equations then reduce to

$$\begin{aligned}
\frac{\partial \lambda}{\partial t} &= \Delta\lambda + e^{2\lambda} - e^{2\nu}, \\
\frac{\partial \mu}{\partial t} &= \Delta\mu + e^{2\mu} - e^{2\nu}, \\
\frac{\partial \nu}{\partial t} &= \Delta\nu + e^{\nu-\lambda} + e^{\nu-\mu}.
\end{aligned}$$

This system describes the coupled evolution of the metric components. For special choices of the metric, Ricci flow reduces to known integrable equations.

If the metric takes a conformal form, $e^{2\lambda} = e^{2\mu}$, then the equation for λ simplifies to $\frac{\partial \lambda}{\partial t} = \Delta\lambda + e^{\lambda}$, which is the classical Toda equation [10]. Toda equations appear in various contexts, including soliton theory and gravitational instantons.

The connection between the algebraic structures underlying Toda equations and the solutions of the Ricci flow provides valuable insights into the integrability and geometric properties of the flow. Toda lattices are well-known for their connection to Lie algebraic structures and integrable systems, raising the question of whether these algebraic techniques contribute to generating new exact solutions or primarily serve as a conceptual tool for analyzing the flow's behavior. Our analysis suggests that while the algebraic formulation facilitates the classification of special solutions and provides a structured approach to understanding the evolution equations, it does not always guarantee the construction of novel explicit solutions. However, in specific cases where the Ricci flow equations admit reductions to known integrable forms, such as Toda-type systems, these techniques can indeed yield exact solutions. Further exploration of this connection could enhance the analytical toolkit available for solving and interpreting Ricci flow dynamics.

For solutions where $e^{2\lambda} = e^{\phi}(x, y)$, the equation for ϕ becomes $\Delta\phi = ke^{\phi}$. This equation describes surfaces of constant negative curvature, fundamental in two-dimensional gravity.

A key question is whether the Toda-based approach provides fundamentally new solutions to the Ricci flow equations or merely reinterprets existing solutions in an alternative mathematical framework. Our analysis reveals that

- In certain symmetric cases, Toda structures enable a systematic classification of self-similar Ricci flow solutions, revealing families of metrics that were previously difficult to obtain through direct PDE analysis.
- The connection to Toda dynamics facilitates the identification of conserved quantities and Hamiltonian structures, which in turn help characterize long-term behavior under

Ricci flow.

- However, in more general, non-integrable cases, the Toda representation does not necessarily yield closed-form solutions but still provides valuable insight into qualitative aspects of curvature evolution.

By framing the Ricci flow equations in terms of Toda systems, we gain access to a wealth of mathematical techniques from integrable systems theory. While not all cases lead to explicit new solutions, the ability to apply algebraic and Hamiltonian structures provides a deeper understanding of the solution space and stability properties of evolving geometries. This suggests that the Toda formulation is more than a mere conceptual tool; it offers tangible computational advantages in specific settings where traditional PDE approaches are less effective.

Assuming that the metric remains homogeneous under the flow, the equations reduce to ODEs. In this case, Ricci flow drives the metric toward constant curvature spaces, in agreement with Thurston's conjecture.

For solutions of the form $e^{2\lambda} = e^{\phi(x,y)}$, the Liouville equation governs the evolution $\Delta\phi = ke^{\phi}$. This describes hyperbolic geometries relevant in conformal field theory.

A Killing vector field ξ^{μ} satisfies $\nabla_{(\mu}\xi_{\nu)} = 0$. These fields generate isometries of the manifold and simplify the Ricci flow equations. Below, we provide some important examples:

***Example* 2.1** In cylindrical coordinates (r, θ, z), the vector field $\xi = \frac{\partial}{\partial\theta}$ is a Killing vector field representing rotational symmetry.

***Example* 2.2** In Euclidean space $\mathbb{R}^3$, the vector fields

$$\xi_1 = \frac{\partial}{\partial x}, \quad \xi_2 = \frac{\partial}{\partial y}, \quad \xi_3 = \frac{\partial}{\partial z}$$

generate translations in the respective directions and preserve the flat metric.

***Example* 2.3** On the hyperbolic space H^2 with metric $ds^2 = \frac{dx^2+dy^2}{y^2}$, the vector fields

$$\xi_1 = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}, \quad \xi_2 = \frac{\partial}{\partial x}$$

are Killing vector fields, generating hyperbolic translations.

The non-uniqueness of global solutions to the transformed Ricci flow equations arises due to several geometric and analytical factors. The use of Killing vector fields in the transformation can introduce additional symmetries that may lead to multiple admissible solutions. Furthermore, while the standard Ricci flow admits unique solutions under appropriate initial conditions, the transformed system may alter these conditions by modifying curvature evolution or introducing singular behavior. In particular, changes in boundary conditions or degeneracies in the parabolic structure of the equations can lead to solution branching. This non-uniqueness can be further understood by analyzing the impact of the transformation on energy estimates and maximum principles, which are crucial for ensuring well-posedness in parabolic PDEs. A deeper investigation into these factors reveals that the transformed Ricci flow may not always satisfy the conditions necessary for global uniqueness.

3. Main Results

The Liouville equation is a fundamental equation in differential geometry and mathematical physics $\Delta\phi = ke^{\phi}$. This equation appears in the study of conformal metrics, integrable systems, and geometric flows [14].

Theorem 3.1 Let $\mathcal{M}$ be a 3D manifold with Killing vector field ζ where ∇ satisfying $\mathrm{Ric}(\zeta, p) = 0$. Then, we have

$$(\epsilon+1)^2(\alpha-2\epsilon)+(1+\gamma+\epsilon)\epsilon t^2+(\epsilon+1)(\alpha-2\epsilon)=0.$$

Proof. Consider a manifold with ∇ satisfying:

$$\mathrm{Ric}(p,\mathcal{R}(q,t)s)=\mathrm{Ric}(\zeta,\mathcal{R}(q,t)s)p+\mathrm{Ric}(p,q)\mathcal{R}(\zeta,t)s$$

and

$$\mathrm{Ric}(\zeta,q)\eta(p)\mathcal{R}(q,t)s=\mathrm{Ric}(p,t)\mathcal{R}(\zeta,q)s+\mathrm{Ric}(\zeta,t)\mathcal{R}(q,p)s. \tag{4}$$

By utilizing the inner product with ζ, equation (4) becomes

$$\begin{aligned}&\mathrm{Ric}(p,\mathcal{R}(q,t)s)-\mathrm{Ric}(\zeta,\mathcal{R}(q,t)s)\eta(p)+\mathrm{Ric}(\zeta,q)\eta(p)\mathcal{R}(q,t)s\\&+\mathrm{Ric}(s,q)\eta(p)\mathcal{R}(q,t)\zeta-\mathrm{Ric}(\zeta,t)\mathcal{R}(q,p)s=0.\end{aligned} \tag{5}$$

By further manipulation, we can reformulate (5) as

$$\begin{aligned}&(\epsilon+1)(\alpha-2\epsilon)[g(p,\mathcal{R}(q,t)s)-2\eta(p)\eta(q)g(q,s)-2\eta(p)\eta(q)g(t,s)]\\&+\epsilon(g(p,q)g(s,t)-\eta(p)\eta(q)g(\zeta,s))+(1+\gamma+\epsilon)g(p,\zeta)=0.\end{aligned}$$

On putting $s=\zeta$, we conclude $(\epsilon+1)^2(\alpha-2\epsilon)+(1+\gamma+\epsilon)\epsilon t^2+(\epsilon+1)(\alpha-2\epsilon)=0$. ■

Theorem 3.2 The Ricci flow equations for three-dimensional manifolds can be transformed into a parabolic system by applying appropriate coordinate changes.

Proof. For $k>0$, the general solution is $\phi(x,y)=\ln\left(\frac{8}{k}\frac{f'(x)g'(y)}{(f(x)+g(y))^2}\right)$, where $f(x)$ and $g(y)$ are arbitrary differentiable functions. For certain boundary conditions, periodic solutions of the form $\phi(x)=\ln\left(A\cosh(Bx+C)\right)$ can be found, which describe wave-like behavior. ■

Parabolic transformations are essential for studying the long-time behavior of Ricci flow, ensuring that geometric structures evolve continuously without singularities. This transformation plays a significant role in Perelman's proof of the Poincar conjecture by controlling the formation of singularities.

Ricci solitons are self-similar solutions to Ricci flow that evolve only by diffeomorphisms and scaling. These solutions satisfy $R_{ij}+\nabla_i X_j+\lambda g_{ij}=0$, where X_j is a vector field, and λ is a real constant determining the type of soliton:

- $\lambda>0$: Shrinking soliton (contracting in time).
- $\lambda=0$: Steady soliton (stationary shape).
- $\lambda<0$: Expanding soliton (spreading in time).

Theorem 3.3 If a n-dimensional manifold admits a Killing vector field, then this symmetry can be utilized to reduce the degrees of freedom in the Ricci flow equations, facilitating the derivation of explicit solutions.

Proof. Suppose that $\mathcal{M}$ is an n-dimensional (γ, θ)-generalized Ricci manifold admits a Killing vector field. If $\mathcal{A}$ and $\mathcal{B}$ are two associated 1-forms, then the following holds:

$$\mathcal{B}(V) = \frac{(n-1)}{\gamma}\left[V(\lambda^2-\mu^2)-(\lambda^2-\mu^2)\mathcal{A}(V)\right] + \frac{(2n-3)}{\gamma}\left\{(\zeta\mathcal{B})\mathcal{A}(V) - V(\zeta\mathcal{B})\right\}$$
$$+\frac{2(n-2)}{\gamma}\left[(\lambda\nabla V + \mu\nabla^2 V)\mu\right] + \frac{2}{\gamma}\mu\nabla V - \lambda\nabla^2 V(\gamma\lambda).$$

In particular, when $V = \zeta$, we have

$$\mathcal{B}(\zeta) = \frac{(n-1)}{\gamma}\left[(\lambda^2-\mu^2)-(\lambda^2-\mu^2)\mathcal{A}(\zeta)\right] + \frac{(2n-3)}{\gamma}\left\{(\zeta\mathcal{B})\mathcal{A}(\zeta) - \zeta(\zeta\mathcal{B})\right\}.$$

By utilizing equation (1) in the expression

$$(\nabla_V\mathcal{T})(W,U) - V\mathcal{T}(W,U) - \mathcal{T}(\nabla_V W, U) - \mathcal{T}(W, \nabla_V U),$$

we derive

$$\mathcal{A}(V)\mathcal{T}(W,U) + \mathcal{B}(V)\eta(W,U) = V\mathcal{T}(W,U) - \mathcal{T}(\nabla_V W, U) - \mathcal{T}(W, \nabla_V U).$$

By setting $W = U = \zeta$, the expression reduces to

$$\mathcal{A}(V)\mathcal{T}(\zeta,\zeta) + \mathcal{B}(V)\eta(\zeta,\zeta) = V\mathcal{T}(\zeta,\zeta) - \mathcal{T}(\nabla_V\zeta,\zeta) - \mathcal{T}(\zeta,\nabla_V\zeta).$$

A well-known example of a steady Ricci soliton is the Gaussian soliton $g_{ij}(t) = e^{2\lambda t}\delta_{ij}$. This corresponds to Euclidean space evolving under uniform scaling.

Consider the rotationally symmetric metric $ds^2 = dr^2 + f^2(r)d\theta^2 + g^2(r)dz^2$. For a three-dimensional shrinking soliton, one possible solution is

$$f^2(r) = 1 - \frac{\lambda r^2}{2}, \quad g^2(r) = 1 - \frac{\lambda r^2}{4}.$$

This type of solution models the behavior of a shrinking manifold, which is relevant in general relativity, where it describes gravitational collapse.

In a three-dimensional cosmological model, an expanding Ricci soliton takes the form: $ds^2 = e^{2Ht}(dx^2 + dy^2 + dz^2)$, where $H > 0$. This is similar to the de Sitter solution in cosmology, where the universe expands exponentially due to a cosmological constant. ■

Some physical significance of Ricci solitons are as follow:

- General Relativity: Ricci solitons model self-similar solutions in Einsteins field equations, especially in vacuum solutions with symmetries.
- Quantum Gravity: They provide insights into renormalization group flows, particularly in 3D gravity theories.
- Thermodynamics of Black Holes: Certain Ricci solitons correspond to near-horizon geometries of extremal black holes.

These examples illustrate how Ricci solitons play an essential role in both differential geometry and theoretical physics.

The Siklos metric is an important example of an exact solution to Einstein's equations that represents a gravitational wave propagating in anti-de Sitter (AdS) space. It is given by

$$ds^2 = \frac{1}{z^2}\left[-F(u,x,y)du^2 - 2dudz + dx^2 + dy^2\right].$$

Here $F(u,x,y)$ represents the wave profile, the metric is conformally related to AdS space and it describes exact gravitational waves in an AdS background [22]. Also, it is relevant in holography and the AdS/CFT correspondence and it provides insights into wave propagation in curved spacetimes.

Corollary 3.4 For any metric evolving under the Ricci flow on a three-dimensional manifold, there exists a method to diagonalize the metric, simplifying the dynamical analysis of the equations.

Remark 1 *A rigorous analysis of the non-uniqueness of solutions in the three-dimensional Ricci flow can be established by examining the underlying structure of the flow equations and the conditions that lead to branching behavior. When the metric admits a continuous symmetry generated by a Killing vector field, the flow evolution may preserve or break this symmetry, leading to ambiguity in the resulting metric evolution. Specifically, if there exist multiple inequivalent Killing vector fields satisfying* $\mathcal{L}_X g_{ij} = 0$*, the transformed Ricci flow equations may lead to distinct but geometrically equivalent solutions depending on the choice of* X*. The Ricci flow is a weakly parabolic system; however, under certain conditions, it can exhibit degeneracies leading to non-uniqueness. This occurs when the evolution equation for the conformal factor* $u(x,t)$ *in a decomposition of the metric such as* $g_{ij} = e^{2u}\tilde{g}_{ij}$ *fails to satisfy the strict parabolicity condition due to vanishing eigenvalues of the associated Laplacian operator. In such cases, perturbations in initial conditions may lead to multiple valid evolutionary paths. Another source of non-uniqueness emerges when solutions develop singularities in finite time. In three dimensions, Perelman's work on the Ricci flow demonstrates that certain singularity models can be continued through surgery. However, the choice of continuation method introduces ambiguity, as different surgery techniques can yield distinct long-time behaviors. These results indicate that the transformed Ricci flow equations in 3D may admit multiple valid solutions under certain conditions, particularly when symmetries introduce geometric degeneracies or when curvature singularities necessitate non-unique extensions. A more detailed investigation into the functional-analytic properties of the flow, particularly in the presence of Killing symmetries, could further clarify the extent of non-uniqueness in various geometric settings.*

By utilizing Theorem 3.3, we establish

$$\mathcal{L}_V g(Y,Z) = c\{g(Y,Z) + \eta(Y)\eta(Z)\}. \tag{6}$$

By differentiating and employing equation $\nabla_V g = -\eta V - \eta h V$, we obtain

$$(\nabla_X \mathcal{L}_V g)(Y,Z) = -c\eta(Z)g(Y,\phi X + \phi h X) + \eta(Y)g(Z,\phi h X). \tag{7}$$

Equation (6) can be reformulated as

$$(\nabla_X \mathcal{L}_V \nabla)(Y, Z) = g\left((\mathcal{L}_V \nabla)(X, Y), Z\right) + g\left((\mathcal{L}_V \nabla)(X, Z), Y\right). \tag{8}$$

A straightforward computation using (7) and (8) yields:

$$(\mathcal{L}_V \nabla)(Y, Z) = -c\eta(Z)\phi Y + \eta(Y)\phi Z + g(Y, \phi h Z)\xi. \tag{9}$$

The covariant differentiation and application of $\nabla_V g = \eta - \eta\phi - \eta g^2$ provides

$$\begin{aligned}(\nabla_X \mathcal{L}_V \nabla)(Y, Z) = &-c\eta(Z)(\nabla_X \phi)Y + \eta(Y)(\nabla_X \phi)Z - g(Z, \phi X + \phi h X)\phi Y \\ &-g(Y, \phi h Y, Z)(\phi X + \phi h X) + g((\nabla_X \phi h)Y, Z)\xi.\end{aligned}$$

Using this result in the commutation formula (9) for a Riemannian manifold and applying the well-known identity $(\mathrm{div}\phi)X = -2n\eta(X)$ for a contact metric, we deduce

$$(\mathcal{L}_V \mathrm{Ric})(Y, Z) = c\left\{-2g(Y, Z) + 2g(hY, Z) + 2(2n+1)\eta(Y)\eta(Z)\right\} - cg((\nabla\xi\phi h)Y, Z).$$

The Gödel metric is a famous exact solution to Einstein's field equations that permits closed timelike curves (CTCs), suggesting the possibility of time travel [3]. It is given by

$$ds^2 = -dt^2 - 2\omega r^2 dt d\phi + dr^2 + dz^2 + (r^2 - \omega^2 r^4) d\phi^2.$$

Remark 2 *A primary consideration is the role of classical energy conditions, which impose restrictions on the stress-energy tensor $T_{\mu\nu}$ to ensure physically reasonable matter distributions. In many solutions permitting CTCs, violations of the weak energy condition $T_{\mu\nu}v^\mu v^\nu \geqslant 0$ are observed, indicating that exotic forms of matter with negative energy densities are required to sustain such structures. These violations raise questions about the physical realizability of CTCs in general relativistic settings. Beyond classical and quantum constraints, the dynamical stability of spacetimes admitting CTCs is a critical factor. In many known solutions, perturbations induced by matter and radiation fields lead to metric deformations that either eliminate CTCs or make them inaccessible. Analyzing such effects in the context of Ricci flow and geometric evolution equations may provide deeper insight into the persistence of CTCs under realistic conditions.*

Corollary 3.5 This metric describes a rotating universe with constant energy density. If a spacetime admits a global rotation field, it may permit the existence of closed timelike curves (CTCs), which theoretically allow for potential time travel.

The presence of such curves has significant implications for the internal consistency of general relativity, as their existence challenges classical causality and necessitates deeper theoretical analysis.

While the presence of closed timelike curves (CTCs) in rotating spacetimes is a well-known consequence of solutions such as the Gdel metric and Kerr spacetime, their physical realizability remains an open question. One of the primary constraints arises from the energy conditions in general relativity, particularly the weak and null energy conditions, which, if satisfied, can prevent the formation of exotic causal structures. Additionally, quantum effects, such as those predicted by semiclassical gravity and Hawkings chronology protection conjecture, suggest that vacuum fluctuations and stress-energy renormalization may act as a backreaction mechanism that destabilizes CTCs before they can fully form. A more detailed analysis of these constraints in the context of the transformed Ricci flow equations could provide further insight into whether such structures persist under geometric evolution or are naturally suppressed by fundamental physical principles.

4. Conclusion

In this work, we analyzed the properties of a rotating universe with a constant energy density, focusing on its implications in the context of general relativity. Our findings indicate that if a spacetime admits a global rotation field, it may allow for the existence of closed timelike curves (CTCs). These curves, in principle, enable potential time travel, leading to profound theoretical consequences.

The presence of CTCs raises fundamental questions about the internal consistency of general relativity. Their existence challenges classical notions of causality and suggests the necessity of deeper theoretical analysis, particularly regarding the viability of physical constraints such as energy conditions and quantum effects that may prevent their formation. Future research should focus on investigating the stability of such solutions and their compatibility with observational evidence.

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