Journal of Linear and Topological Algebra
Vol. 13*, No.* 04*,* 2024*,* 241- 249
DOR:
DOI: 10.71483/JLTA.2024.1201923

Coincidence point theorems in quasi-ordered $\mathcal{F}$-metric spaces and its application

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Received 12 December 2024; Revised 13 March 2025; Accepted 15 March 2025.

Communicated by Hamidreza Rahimi

Abstract. The main goal of this article is to demonstrate the existence of a (couple) coincidence point for an infinite family of mappings in quasi-ordered $\mathcal{F}$-metric spaces. Some consequences are also added, along with an example and an application, to show the efficiency of the obtained results.

Keywords: quasi-ordered $\mathcal{F}$-metric spaces, coincidence point, coupled coincidence point, mixed g-monotone property, integral equation.

2010 AMS Subject Classification: 47B60, 54H25

1. Introduction and preliminaries

In the last century, nonlinear functional analysis has been experienced many advances. One of them is the introduction of various metric spaces and the existence of fixed points in such spaces along with their applications in other sciences. Among metric spaces, a new space named $\mathcal{F}$-metric space has been introduced by Jleli and Samet [10]. After that, other researchers defined various cases of these spaces and proved some fixed point theorems therein (for example, see [2–4, 6]). On the other hand, in 2004, Ran and Reurings [13] considered a partial order in metric spaces and proved some fixed point theorems for the comparable elements of metric spaces. Continuing their work, in 2006, Bhaskar and Lakshmikantham [5] defined the concept of a coupled fixed point and presented some theorems for mixed monotone mappings in partial order metric spaces. They also investigated the existence and uniqueness of a solution of a periodic boundary value problem.

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Print ISSN: 2252-0201
Online ISSN: 2345-5934
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After that, other researchers extended Bhaskar and and Lakshmikantham's results in other spaces and for many contractions (see [1, 8, 9, 11, 14, 15] and references therein).

In the present article, we prove the existence of a (couple) coincidence point for an infinite family of mappings in quasi-ordered $\mathcal{F}$-metric spaces. Some consequences are also added, along with an examples and an application. To do this, the following definitions and notations will be needed.

Definition 1.1 [2, 10] A function $f : (0, \infty) \to \mathbb{R}$ is called nondecreasing if, for all $s_1, s_2 \in (0, \infty)$, $0 < s_1 < s_2$ implies $f(s_1) \leqslant f(s_2)$. Also, f is said to be logarithmic-like when for each positive sequence $\{t_n\}$, we have $\lim_{n\to\infty} t_n = 0$ iff $\lim_{n\to\infty} f(t_n) = -\infty$.

In the sequel, assume $\mathcal{F}$ is set of all nondecreasing and logarithmic-like function.

Definition 1.2 [10] Presume $B \in [0, \infty)$ and the mapping $f \in \mathcal{F}$ is constant. Then $\omega : X \times X \to [0, \infty)$ is called a $\mathcal{F}$-metric on X if, for all $x, y \in X$, it satisfies the following conditions:

$(D1)$ $\omega(x, y) = 0 \Leftrightarrow x = y$;
$(D2)$ $\omega(x, y) = \omega(y, x)$;
$(D3)$ for each integer $N \geqslant 2$, we have

$$\omega(x, y) > 0 \Rightarrow f(\omega(x, y)) \leqslant f(\sum_{i=1}^{N-1} \omega(v_i, v_{i+1})) + B$$

for all $(v_i)_{i=1}^{N} \subset X$ with $(v_1, v_N) = (x, y)$.

In this case, (X, ω) is called a $\mathcal{F}$-metric space.

Definition 1.3 [2] Assume that (X, ω) is a $\mathcal{F}$-metric space. A sequence $\{x_n\}$ in X is said to be convergent to a $x \in X$ when $\lim_{n,\to\infty} \omega(x_n, x) = 0$. Also, it is called a Cauchy sequence if $\lim_{n,m\to\infty} \omega(x_n, x_m) = 0$.

A $\mathcal{F}$-metric space is complete when each Cauchy sequence is convergent to a $x \in X$.

Definition 1.4 [11] Suppose that $(X, \preceq)$ is an ordered set and $T : X \to X$ and $g : X \to X$ are two given mappings. Then we say T has nondecreasing g-monotone property when, for $x_1, x_2 \in X$, we have

$$gx_1 \preceq gx_2 \Rightarrow Tx_1 \preceq Tx_2.$$

Also, if $F : X \times X \to X$ and $g : X \to X$ are two arbitrary mappings, then F is said to be has mixed g-monotone property when, for each $x, y \in X$, we have

$$x_1, x_2 \in X, gx_1 \preceq gx_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),$$
$$y_1, y_2 \in X, gy_1 \preceq gy_2 \Rightarrow F(x, y_2) \preceq F(x, y_1).$$

Note that a quasi-order is a binary relation "$\preceq$" on a set X that is reflexive and transitive. A partial order is an antisymmetric quasi-order. We say $(X, \preceq)$ is an ordered set, when "$\preceq$" is a partial order on X.

Definition 1.5 [11] $(x, y) \in X \times X$ is said to be a coupled coincidence point of two mappings $F : X \times X \to X$ and $g : X \to X$ whenever $F(x, y) = gx$ and $F(y, x) = gy$.

Note that if g is an identity mapping, then $F(x, y) = x$ and $F(y, x) = y$, and in this case, (x, y) is called a coupled fixed point for the mapping F.

Definition 1.6 [11] Assume that X is a nonempty set, and $F : X \times X \to X$ and $g : X \to X$ are two given mappings. Then F and g are said to be commutative if, for all $x, y \in X$, we have $g(F(x, y)) = F(gx, gy)$. Also, $T : X \to X$ and $g : X \to X$ are commutative if, for each $x \in X$, we have $g(Tx) = T(gx)$.

In the sequel, take Ψ as a family of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(1) $\psi(t) \leqslant t$;
(2) $\lim\limits_{n\to\infty} \psi^n(t) = 0$.

Also, take Φ as a familty of all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(1) ϕ is lower semi-continuous;
(2) $\phi(t) = 0$ iff $t = 0$.

2. Main results

The following theorem is the first main result.

Theorem 2.1 Presume $(X, \omega, \preceq)$ is a complete quasi-ordered $\mathcal{F}$-metric space, and $\{T_\alpha : X \to X : \alpha \in \Lambda\}$ and $g : X \to X$ two mappings such that for a $\alpha_0 \in \Lambda$, T_{α_0} has a nondecreasing g-monotone property and commutes with g on X. Also, assume that there exists $x_0 \in X$ such that $gx_0 \preceq T_{\alpha_0}x_0$, and $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$\omega(T_{\alpha_0}x, T_\alpha u) \leqslant \psi(M(x, u)) - \phi(M(x, u)), \tag{1}$$

where

$$M(x, u) = \max\{\omega(gx, gu), \omega(gx, T_\alpha x), \omega(gu, T_{\alpha_0}u)\}$$

for each $\alpha \in \Lambda$ and $x, u \in X$ with $gx \preceq gu$. Moreover, suppose that $T_{\alpha_0}(X) \subset g(X)$, and g and T_{α_0} are two continuous mappings. Then $\{T_\alpha : \alpha \in \Lambda\}$ and g have a coincidence point in X.

Proof. By hypothesis, there exists $x_0 \in X$ such that $gx_0 \preceq T_{\alpha_0}x_0$. Since $T_{\alpha_0}(X) \subset g(X)$, there is $x_1 \in X$ so that $gx_1 = T_{\alpha_0}x_0$. Hence, $gx_0 \preceq T_{\alpha_0}x_0 = gx_1$. Similarly, there exists $x_2 \in X$ such that $gx_2 = T_{\alpha_0}x_1$. Since T_{α_0} has nondecreasing g-monotone property, we have $gx_1 = T_{\alpha_0}x_0 \preceq T_{\alpha_0}x_1 = gx_2$. Continuing this process, we can construct $\{x_n\}$ in X with $gx_{n+1} = T_{\alpha_0}x_n$ for $n = 0, 1, 2, 3, \cdots$ such that

$$gx_0 \preceq gx_1 \preceq gx_2 \cdots \preceq gx_n \preceq gx_{n+1} \preceq \cdots \tag{2}$$

Using (1) and (2), we have from $(x, u) = (x_n, x_{n+1})$ that

$$\omega(T_{\alpha_0}x_n, T_{\alpha_0}x_{n+1}) = \omega(gx_{n+1}, gx_{n+2}) \leqslant \psi(M(x_n, x_{n+1})) - \phi(M(x_n, x_{n+1})) \tag{3}$$

in which

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{\omega(gx_n, gx_{n+1}), \omega(gx_n, T_{\alpha_0}x_n), \omega(gx_{n+1}, T_{\alpha_0}x_{n+1})\} \\ &= \max\{\omega(gx_n, gx_{n+1}), \omega(gx_{n+1}, gx_{n+2})\}. \end{aligned}$$

It follows from (3) that

$$\begin{aligned} \omega(gx_{n+1}, gx_{n+2}) \leqslant \psi(&\max\{\omega(gx_n, gx_{n+1}), \omega(gx_{n+1}, gx_{n+2})\}) \\ &-\phi(\max\{\omega(gx_n, gx_{n+1}), \omega(gx_{n+1}, gx_{n+2})\}). \end{aligned} \tag{4}$$

Now, we have two cases:
Case 1. if $\max\{\omega(gx_n, gx_{n+1}), \omega(gx_{n+1}, gx_{n+2})\} = \omega(gx_{n+1}, gx_{n+2})$, then we have by (4) that

$$\omega(gx_{n+1}, gx_{n+2}) \leqslant \psi(\omega(gx_{n+1}, gx_{n+2})) - \phi(\omega(gx_{n+1}, gx_{n+2})) < \psi(\omega(gx_{n+1}, gx_{n+2})), \tag{5}$$

which is a contradiction as $\psi(t) \leqslant t$.
Case 2. if $\max\{\omega(gx_n, gx_{n+1}), \omega(gx_{n+1}, gx_{n+2})\} = \omega(gx_n, gx_{n+1})$, then we have by (4) that

$$\omega(gx_{n+1}, gx_{n+2}) \leqslant \psi(\omega(gx_n, gx_{n+1})) - \phi(\omega(gx_n, gx_{n+1})) < \psi(\omega(gx_n, gx_{n+1})). \tag{6}$$

By induction, we have $\omega(gx_{n+1}, gx_{n+2}) < \psi^n(\omega(gx_0, gx_1))$ for each $n \in \mathbb{N}$. Thus,

$$\sum_{i=n}^{m-1} \omega(gx_i, gx_{i+2}) < \sum_{i=n}^{m-1} \psi^n(\omega(gx_0, gx_1)).$$

Suppose that there exists $(f, B) \in \mathcal{F} \times [0, +\infty)$ so that $(D3)$ is held. Thus, for given $\epsilon > 0$ there exists $\gamma > 0$

$$0 < t < \gamma \Rightarrow f(t) < f(\epsilon) - B. \tag{7}$$

On the other hand, since $\lim_{n\to\infty} \psi^n(t) = 0$, we have

$$\lim_{n\to\infty} \psi^n(\omega(gx_0, gx_1)) = 0.$$

Hence, there exists a $N \in \mathbb{N}$ such that $0 < \psi^n(\omega(gx_0, gx_1)) < \gamma$ for $n \geqslant N$. Now, using (7) and $(D3)$, we have for $m > n \geqslant N$ that

$$f(\sum_{i=n}^{m-1} \omega(gx_i, gx_{i+2})) < f(\sum_{i=n}^{m-1} \psi^n(\omega(gx_0, gx_1))) < f(\epsilon) - B. \tag{8}$$

Now, using $(D3)$ and (8), we obtain

$$\omega(gx_n, gx_m) > 0 \Rightarrow f(\omega(gx_n, gx_m)) \leqslant f(\sum_{i=n}^{m-1} \omega(gx_i, gx_{i+2})) + B < f(\epsilon),$$

which implies that $\omega(gx_n, gx_m) < \epsilon$. Thus, $\{gx_n\}$ is a Cauchy sequence. As X is complete, there is $x \in X$ such that

$$\lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} T_{\alpha_0}x_n = x.$$

On the other hand, since g commutes with T_{α_0}, we conclude that $g(gx_{n+1}) = g(T_{\alpha_0}x_n) = T_{\alpha_0}(gx_n)$, which induces that

$$g(x) = g(\lim_{n\to\infty} gx_{n+1}) = g(\lim_{n\to\infty} T_{\alpha_0}x_n) = \lim_{n\to\infty} T_{\alpha_0}(gx_n).$$

Hence, $g(x) = T_{\alpha_0}x$. Now, we show that $gx = T_\alpha x$ for each $\alpha \in \Lambda$. To contrary, assume that it doesn't valid. Then $\alpha_1 \in \Lambda$ exists that $r = \omega(gx, T_{\alpha_1}x) > 0$. Now, using (1), we have

$$r = \omega(gx, T_{\alpha_1}x) \leqslant \psi(M(x,x)) - \phi(M(x,x)),$$

where

$$M(x,x) = \max\{\omega(gx, gx), \omega(gx, T_{\alpha_1}x), \omega(gx, gx)\}.$$

Then we have

$$r = \omega(gx, T_{\alpha_1}x) < \psi(M(x,x)) < \psi(\omega(gx, T_{\alpha_1}x)),$$

which is a contradiction as $\psi(t) \leqslant t$. This completes the proof. ∎

Following the idea of Soleimani Rad et al. [15], Ghasab et al. [6, 2020] proved a lemma showing the relation between fixed point and n-tuple fixed point in $\mathcal{F}$-metric spaces. Similarly, their lemma can be proved for quasi-ordered $\mathcal{F}$-metric spaces. In the sequel, we denote $X \times X \times \cdots \times X$ by X^n, where X is a nonempty set and $n \in \mathbb{N}$.

Lemma 2.2 Assume that $(X, \omega, \preceq)$ is a quasi-ordered $\mathcal{F}$-metric space. Then the followings are held:

(1) $(X^n, \Omega, \preceq)$ also is a quasi-ordered $\mathcal{F}$-metric space.

$$\Omega((u_1, \cdots, u_n), (v_1, \cdots, v_n)) = \max[\omega(u_1, v_1), \omega(u_2, v_2), \cdots, \omega(u_n, v_n)].$$

(2) $f : X^n \to X$ and $g : X \to X$ have n-tuple coincidence point iff $F : X^n \to X^n$ and $G : X^n \to X^n$ have a coincidence point in X^n.

$$F(u_1, u_2, \cdots, u_n) = (f(u_1, u_2, \cdots, u_n), f(u_2, \cdots, u_n, u_1), \cdots, f(u_n, u_1 \cdots, u_{n-1})),$$
$$G(u_1, u_2, \cdots, u_n) = (gu_1, gu_2, \cdots, gu_n).$$

(3) $(X, \omega, \preceq)$ is complete iff $(X^n, \Omega, \preceq)$ is complete.

Note that Lemma 2.2 is a two-sided lemma; that is, a n-tuple coincidence point can be obtained from a coincidence point and conversely. Using this lemma, we can obtain next main result from Theorem 2.1.

Theorem 2.3 Let $(X, \omega, \preceq)$ be a complete quasi-ordered $\mathcal{F}$-metric, and $\{T_\alpha : X \times X \to X : \alpha \in \Lambda\}$ and $g : X \to X$ be two mappings such that for a $\alpha_0 \in \Lambda$, T_{α_0} has mixed g-monotone and commutes with g on X. Also, suppose that there exist $x_0, y_0 \in X$ such that $gx_0 \preceq T_{\alpha_0}(x_0, y_0)$ and $gy_0 \succeq T_{\alpha_0}(y_0, x_0)$. Further, assume that there exist two mappings $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$\omega(T_{\alpha_0}(x,y), T_\alpha(u,v)) \leqslant \psi(M(x,y,u,v)) - \phi(M(x,y,u,v)),$$

where

$$\begin{aligned} M(x,y,u,v) = \max\{&\omega(gx,gu), \omega(gy,gv), \omega(gx, T_\alpha(x,y)), \omega(gu, T_{\alpha_0}(u,v)),\\ &\omega(gy, T_\alpha(y,x)), \omega(gv, T_{\alpha_0}(v,u))\} \end{aligned}$$

for each $x, y, u, v \in X, \alpha \in \Lambda$ with $gx \preceq gu$ and $gy \succeq gv$. Moreover, presume that $T_{\alpha_0}(X \times X) \subset g(X)$ and both mappings g and T_{α_0} are continuous. Then $\{T_\alpha : \alpha \in \Lambda\}$ and g possesses a coupled coincidence point in X.

Proof. First, we define $\Omega : X^2 \times X^2 \to [0, \infty)$ by

$$\Omega((x_1, x_2), (y_1, y_2)) = \max[\omega(x_1, y_1), \omega(x_2, y_2)]$$

for all $(x_1, x_2), (y_1, y_2) \in X^2$. Then, take the mappings $F_\alpha : X^2 \to X^2$ and $G : X^2 \to X^2$ by

$$F_\alpha(x,y) = (T_\alpha(x,y), T_\alpha(y,x)) \quad \text{and} \quad G(x,y) = (gx, gy).$$

Now, by Lemma 2.2, (X^2, Ω) is a complete $\mathcal{F}$-metric space. Moreover, $(x,y) \in X^2$ is a coupled coincidence point of T and g iff it is a coincidence point of F and G.

On the other hand, we have

$$\begin{aligned} \Omega(F_{\alpha_0}(x,y), F_\alpha(u,v)) &= \Omega((T_{\alpha_0}(x,y), T_{\alpha_0}(y,x)), (T_\alpha(u,v), T_\alpha(v,u)))\\ &= \max[\omega(T_{\alpha_0}(x,y), T_\alpha(u,v)), \omega(T_{\alpha_0}(y,x), T_\alpha(v,u))]\\ &= \omega(T_{\alpha_0}(x,y), T_\alpha(u,v))\\ &\leqslant \psi(M(x,y,u,v)) - \phi(M(x,y,u,v))\\ &\leqslant \psi(\max\{\omega(gx,gu), \omega(gy,gv), \omega(gx, T_\alpha(x,y)), \omega(gu, T_{\alpha_0}(u,v)),\\ &\quad \omega(gy, T_\alpha(y,x)), \omega(gv, T_{\alpha_0}(v,u))\})\\ &\quad - \phi(\max\{\omega(gx,gu), \omega(gy,gv), \omega(gx, T_\alpha(x,y)), \omega(gu, T_{\alpha_0}(u,v)),\\ &\quad \omega(gy, T_\alpha(y,x)), \omega(gv, T_{\alpha_0}(v,u))\})\\ &\leqslant \psi(\max\{\Omega(G(x,y), G(u,v)), \Omega(G(x,y), F_\alpha(x,y)),\\ &\quad \Omega(G(u,v), F_{\alpha_0}(u,v))\})\\ &\quad - \phi(\max\{\Omega(G(x,y), G(u,v)), \Omega(G(x,y), F_\alpha(x,y)),\\ &\quad \Omega(G(u,v), F_{\alpha_0}(u,v))\}) \end{aligned}$$

or

$$\begin{aligned}
\Omega(F_{\alpha_0}(x,y),F_\alpha(u,v)) &= \Omega((T_{\alpha_0}(x,y),T_{\alpha_0}(y,x)),(T_\alpha(u,v),T_\alpha(v,u)))\\
&= \max[\omega(T_{\alpha_0}(x,y),T_\alpha(u,v)),\omega(T_{\alpha_0}(y,x),T_\alpha(v,u))]\\
&= \omega(T_{\alpha_0}(y,x),T_\alpha(v,u))\\
&\leqslant \psi(M(x,y,u,v)) - \phi(M(x,y,u,v)) + L\theta(N(x,y,u,v))\\
&\leqslant \psi(\max\{\omega(gy,gv),\omega(gx,gu),\omega(gy,T_\alpha(y,x)),\omega(gv,T_{\alpha_0}(v,u)),\\
&\quad \omega(gx,T_\alpha(x,y)),\omega(gu,T_{\alpha_0}(u,v))\})\\
&\quad - \phi(\max\{\omega(gy,gv),\omega(gx,gu),\omega(gy,T_\alpha(y,x)),\omega(gv,T_{\alpha_0}(v,u)),\\
&\quad \omega(gx,T_\alpha(x,y)),\omega(gu,T_{\alpha_0}(u,v))\})\\
&\leqslant \psi(\max\{\Omega(G(y,x),G(v,u)),\Omega(G(y,x),F_\alpha(y,x)),\\
&\quad \Omega(G(v,u),F_{\alpha_0}(v,u))\})\\
&\quad - \phi(\max\{\Omega(G(y,x),G(v,u)),\Omega(G(y,x),F_\alpha(y,x)),\\
&\quad \Omega(G(v,u),F_{\alpha_0}(v,u))\}).
\end{aligned}$$

Consequently, by Theorem 2.1, F and G have a coincidence point and by Lemma 2.2, T and g have a coupled coincidence point. ■

***Example* 2.4** Consider the mapping $\omega : [0,1]\times[0,1] \to [0,\infty)$ by $\omega(x,y) = |x-y|$ for each $x,y \in X$. It is clear that $([0,1],\omega)$ is a complete $\mathcal{F}$-metric with $f(t) = \ln t$ and $B = 0$. Also, presume $\{T_\alpha : X \to X : \alpha \in \mathcal{N}\}$ and $g : X \to X$ are defined by $T_\alpha(x) = \alpha x$ and $g(x) = 2x$, respectively. Clearly, $0 = g0 \leqslant T_1(0) = 0$ for $0 \in [0,1]$. Moreover,

$$\begin{aligned}
\omega(T_1(x),T_\alpha(x^*)) = (|x-\alpha x^*|) &\leqslant (|x-x^*|)\\
&= \frac{1}{2}(|2x-2x^*|)\\
&= \frac{1}{2}\omega(gx,gx^*)\\
&\leqslant \frac{1}{2}(\max\{\omega(gx,gx^*),\omega(gx,T_\alpha x),\omega(gx^*,T_1gx^*)\}).
\end{aligned}$$

Thus, all the condition of Theorem 2.1 are held by setting $\phi(t) = 0$ and $\psi(t) = \frac{1}{2}$. Hence, T_α and g have a coincidence point in X.

3. Application

It should be noted that many applications related to the existence and uniqueness of solution of integral equations have been presented by many researcher as an application of coupled fixed point theory ([5–7, 11, 12]). Here, to support Theorem 2.1, we show the existence of a solution for the following integral equation:

$$x(t) = \int_0^t K(t,s,x(s))ds + v(t) \tag{9}$$

in which $t \in I = [0,1]$, $K \in C(I \times I \times \mathbb{R}, \mathbb{R})$ and $v \in C(I, \mathbb{R})$. Assume $C(I, \mathbb{R})$ is the space of all continuous real functions on I along with the norm $||x||_\infty = \max_{t\in I} |x(t)|$, where $x \in C(I, \mathbb{R})$, and $C(I \times I \times C(I, \mathbb{R}), \mathbb{R})$ is the space of all continuous real functions on $I \times I \times C(I, \mathbb{R})$. Also, take $C(I, \mathbb{R})$ with the norm $||x||_B = \sup_{t\in I}\{|x(t)|e^{-\tau t}\}$ for all $x \in C(I, \mathbb{R})$ and $\tau > 0$. Then, the metric induced by this norm on $C(I, \mathbb{R})$ is defined $\omega_B(x, y) = ||x - y||_B$ for all $x, y \in C(I, \mathbb{R})$. Now, consider the mapping $\omega : C(I, \mathbb{R}) \times C(I, \mathbb{R}) \to [0, \infty)$

$$\omega(x, y) = \sup_{t\in I}\{|x(t) - y(t)|e^{-\tau t}\}.$$

for each $x, y \in C(I, \mathbb{R})$. Further, define the quasi-order " $\leqslant$ " on $C(I, \mathbb{R})$ by

$$x \leqslant y \Leftrightarrow ||x||_\infty \leqslant ||y||_\infty$$

for all $x, y \in C(I, \mathbb{R})$. Then $(C(I, \mathbb{R}), \omega, \leqslant)$ is a complete quasi-ordered $\mathcal{F}$-metric space.

Theorem 3.1 Consider a complete quasi-ordered $\mathcal{F}$-metric space $(C(I, \mathbb{R}), \omega, \leqslant)$ with $f(t) = Ln(t)$. Presume $T : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ is an operator defined by

$$Tx(t) = \int_0^t K(t, s, x(s))ds + v(t), \quad v \in C(I, \mathbb{R})$$

and $gx = Ix = x$. Also, suppose $K \in C(I \times I \times \mathbb{R}, \mathbb{R})$ is an operator satisfying the following properties:

(i) K is continuous;
(ii) for all $t, s \in I$, $\int_0^t K(t, s, \cdot)$ is increasing;
(iii) there exists $\tau > 0$ so that

$$|K(t, s, x(s)) - K(t, s, y(s))| \leqslant \tau|x(s) - y(s)|$$

for all $x, y \in C(I, \mathbb{R})$ and $t, s \in I$.

Then integral equation (9) possesses a solution in $C(I, \mathbb{R})$.

Proof. Using (i)-(iii), we have by the definition of T that

$$\begin{aligned}
\omega(Tx, Ty) &= \sup_{t\in I}\{|\int_0^t K(t, s, x(s))ds - \int_0^t K(t, s, y(s))ds|\ e^{-\tau s}\} \\
&\leqslant \sup_{t\in I}\{\int_0^t |K(t, s, x(s)) - K(t, s, y(s))|e^{-\tau s}\ ds\} \\
&\leqslant \sup_{t\in I}\{\int_0^t \tau|x(s) - y(s)|e^{-\tau s}\ ds\} \\
&\leqslant (\tau||x - y||_B) \sup_{t\in I}\{\int_0^t e^{-\tau s}ds\} \\
&\leqslant (1 - e^{-\tau})(||x - y||_B).
\end{aligned}$$

Then all conditions of Theorem 2.1 are held by taking $f(t) = \ln(t)$ for each $t \in I$, $B = 0$,

$\phi(t) = 0$ and $\psi(t) = (1 - e^{-\tau})$. Thus, T and g have a coincidence point in $C(I, \mathbb{R})$ (and as a results, since $g(x) = x$, T has a fixed point) which implies that the integral equation (9) has a solution in $C(I, \mathbb{R})$. ∎

Acknowledgment

We are grateful to the Research Council of Shahid Chamran University of Ahvaz for financial support (Grant number: SCU.MM1403.25894). Also, the authors wish to thank the Editorial Board and referees for their helpful suggestions to improve this manuscript.

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