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A study on $Ivar(G)$ of a p-group fixing certain subgroups

S. Barin[a,*], M. M. Nasrabadi[a]
[a]*Department of Mathematics, University of Birjand, Birjand, Iran.*

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Abstract. Let G be a group, and M and N be two normal subgroups of G. In this paper, we first introduce a subgroup $\mathcal{E}(G)$ and consider the set of all automorphisms of G which centralize G/M and N. Then we investigate the conditions in which this set of automorphisms with different M and N is equal or to be equal with $Ivar(G)$.

Keywords: IA-group, IA-central subgroup, $Ivar(G)$, normal subgroups.

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1. Introduction and preliminaries

Let G be a group and p be a prime. Let us denote by C_n, G', $exp(G)$, $Ker(G)$, $Hom(G,H)$ and $Aut(G)$, respectively, the cyclic group of order n, the commutator subgroup, the exponent, the kernel, the group of homomorphisms of G into an abelian group H and the full automorphism group. Let $G^{p^n} = \langle g^{p^n} \mid g \in G \rangle$. The group G is called a purely non-abelian if it does not have an abelian direct factor.

Bachmuth [3] defined an IA-automorphism of a group G as

$$IA(G) = \left\{ \alpha \in Aut(G) \;\middle|\; g^{-1}\alpha(g) = [g, \alpha] \in G',\ \forall\, g \in G \right\}.$$

The group of IA-automorphisms is of great importance in the study of $Aut(G)$ and has been investigated by many authors (see, for instance, [1, 3, 6–8, 10]). Ghumde and Ghate

*Corresponding author.
E-mail address: s_barin10@yahoo.com (S. Barin); dr.mm.Nasrabadi@gmail.com (M. M. Nasrabadi).

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[8] introduced the $S(G)$ and $Ivar(G)$ subgroups as follows:

$$S(G) = \left\{g \in G \mid g^{-1}\alpha(g) = 1,\ \alpha \in IA(G)\right\},$$
$$Ivar(G) = \left\{\alpha \in IA(G) \mid g^{-1}\alpha(g) \in S(G),\ \forall g \in G\right\}.$$

They proved, for a finite polycyclic group G with derived series that has cyclic factors of prime order, the group $IA(G)$ is polycyclic. Ghumde [6] gave a similar result without using the condition of finiteness. Moreover, he obtained a sufficient condition for $IA(G)$ to be solvable. Let

$$C_{IA(G)}\big(Ivar(G)\big) = \{\alpha \in IA(G) \mid \sigma\alpha = \sigma\alpha,\ \forall\, \sigma \in Ivar(G)\}$$

be the centralizer of $Ivar(G)$ in $IA(G)$. We define $\mathcal{E}(G) = [G, C_{IA(G)}\big(Ivar(G)\big)]$. For any group G, $G' \leqslant \mathcal{E}(G) \overset{ch}{\leqslant} G$ and $Ivar(G)$ acts trivially on $\mathcal{E}(G)$. See [4] for more details on this concept.

Let $M, N \trianglelefteq G$. Azhdari [2] defined $Aut^M(G)$ as the subgroup of $Aut(G)$ consisting of all the automorphisms which centralize G/M and $Aut_N(G)$ as the subgroup of $Aut(G)$ consisting of all the automorphisms which centralize N. Also, $Aut^M_N(G) = Aut^M(G) \cap Aut_N(G)$.

2. Main results

Let G be a p-group. In this section, we consider $Aut^{\bigcirc}_{\bigcirc}(G)$ with various subgroups of G that are replaced by "$\bigcirc$". These subgroups are normal and relate to $S(G)$, $\mathcal{E}(G)$, or both of them. Then, we provide our results on the conditions in which two automorphisms like $Aut^{\bigcirc}_{\bigcirc}(G)$ with different subgroups are equal, especially to be equal with $Ivar(G)$. We first have some notations that characterize these subgroups.

Let G be a finite p-group and M, M_1 and M_2 be a central subgroups of G. Then

$$\begin{aligned}
M &= C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_m}}, && a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m > 0\\
M_1 &= C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_{m_1}}}, && a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{m_1} > 0\\
M_2 &= C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_{m_2}}}, && b_1 \geqslant b_2 \geqslant \cdots \geqslant b_{m_2} > 0\\
S(G) &= C_{p^{t_1}} \times C_{p^{t_2}} \times \cdots \times C_{p^{t_{s_1}}}, && t_1 \geqslant t_2 \geqslant \cdots \geqslant t_{s_1} > 0\\
S(G) \cap G' &= C_{p^{q_1}} \times C_{p^{q_2}} \times \cdots \times C_{p^{q_{s_2}}}, && q_1 \geqslant q_2 \geqslant \cdots \geqslant q_{s_2} > 0\\
\frac{G}{\mathcal{E}(G)} &= C_{p^{d_1}} \times C_{p^{d_2}} \times \cdots \times C_{p^{d_r}}, && d_1 \geqslant d_2 \geqslant \cdots \geqslant d_r > 0.
\end{aligned}$$

Also, Let k be the smallest integer between 1 and m_1 such that $a_i = b_i$ for all $k+1 \leqslant i \leqslant m_1$ and $k \neq m_1$.

Theorem 2.1 Let G be a finite p-group, $C_{IA(G)}\big(Ivar(G)\big) = Inn(G)$ and $M_1, M_2, N_1, N_2 \trianglelefteq G$ such that $M_1 \leqslant M_2$, $N_2 \leqslant N_1$ and $M_i \leqslant S(G) \cap N_i$, for $i = 1, 2$. Then $Aut^{M_1}_{N_1}(G) = Aut^{M_2}_{N_2}(G)$ if and only if one of the following conditions holds:

a) $M_1 = M_2$, $N_1 \leqslant \mathcal{E}(G)G^{p^l}N_2$ and $p^l = exp(M_1)$.
b) $N_1 = N_2$, $m_1 = m_2$ and $exp\big(G/\mathcal{E}(G)N_1\big) \leqslant p^{a_k}$.

Before proving the theorem, we need the following three lemmas.

Lemma 2.2 Let $C_{IA(G)}\big(Ivar(G)\big) = Inn(G)$, $M \leqslant S(G) \cap G'$, $exp(M) = p^l$ and $N \trianglelefteq G$. Then $Ker(f) = \mathcal{E}(G)G^{p^l}N/N$ for every $f \in Hom(G/N, M)$.

Proof. Obviously, $\mathcal{E}(G)N/N \leqslant Ker(f)$ and $G^{p^l}N/N \leqslant Ker(f)$ for every $f \in Hom(G/N, M)$. Thus, $\mathcal{E}(G)G^{p^l}N/N \leqslant Ker(f)$. To prove $Ker(f) \leqslant \mathcal{E}(G)G^{p^l}N/N$, assume by way of contradiction that $h \in Ker(f)$ and $h \notin \mathcal{E}(G)G^{p^l}N$. As $M \leqslant S(G)$, we have

$$Hom\Big(\frac{G}{N}, M\Big) \cong Hom\Big(\frac{G}{\mathcal{E}(G)N}, M\Big).$$

Put $H = G/\mathcal{E}(G)N$ and $h = g\mathcal{E}(G)N \in G/\mathcal{E}(G)N$. H is a finite abelian p-group. Therefore, by Lemma 2.2 [9], there exist non-trivial elements $h_1, h_2, \ldots, h_k \in G$ such that $H = \langle h_1\rangle \times \langle h_2\rangle \times \cdots \times \langle h_k\rangle$ and

$$g\mathcal{E}(G)N = h_1^{p^{m_1}} \times h_2^{p^{m_2}} \times \cdots \times h_k^{p^{m_k}}\mathcal{E}(G)N, \quad m_i \geqslant 0.$$

Since $h \notin \mathcal{E}(G)G^{p^l}N/\mathcal{E}(G)N$, it follows that $h_i^{p^{m_i}} \notin G^{p^l}$ for $1 \leqslant i \leqslant k$. Thus, $m_i < l$. Choose element $x \in M$ such that $|x| = min\{|h_i|, p^l\}$. We define a homomorphism $\varphi : \ H \longrightarrow M$ by $h_i \longmapsto x$ and put $K = Ker(\varphi)$. Because $M \leqslant S(G)$, if K is a direct factor of H, then each $\varphi \in Hom(K, M)$ induces an element $\varphi' \in Hom(H, M)$, which is trivial on the complement of K in H. To simplify the notion, we will consider f_x with the corresponding homomorphism from H to M, then $f_x(h) = f_x(h_i^{p^{m_i}}) = x^{p^{m_i}} \neq 1$. Consequently, $h \notin \cap_{f\in Hom(G/N,M)}Ker(f)$, a contradiction. Hence, $Ker(f) = \mathcal{E}(G)G^{p^l}N/N$. ■

Lemma 2.3 Let $C_{IA(G)}\big(Ivar(G)\big) = Inn(G)$ and $N_1, N_2 \trianglelefteq G$ such that $N_2 \leqslant N_1$ and $M \leqslant S(G) \cap N_i$ for $i = 1, 2$. Then $Aut^M_{N_1}(G) = Aut^M_{N_2}(G)$ if and only if $N_1 \leqslant \mathcal{E}(G)G^{p^l}N_2$, where $p^l = exp(M)$.

Proof. Let $Aut^M_{N_1}(G) = Aut^M_{N_2}(G)$. Then $\alpha(n_1) = n_1$ for all $n_1 \in N_1$ and $\alpha \in Aut^M_{N_2}(G)$. By Lemma 2.1 [2], $\alpha^* \in Hom(G/N_2, M)$ for all $\alpha \in Aut^M_{N_2}(G)$, where $\alpha^*(gN_2) = g^{-1}\alpha(g)$ for every $g \in G$. Hence, $\alpha^*(n_1N_2) = 1$ for any $n_1N_2 \in N_1N_2/N_2$. Thus, by Lemma 2.2,

$$\begin{aligned} n_1N_2 \in Ker(f) =& \frac{\mathcal{E}(G)G^{p^l}N_2}{N_2}, \qquad f \in Hom(G/N_2, M) \\ \Longrightarrow \quad \frac{N_1N_2}{N_2} \leqslant& \frac{\mathcal{E}(G)G^{p^l}N_2}{N_2} \\ \Longrightarrow \qquad N_1 \leqslant& \mathcal{E}(G)G^{p^l}N_2. \end{aligned}$$

Conversely, because $N_2 \leqslant N_1$, $Aut^M_{N_1}(G) \leqslant Aut^M_{N_2}(G)$. Let $N_1 \leqslant \mathcal{E}(G)G^{p^l}N_2$. Then, by Lemma 2.2, we have $\dfrac{N_1N_2}{N_2} \leqslant \dfrac{\mathcal{E}(G)G^{p^l}N_2}{N_2} = Ker(f)$ for all $f \in Hom(G/N_2, M)$. Again, since $N_2 \leqslant N_1$, we have

$$Hom\Big(\frac{G}{N_2}, M\Big) \cong Hom\Big(\frac{G}{N_1N_2}, M\Big) \cong Hom\Big(\frac{G}{N_1}, M\Big).$$

Whence

$$Aut^M_{N_1}(G) \cong Aut^M_{N_2}(G) \Longrightarrow \quad |Aut^M_{N_1}(G)| = |Aut^M_{N_2}(G)| \Longrightarrow \quad Aut^M_{N_1}(G) = Aut^M_{N_2}(G).$$

■

Lemma 2.4 Let $C_{IA(G)}\big(Ivar(G)\big) = Inn(G)$, $N \trianglelefteq G$ and $M_1 \leqslant M_2 \trianglelefteq S(G)N$. Then $Aut^{M_1}_N(G) = Aut^{M_2}_N(G)$ if and only if $m_1 = m_2$ and $exp\big(G/\mathcal{E}(G)N\big) \leqslant p^{a_k}$.

Proof. As $M_1 \leqslant M_2$, $Aut^{M_1}_N(G) \leqslant Aut^{M_2}_N(G)$. By Lemma 2.1 [2],

$$Aut^{M_1}_N(G) = Aut^{M_2}_N(G) \iff Hom(\frac{G}{N}, M_1) \cong Hom(\frac{G}{N}, M_2).$$

Let $m_1 = m_2$ and $exp\big(G/\mathcal{E}(G)N\big) \leqslant p^{a_k}$. Then

$$\Big|Hom\Big(\frac{G}{N}, M_1\Big)\Big| = \Big|Hom\Big(\frac{G}{N}, M_2\Big)\Big| \Longrightarrow \quad Aut^{M_1}_N(G) = Aut^{M_2}_N(G).$$

To prove the converse, let $Aut^{M_1}_N(G) = Aut^{M_2}_N(G)$. Then $m_1 = m_2$, and by Lemma 2.5 [11], we have $exp(G/\mathcal{E}(G)N) \leqslant p^{a_k}$. Otherwise if $exp(G/\mathcal{E}(G)N) \geqslant p^{a_k}$, then

$$\Big|Hom\Big(\frac{G}{N}, M_1\Big)\Big| < \Big|Hom\Big(\frac{G}{N}, M_2\Big)\Big|,$$

a contradiction. ■

Proof of Theorem 2.1.
Let $Aut^{M_1}_{N_1}(G) = Aut^{M_2}_{N_2}(G)$. Then, by Lemma [2, 2.1], $Hom\Big(\frac{G}{N_1}, M_1\Big) \cong Hom\Big(\frac{G}{N_2}, M_2\Big)$. If $M_1 < M_2$ and $N_2 < N_1$, we have by Lemma [5, D] $Hom\Big(\frac{G}{N_1}, M_1\Big) < Hom\Big(\frac{G}{N_2}, M_2\Big)$, a contradiction. Thus, $M_1 = M_2$ or $N_1 = N_2$.

If $M_1 = M_2$, then we have by Lemma 2.3 that $N_1 \leqslant \mathcal{E}(G)G^{p^l}N_2$ and it gives part (a). If $M_1 \neq M_2$ and $N_1 = N_2$, then $m_1 = m_2$ by Lemma 2.4, and $exp\big(G/\mathcal{E}(G)N_1\big) \leqslant p^{a_k}$, whence part (b) follows. To prove the converse, let part (a) or (b) holds. Then $Hom\Big(\frac{G}{N_1}, M_1\Big) \cong Hom\Big(\frac{G}{N_2}, M_2\Big)$. On the other hand, because $M_1 \leqslant M_2$ and $N_2 \leqslant N_1$. So, $Aut^{M_1}_{N_1}(G) \leqslant Aut^{M_2}_{N_2}(G)$. Therefore, $Aut^{M_1}_{N_1}(G) = Aut^{M_2}_{N_2}(G)$.

Remark 1 *In the proof of Theorem 2.1, we use the conditions $M_i \leqslant S(G) \cap N_i$ for $i = 1, 2$ only to prove the equality $|Aut^{M_i}_{N_i}(G)| = \Big|Hom\Big(\frac{G}{N_i}, M_i\Big)\Big|$. Hence, by Lemma 2.1[2], we may substitute this condition by "G be a purely non-abelian group" and we have the next Theorem.*

Theorem 2.5 Let G be a finite purely non-abelian p-group, $M_1, M_2 \leqslant S(G)$ and $N_1, N_2 \trianglelefteq G$ such that $M_1 \leqslant M_2$ and $N_2 \leqslant N_1$. Also, let $C_{IA(G)}\big(Ivar(G)\big) = Inn(G)$. Then $Aut^{M_1}_{N_1}(G) \leqslant Aut^{M_2}_{N_2}(G)$ if and only if one of the following conditions holds:

a) $M_1 = M_2$, $N_1 \leqslant \mathcal{E}(G)G^{p^l}N_2$ and $p^l = exp(M_1)$.
b) $N_1 = N_2$, $m_1 = m_2$ and $exp\big(G/\mathcal{E}(G)N_1\big) \leqslant p^{a_k}$.

Corollary 2.6 Let G be a finite p-group and $M, N \trianglelefteq G$ such that $M \leqslant S(G) \cap G' \leqslant N$ and $C_{IA(G)}\big(Ivar(G)\big) = Inn(G)$. Then

a) $Aut_N^M(G) = Aut_{S(G)}^{S(G)\cap G'}(G)$ if and only if one of the following conditions holds:
   1) $M = S(G) \cap G'$, $N \leqslant \mathcal{E}(G)G^{p^r}S(G)$ and $p^r = exp\big(S(G) \cap G'\big)$.
   2) $N = S(G)$, $m = s_2$ and $exp\big(G/\mathcal{E}(G)N\big) \leqslant p^{a_k}$.

b) $Aut_N^M(G) = Ivar(G)$ if $N \leqslant \mathcal{E}(G)$, $m = s_2$ and $exp\big(G/\mathcal{E}(G)\big) \leqslant p^{a_k}$.

Proof. Put $M_1 = M$, $N_1 = N$, $M_2 = S(G) \cap G'$ and $N_2 = S(G)$. Now, by Theorem 2.1, the results of part (a) are obtained.

b) By the assumptions, we have

$$Aut_N^M(G) = Hom\Big(\frac{G}{N}, M\Big) \cong Hom\Big(\frac{G}{\mathcal{E}(G)N}, S(G) \cap G'\Big) \cong Hom\Big(\frac{G}{\mathcal{E}(G)}, S(G) \cap G'\Big) = Ivar(G).$$

Thus, $|Aut_N^M(G)| = |Ivar(G)|$. Now, from $M \leqslant S(G) \cap G' \leqslant N\mathcal{E}(G)$, it follows that $Aut_N^M(G) \leqslant Aut_{\mathcal{E}(G)}^{S(G)\cap G'}(G) \leqslant Ivar(G)$. Therefore, $Aut_N^M(G) = Ivar(G)$. ■

Let G be a finite p-group, and M_1 and M_2 be two central subgroups of G. Then we can write

$$\begin{aligned} M_1 &= C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \cdots \times C_{p^{\alpha_{m_1}}}, & \alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_{m_1} > 0,\\ M_2 &= C_{p^{\beta_1}} \times C_{p^{\beta_2}} \times \cdots \times C_{p^{\beta_{m_2}}}, & \beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_{m_2} > 0,\\ M_1 \cap M_2 &= C_{p^{\gamma_1}} \times C_{p^{\gamma_2}} \times \cdots \times C_{p^{\gamma_{m_3}}}, & \gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_{m_3} > 0. \end{aligned}$$

Let l_1 be a smallest integer between 1 and m_3 such that $\alpha_i = \beta_i$ for all $l_1 + 1 \leqslant i \leqslant m_3$ and l_2 be a smallest integer between 1 and m_3 such that $\alpha_i = \gamma_i$ for all $l_2 + 1 \leqslant i \leqslant m_3$. Also, $exp(M_i) = p^{n_i}$ for $i = 1, 2$.

The following result shows that we can remove the two conditions $M_1 \leqslant M_2$ and $N_2 \leqslant N_1$ from Theorem 2.1.

Corollary 2.7 Let G be a finite p-group, $C_{IA(G)}\big(Ivar(G)\big) = Inn(G)$ and $M_1, M_2, N_1, N_2 \trianglelefteq G$ such that $M_i \leqslant S(G) \cap N_i$ for $i = 1, 2$. Then $Aut_{N_1}^{M_1}(G) = Aut_{N_2}^{M_2}(G)$ if and only if one of the following conditions holds:

a) $M_1 = M_2$ and $N_i \leqslant \mathcal{E}(G)G^{p^{n_j}}N_j$ for $i, j = 1, 2$ and $i \neq j$.
b) $M_1 \leqslant M_2$, $m_1 = m_2 = m_3$, $N_1 \leqslant N_2 \leqslant \mathcal{E}(G)G^{p^{n_1}}N_1$ and $exp\big(G/\mathcal{E}(G)N_2\big) \leqslant p^{\alpha_{l_2}}$.
c) $M_2 \leqslant M_1$, $m_1 = m_2 = m_3$, $N_2 \leqslant N_1 \leqslant \mathcal{E}(G)G^{p^{n_2}}N_2$ and $exp\big(G/\mathcal{E}(G)N_1\big) \leqslant p^{\alpha_{l_1}}$.
d) $N_1 = N_2$, $m_1 = m_2 = m_3$ and $exp\big(G/\mathcal{E}(G)N_1\big) \leqslant p^{\alpha_{l_i}}$ for $i = 1, 2$.

Proof. Let $Aut_{N_1}^{M_1}(G) = Aut_{N_2}^{M_2}(G)$. Because $M_1 \cap M_2 \leqslant M_i$ and $N_i \leqslant N_1N_2$ for $i = 1, 2$, $Aut_{N_i}^{M_i}(G) = Aut_{N_1N_2}^{M_1\cap M_2}(G)$ and by Theorem 2.1, one of the following cases happens:

Case 1) $M_i = M_1 \cap M_2$ and $N_1N_2 \leqslant \mathcal{E}(G)G^{p^{n_i}}N_i$. Thus, $M_i \leqslant M_j$ and $N_j \leqslant \mathcal{E}(G)G^{p^{n_i}}N_i$ for $i \neq j$.

Case 2) $N_i = N_1N_2$, $m_3 = m_i$ and $exp\big(G/\mathcal{E}(G)N_i\big) \leqslant p^{\alpha_{l_i}}$.

Therefore, $N_j \leqslant N_i$, $m_3 = m_i$ and $exp\big(G/\mathcal{E}(G)N_i\big) \leqslant p^{\alpha_{l_i}}$ for $i \neq j$. Hence, we have the following four states:

I) If **Case (1)** holds for $i = 1, 2$, then $M_1 = M_2$ and $N_i \leqslant \mathcal{E}(G)G^{p^{n_j}}$ for $i, j = 1, 2$ and $i \neq j$ whence part (a) follows.

II) If **Case (1)** for $i = 1$ and **Case (2)** for $i = 2$ happen, then $M_1 \leqslant M_2$ and $N_2 \leqslant \mathcal{E}(G)G^{p^{n_1}}N_1$. Consequently, $m_3 = m_2$, $N_1 \leqslant N_2$ and $exp\big(G/\mathcal{E}(G)N_2\big) \leqslant p^{\alpha_{l_2}}$. Therefore, $\mathcal{E}(G)N_2 \leqslant \mathcal{E}(G)G^{p^{n_1}}N_1$. Since $\alpha_{l_2} \leqslant n_1$, we have $G^{p^{n_1}} \leqslant \mathcal{E}(G)N_2$ and from $N_1 \leqslant N_2$, it follow that $\mathcal{E}(G)G^{p^{n_1}}N_1 \leqslant \mathcal{E}(G)N_1N_2 = \mathcal{E}(G)N_2$. Thus, $\mathcal{E}(G)N_2 = \mathcal{E}(G)G^{p^{n_1}}N_1$. Also, since $M_1 \leqslant M_2$, we have $m_3 = m_1 = m_2$, $N_1 \leqslant N_2 \leqslant \mathcal{E}(G)G^{p^{n_1}}N_1$ and $exp\big(G/\mathcal{E}(G)N_2\big) \leqslant p^{\alpha_{l_2}}$. Therefore, part (b) holds.

III) If **Case (2)** for $i = 1$ and **Case (1)** for $i = 2$ happen, then similar to the proof of (II), we can conclude that part (c) holds.

IV) If **Case (2)** holds for $i = 1, 2$, then $N_1 = N_2 = N_1N_2$, $m_1 = m_2 = m_3$, and for $i = 1, 2$, $exp\big(G/\mathcal{E}(G)N_1\big) \leqslant p^{\alpha_{l_i}}$, the same part (d).

To prove the converse, suppose that part (a) holds and put $M = M_1 = M_2$. Since $N_i \leqslant \mathcal{E}(G)G^{p^{n_j}}N_j$,

$$Aut_{N_j}^{M}(G) = Hom\Big(\frac{G}{N_j}, M\Big) \cong Hom\Big(\frac{G}{N_1N_2}, M\Big) = Aut_{N_1N_2}^{M}(G),$$

for $i, j = 1, 2$ and $i \neq j$ forcing $Aut_{N_1}^{M}(G) = Aut_{N_2}^{M}(G)$. If part (b) holds, then

$$Hom\Big(\frac{G}{N_2}, M_2\Big) \cong Hom\Big(\frac{G}{N_1N_2}, M_2\Big) \cong Hom\Big(\frac{G}{N_1N_2}, M_1 \cap M_2\Big)$$

$$\Longrightarrow \qquad Aut_{N_2}^{M_2}(G) = Aut_{N_1N_2}^{M_2}(G) = Aut_{N_1N_2}^{M_1\cap M_2}(G).$$

Also, $Aut_{N_1}^{M_1}(G) = Aut_{N_1N_2}^{M_1}(G) = Aut_{N_1N_2}^{M_1\cap M_2}(G)$. Hence, $Aut_{N_1}^{M_1}(G) = Aut_{N_2}^{M_2}(G)$. If part (c) holds, the above relation is proved similar. Lastly, assume part (d) holds and put $m = m_1 = m_2$ and $N = N_1 = N_2$. Then $Aut_{N}^{M_i}(G) = Aut_{N}^{M_1\cap M_2}(G)$ for $i = 1, 2$. Consequently, $Aut_{N_1}^{M_1}(G) = Aut_{N_2}^{M_2}(G)$. ■

Remark 2 *In the previous corollary, the condition "$M_i \leqslant S(G) \cap N_i, i = 1, 2$" can be replaced by the condition "G is a purely non-abelian group" and the following corollary concludes.*

Corollary 2.8 Let G be a finite purely non-abelian p-group and $M_1, M_2, N_1, N_2 \trianglelefteq G$ such that $M_i \leqslant S(G)$ for $i = 1, 2$ and $C_{IA(G)}\big(Ivar(G)\big) = Inn(G)$. Then $Aut_{N_1}^{M_1}(G) = Aut_{N_2}^{M_2}(G)$ if and only if one of the following conditions holds:

a) $M_1 = M_2$ and $N_i \leqslant \mathcal{E}(G)G^{p^{n_j}}N_j$ for $i, j = 1, 2$ and $i \neq j$.
b) $M_1 \leqslant M_2$, $m_1 = m_2 = m_3$, $N_1 \leqslant N_2 \leqslant \mathcal{E}(G)G^{p^{n_1}}N_1$ and $exp\big(G/\mathcal{E}(G)N_2\big) \leqslant p^{\alpha_{l_2}}$.
c) $M_2 \leqslant M_1$, $m_1 = m_2 = m_3$, $N_2 \leqslant N_1 \leqslant \mathcal{E}(G)G^{p^{n_2}}N_2$ and $exp\big(G/\mathcal{E}(G)N_1\big) \leqslant p^{\alpha_{l_1}}$.
d) $N_1 = N_2$, $m_1 = m_2 = m_3$ and $exp\big(G/\mathcal{E}(G)N_1\big) \leqslant p^{\alpha_{l_i}}$ for $i = 1, 2$.

The last equality is shown by the next theorem.

Theorem 2.9 Let G be a finite purely non-abelian p-group, $M, N_1, N_2 \trianglelefteq G$, $M \leqslant S(G)$ and $C_{IA(G)}\big(Ivar(G)\big) = Inn(G)$. If the invariants of M (in the cyclic decomposition) are greater than or equal to $exp\big(G/\mathcal{E}(G)N_i\big)$ for $i = 1, 2$, then $Aut_{N_1}^{M}(G) = Aut_{N_2}^{M}(G)$ if and only if $\mathcal{E}(G)N_1 = \mathcal{E}(G)N_2$.

Proof. Let $M = C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_m}}$ and $exp\Big(\frac{G}{\mathcal{E}(G)N_i}\Big) \leqslant p^{n_i}$ for $i = 1, 2$. First, suppose that $N_2 \leqslant N_1$. Since $a_j \geqslant n_i$ for $i = 1, 2$ and $1 \leqslant j \leqslant m$. So, for $i = 1, 2$,

$$\begin{aligned} Hom\Big(\frac{G}{N_i}, M\Big) &\cong Hom\Big(\frac{G}{N_i}, C_{p^{a_1}}\Big) \times \cdots \times Hom\Big(\frac{G}{N_i}, C_{p^{a_m}}\Big) \\ &\cong Hom\Big(\frac{G}{\mathcal{E}(G)N_i}, C_{p^{a_1}}\Big) \times \cdots \times Hom\Big(\frac{G}{\mathcal{E}(G)N_i}, C_{p^{a_m}}\Big) \\ &\cong \Big(\frac{G}{\mathcal{E}(G)N_i}\Big)^m. \end{aligned}$$

Thus,

$$Aut_{N_1}^M(G) = Aut_{N_2}^M(G) \iff \frac{G}{\mathcal{E}(G)N_1} = \frac{G}{\mathcal{E}(G)N_2} \iff \mathcal{E}(G)N_1 = \mathcal{E}(G)N_2.$$

The case $N_1 \leqslant N_2$ is proved by the same argument. ■

3. Conclusion

In this paper, we studied the relationship between automorphisms like $Aut_N^M(G)$ with different M and N and the relationship between $Ivar(G)$ and $Inn(G)$ in special cases in which for $\alpha \in IA(G)$, $\sigma\alpha = \sigma\alpha$ for all $\sigma \in Ivar(G)$. We concluded that writing any two automorphisms in form $Aut_N^M(G)$ and adding suitable conditions, we observe that they are equal. Although all the developments in this paper cover various automorphisms, it is worth exploring the results obtained here for special groups, which seems to be a promising research line. The first step in this direction would be to investigate the p-groups and nilpotent groups to reduce conditions.

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