Journal of Linear and Topological Algebra
Vol. 13*, No.* 04*,* 2024*,* 251- 259
DOR:
DOI: 10.71483/JLTA.2024.1198091

Soft pseudo-topological (Tallini) hypervector spaces

F. Farzanfar[a,*]

[a] *Department of Computer Engineering and Information Technology, Payame Noor University, P.O. Box 19395-3697 Tehran, Iran.*

Received 30 January 2025; Revised 15 March 2025; Accepted 19 March 2025.

Communicated by Hamidreza Rahimi

Abstract. In this paper, we study the pseudo-topological (Tallini) hypervector spaces over the topological field and introduce the notions of soft hypervector spaces, soft pseudo-topological hypervector spaces, and soft pseudo-topological subhyperspaces. Also, we show that the product of two soft pseudo-topological hypervector spaces is a soft pseudo-topological hypervector space.

Keywords: Soft set, soft hypervector space, pseudo-topological hypervector space, soft pseudo-topological hypervector space.

2010 AMS Subject Classification: 46A19, 47L07.

1. Introduction and preliminaries

The concept of hypergroup as a generalization of algebraic structures was first introduced in 1934 by Marty [9]. In classical algebraic structures, the synthetic result of two elements is an element, while in the hyper algebraic system, the synthetic result of two elements is a set of elements. Different hyperstructures such as hypergroups, hyperrings, and hyperfields have been widely studied by many mathematicians in both theoretical and applied fields.A hypervector space is a special kind of hyperstructure, which is a generalization of classical vector spaces. There are various kinds of hypervector spaces, but in this paper we consider the hypervector spaces in the sense of Tallini [15, 16].

In modeling with classical mathematics in economics, engineering, environment, sociology, medical science and other scientific fields, there are complications for uncertain data. Although the uncertainties appearing in these areas can be justified to some extent by the

*Corresponding author.
E-mail address: farzad.farzanfar@gmail.com, farzanfar@pnu.ac.ir (F. Farzanfar).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

theory of probability, fuzzy sets [19] and the theory of rough sets [13], but each of them has its own inherent problems as pointed out by Molodtsov [11]. Molodtsov proposed the soft set theory, as a powerful mathematical approach for modeling uncertainties, which has been studied algebraically and topologically by many mathematicians [4, 10, 14]. Yamak et al. [18], Wang et al. [17] and Gulay Oguz et al. [12] introduced the concepts of soft hyperstructure, soft polygroups, soft topological hyperstructures, respectively.

The purpose of this paper is to introduce the notion of soft pseudo-topological hypervector spaces by applying the soft set theory. First, we review some definitions and the results of (topological) hyperstructures and soft set theory that are used throughout the paper. Then, we introduce the notion of soft hypervector spaces, soft pseudo-topological hypervector spaces, soft pseudo-topological subhyperspaces and provide a few examples. Finally, we show that the Cartesian product of two soft pseudo-topological hypervector spaces is a soft pseudo-topological hypervector space. We review some definitions from [1–3, 5, 15, 16].

Let H be a nonempty set and $P^*(H)$ be the family of all nonempty subsets of H, every function $\circ_i : H \times H \longrightarrow P^*(H)$, where $i \in \{1, 2, \ldots, n\}$ and $n \in \mathbb{N}$, is called a hyperoperation and for all x, y of H, $\circ(x, y)$ is called the hyperproduct of x, y.

An algebraic system $(H, \circ_1, \circ_2, \ldots, \circ_n)$ is called a hyperstructure, with hyperoperations $\cdot_i, i = 1, ..., n$; and pair $(H, \circ)$ endowed with only one hyperoperation is called a hypergroupoid. For nonempty subsets A and B of H and $x \in H$:

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b, \quad A \circ x = \bigcup_{a\in A} a \circ x \quad \text{and} \quad x \circ B = \bigcup_{b\in B} x \circ b.$$

Recall that a hypergroupoied $(H, \circ)$ is called a semihypergroup if for any $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$ and semihypergroup $(H, \circ)$ is a hypergroup if for any $x \in H, x \circ H = H \circ x = H$.

Let K be a field and $(V, +)$ be an abelian group. A hypervector space over K (in the sense of Tallini) is a quadruple $(V, +, \circ, K)$, where "$\circ$" is a mapping $\circ : K \times V \to P^*(V)$ such that for all $a, b \in K$ and $x, y \in V$ the following conditions are satisfied:

(H_1) $a \circ (x + y) \subseteq a \circ x + a \circ y$,
(H_2) $(a + b) \circ x \subseteq a \circ x + b \circ x$,
(H_3) $a \circ (b \circ x) = (ab) \circ x$,
(H_4) $a \circ (-x) = (-a) \circ x = -(a \circ x)$,
(H_5) $x \in 1 \circ x$,

where for all $A, B \in P^*(V), A + B = \{a + b \mid a \in A, b \in B\}$. If in (H_1) the equality holds, the hypervector space is called strongly right distributive. If in (H_2) the equality holds, the hypervector space is called strongly left distributive.

We will call a hypervector space is a strongly distributive hypervector space, if it is a strongly left distributive hypervector space and strongly right distributive hypervector space.

***Example* 1.1** [15] In $(V = \mathbb{R}^2, +)$, for any $a \in \mathbb{R}$, $x \in \mathbb{R}^2$ we define the hyperoperation:

$$a \circ x = \begin{cases} \overline{ox} & \text{if } x \neq \underline{0}, \\ \{\underline{0}\} & \text{if } x = \underline{0}, \end{cases}$$

where $\overline{ox}$ is the line through the point x and $\underline{0} = (0, 0)$. Then $(V = \mathbb{R}^2, +, \circ, \mathbb{R})$ is a strongly left distributive hypervector space.

Definition 1.2 [2] Let $(V,+,\circ,K)$ be a hypervector space over K. A nonempty subset W of V is called a subhyperspace of V if, for all $x,y\in W$, $x-y\in W$, and for all $a\in K, x\in W$, $a\circ x\subseteq W$. In this case, we write $W\leqslant V$. If W be a subhyperspace of V then W is a subgroup of the group $(V,+)$.

A topological group is a group G which is also a topological space such that the multiplication map $(g,h)\to gh$ from $G\times G$ to G, and the inverse map $g\to g^{-1}$ from G to G, are both continuous. Similarly, we can define topological rings, topological fields, etc. A topological vector space is a vector space X over a topological field K (most often the real or complex fields with their standard topologies) that is endowed with a topology such that vector addition $+:X\times X\to X$ and scalar multiplication $.:K\times X\to X$ both are continuous functions with respect to the product topologies on $X\times X$, $K\times X$ and topology on X, respectively.

Definition 1.3 [1, 7] Let $(H,\circ)$ be a hypergroupoied and (H,τ) be a topological space. The hyperoperation "$\circ$" is called pseudo-continuous if

$$U_*=\{(x,y)\in H^2\mid x\cdot y\subseteq U\},\ \text{for any } U\in\tau$$

is open in $H\times H$. In terms of open sets, The hyperoperation "$\circ$" is pseudo-continuous (p-continuous) if and only if for any $U\in\tau$ and any pair $(x,y)\in H\times H$ such that $x\circ y\subseteq U$, there exist $V,W\in\tau$, $x\in V$ and $y\in W$ such that $v\circ w\subseteq U$ for any $v\in V$ and $w\in W$.

Definition 1.4 [1, 7] Let $(H,\circ)$ be a hypergroupoied and (H,τ) be a topological space. The triple $(H,\circ,\tau)$ is called pseudo-topological (p-topological) hypergroupoied if the mapping $\circ:H\times H\to P^*(H)$ is pseudo-continuous.

Theorem 1.5 [1, 7] Let (H,τ) be a topological space. Then the family $\mathcal{U}$ consisting of all $S_U=\{W\in P^*(H)|\ W\subseteq U,\ U\in\tau\}$ is a base for a topology on $P^*(H)$. This topology is denoted by $\tau_{P^*(H)}$.

Theorem 1.6 [1, 7] Let $(H,\circ)$ be a hypergroupoied and (H,τ) be a topological space. The triple $(H,\circ,\tau)$ is pseudo-topological hypergroupoied if and only if the mapping $\circ:H\times H\to P^*(H)$ is continuous with respect to topologies $\tau_{P^*(H)}$ and $\tau\times\tau$.

Let $(V,+,\circ,K)$ be a hypervector space over a topological field K and τ be a topology on V. In the following, we use the topology $\tau_{P^*(V)}$ on $P^*(V)$ and the product topology on $V\times V$.

Definition 1.7 [3] Let $(V,+,\circ,K)$ be a hypervector space over a topological field K and (V,τ) be a topological space. Then $(V,+,\circ,K,\tau)$ is called a pseudo-topological hypervector space, if the operations $+:V\times V\to V,(x,y)\mapsto x+y$, $i:V\to V,x\mapsto -x$ and the hyperoperation $\circ:K\times V\to P^*(V),(a,x)\mapsto a\circ x$ are continuous.

Definition 1.8 [11] Let U be an initial universe set and E be a set of parameters. If $A\subseteq E$ and $P(U)$ is the power set of U, then the pair $(\mathcal{F},A)$ is called a soft set over U, where $\mathcal{F}$ is the mapping $\mathcal{F}:A\to P(U)$. Clearly, a soft set over U can be regarded as a parametrized family of subsets of the universe U.

Definition 1.9 In [6], for a soft set $(\mathcal{F},A)$, the set $Supp(\mathcal{F},A)=\{x\in A:\mathcal{F}(x)\neq\emptyset\}$ is called the support of the soft set $(\mathcal{F},A)$. If $Supp(\mathcal{F},A)\neq\emptyset$, then $(\mathcal{F},A)$ is called a non-null soft set.

Definition 1.10 [8] Let $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ be two soft sets over the common universe U. Then, $(\mathcal{F}, A)$ is called a soft subset of $(\mathcal{G}, B)$ if

(1) $A \subseteq B$;
(2) $\mathcal{F}(a)$ and $\mathcal{G}(a)$ are identical approximations for all $a \in A$.

We denote it by $(\mathcal{F}, A) \subseteq (\mathcal{G}, B)$.

2. Main results

In this section, we study the pseudo-topological (p-topological) hypervector spaces over a topological field and introduce the concept of soft pseudo-topological (p-topological) hypervector spaces.

Definition 2.1 Let $(V, +, \cdot, K)$ be a hypervector space over a field $(K, +, \cdot)$ and (V, τ_V) and (K, τ_K) be topological spaces. The hyperoperation "$\circ$" is called pseudo-continuous (p-continuous) if, for any $U \in \tau_V$, $U_* = \{(k, v) \in K \times V \mid k \cdot v \subseteq U\} \in \tau_{K\times V}$.

Theorem 2.2 Let $(V, +, \circ, K)$ be a hypervector space, (V, τ_V) and (K, τ_K) be topological spaces. Then the hyperoperation "$\circ$" is a p-continuous map if and only if for every $U \in \tau_V$ and $v \in V, k \in K$ that $k \circ v \subseteq U$, there exist $W \in \tau_K, W' \in \tau_V$, such that $k \in W, v \in W'$ for all $w \in W$ and $w' \in W'$, we have $w \circ w' \subseteq U$.

Proof. Let a hyperoperation "$\circ$" be a $p-$continuous map and $U \in \tau_V$, then $U_* \in \tau_{K\times V}$. Hence, there exist $W \in \tau_V$ and $W' \in \tau_K$ such that for all $k \in K, v \in V$, we have $(k, v) \in W \times W' \subseteq U_*$. It follows for all $w \in W$ and $w' \in W'$ that $(w, w') \in U_*$. Hence, for all $w \in W$ and $w' \in W'$, we have $w \circ w' \subseteq U$.

Conversely, suppose $U \in \tau_V$ and $(k, v) \in U_*$. Then $k \circ v \subseteq U$ and there exist $W \in \tau_K$, $W' \in \tau_V$, $k \in W$ and $v \in W'$ such that for all $w \in W$ and $w' \in W'$, we get that $w \circ w' \subseteq U$. Thus, for all $w \in W$ and $w' \in W'$, we have $(w, w') \in W \times W' \subseteq U_*$ and so $(k, v) \in W \times W' \subseteq U_*$. Therefore, $U_* \in \tau_{K\times V}$. ∎

Definition 2.3 Let $(V, +, \circ, K)$ be a hypervector space over the field $(K, +, \cdot)$ and $(V, +, \tau_V)$ be a topological group $(K, +, \cdot, \tau_K)$ be a topological field. Then $(V, +, \circ, K, \tau_V, \tau_K)$ is called a pseudo-topological (p-topological) hypervector space if the mapping $\cdot : K \times V \to P^*(V)$ is pseudo-continuous with respect to product topology on $K \times V$ and the topology $\tau_{P(*(V)}$.

Theorem 2.4 Let $(V, +, \circ, K)$ be a hypervector space, (V, τ_V) and (K, τ_K) be topological spaces. Then the hyperoperation "$\circ$" is a continuous map with respect to $\tau_{K\times V}$ and $\tau_{P*(V)}$ if and only if for all $U \in \tau_V$ and $v \in V$, $k \in K$ such that $k \circ v \subseteq U$, there exist $W \in \tau_K, W' \in \tau_V$ such that $k \in W$, $v \in W'$ and $W \circ W' \subseteq U$.

Proof. Let $U \in \tau_V$, and for all $v \in V$ and $k \in K$, $k \circ v \subseteq U$. Then

$$\circ^{-1}(S_U) = \{(k, v) \mid k \circ v \subseteq U\} = U_* \in \tau_{K\times V}.$$

Thus, there exist $W \in \tau_K$ and $W' \in \tau_V$ such that $k \in W$, $v \in W'$ and $(k, v) \in W \times W' \subseteq \circ^{-1}(S_U)$, and so $\circ(W \times W') \subseteq \circ(\circ^{-1}(S_U)) \subseteq S_U$. Thus,

$$W \circ W' = \bigcup_{w\in W, w'\in W'} w.w' = \bigcup_{w\in W, w'\in W'} \circ(w, w') \subseteq U.$$

Conversely, let $U \in \tau_V$ and $(k,v) \in \circ^{-1}(S_U)$. Then $k \circ v \subseteq U$, and so, there exist $W \in \tau_K$ and $W' \in \tau_V$ such that $k \in W$, $v \in W'$ and $W \circ W' \subseteq U$. Hence, $(k,v) \in W \times W' \subseteq \circ^{-1}(S_U)$. Thus, for all $A \in \tau_{P^*(V)}$, we have $A = \bigcup_{U \in \tau_V} S_U$. It follows that

$$\circ^{-1}(A) = \circ^{-1}(\bigcup_{U\in\tau_V} S_U) = \bigcup_{U\in\tau_V} \cdot^{-1}(S_U) \in \tau_{K\times V}.$$

■

***Example* 2.5** Let $V = \mathbb{R}^2$ and $K = \mathbb{R}$. We define the hyperoperation $\circ : \mathbb{R} \times \mathbb{R}^2 \to P^*(\mathbb{R}^2)$ such that for all $a \in \mathbb{R}$ and $(x,y) \in \mathbb{R}^2$,

$$a \circ (x,y) = \begin{cases} \{(ax, ay)\} & \text{if } a \neq 0, \\ \{(0,0)\} & \text{if } a = 0. \end{cases}$$

Let $\mathbb{R}$ be endowed with the discrete topology and $\mathbb{R}^2$ be endowed with the standard topology. Then $(V, +, \circ, K)$ is a pseudo-topological hypervector space over K.

Definition 2.6 Let $(V, +, \circ, K)$ be a hypervector space over the field K and $(\mathcal{F}, A)$ be a non-null soft set over V. Then $(\mathcal{F}, A)$ is called a soft hypervector space over V if $\mathcal{F}(x)$ is a subhyperspace of V for all $x \in Supp(\mathcal{F}, A)$.

***Example* 2.7** Let $K = \mathbb{Z}_3$ and $V = \mathbb{Z}_3$. Then $(V, +, \circ, K)$ is a hypervector space as follows:

$\circ$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{0}$	$\{\overline{0}\}$	$\{\overline{0}\}$	$\{\overline{0}\}$
$\overline{1}$	$\{\overline{0}\}$	$\{\overline{0},\overline{1}\}$	$\{\overline{0},\overline{2}\}$
$\overline{2}$	$\{\overline{0}\}$	$\{\overline{0},\overline{2}\}$	$\{\overline{0},\overline{1}\}$

The subhyperspaces of V are $\{\overline{0}\}$,$\{\overline{0},\overline{1}\}$ and V. Let $(\mathcal{F}, A)$ be a soft set of V, where $A = \{a_1, a_2, a_3\}$ and $\mathcal{F} : A \to P^*(V)$, $\mathcal{F}(a_1) = \{\overline{0}\}$, $\mathcal{F}(a_2) = \{\overline{0},\overline{1}\}$ and $\mathcal{F}(a_3) = V$. Therefore, $(\mathcal{F}, A)$ is a soft hypervector space over V.

***Example* 2.8** In Example 1.1, the subhyperspaces of V are $\{\underline{0}\}$ and all lines passing through the origin. Let $A = \mathbb{R}^2$, we define the soft set F:

$$\forall x \in \mathbb{R}^2,\ \mathcal{F}(x) = \begin{cases} \overline{ox} & \text{if } x \neq \underline{0}, \\ \{\underline{0}\} & \text{if } x = \underline{0}. \end{cases}$$

Therefore, $(\mathcal{F}, A)$ is a soft hypervector space over V.

Definition 2.9 Let $(V, +, \circ, K)$ be a hypervector space over a topological field K and (V, τ) be a topological space. Let $(\mathcal{F}, A)$ be a non-null soft set over V. Then the system $(\mathcal{F}, A, \tau)$ is called a soft pseudo-topological hypervector space over V, if the following conditions are satisfied:

1. $\mathcal{F}(a)$ is a subhyperspace of V for all $a \in Supp(\mathcal{F}, A)$;
2. The mappings

$$+ : \mathcal{F}(a) \times \mathcal{F}(a) \to \mathcal{F}(a), (x,y) \mapsto x + y,\ i : \mathcal{F}(a) \to \mathcal{F}(a), x \mapsto -x$$

and the hyperoperation $\circ : K \times \mathcal{F}(a) \to P^*(\mathcal{F}(a)), (a, x) \mapsto a \circ x$ are continuous with respect to the topologies induced by $\tau, \tau \times \tau$ and $\tau_{P^*(V)} \times \tau_{P^*(V)}$ for all $a \in Supp(\mathcal{F}, A)$.

It is clear that if condition (1) is satisfied in a pseudo-topological hypervector space V, then $(\mathcal{F}, A, \tau)$ is a soft pseudo-topological hypervector. Namely, the soft pseudo-topological hypervector space $(\mathcal{F}, A, \tau)$ can be considered as a parameterized family of subhyperspaces of the pseudo-topological hypervector space V.

***Example* 2.10** Let $V = \mathbb{R}^2$ and $K = \mathbb{R}$. By considering the hyperoperation

$$\circ : \mathbb{R} \times \mathbb{R}^2 \to P^*(\mathbb{R}^2),\ a \circ (x, y) = \mathbb{R} \times a \cdot y,$$

$(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a strongly distributive hypervector space. With standard topology, K is a topological field. Also, the family $\mathcal{B} = \{(x, y)|\ a < y < b, x \in \mathbb{R}\}$ is a base for a topology as τ on $\mathbb{R}^2$. Therefore, $(\mathbb{R}^2, +, \circ, \mathbb{R}, \tau)$ is a pseudo-topological hypervector space. The subhyperspaces of V are $\{(0, 0)\}$, $\{(0, y)|\ y \in \mathbb{R}\}$ and V. Let $(\mathcal{F}, A)$ be a soft set of V where $A = \{a_1, a_2, a_3\}$ and

$$\mathcal{F} : A \to P^*(V),\ \mathcal{F}(a_1) = \{(0, 0)\},\ \mathcal{F}(a_2) = \{(0, y)|\ y \in \mathbb{R}\} \text{ and } \mathcal{F}(a_3) = V.$$

Therefore, $(\mathcal{F}, A, \tau)$ is a soft pseudo-topological hypervector space over V.

Theorem 2.11 Every soft hypervector space on a pseudo-topological hypervector space is a soft pseudo-topological hypervector space.

Proof. Let $(V, +, \circ, K, \tau)$ be a pseudo-topological hypervector space over the topological field K and $(\mathcal{F}, A)$ be a soft hypervector space over V. Then $\mathcal{F}(a)$ is a subhyperspace of V for all $a \in A$. Hence, $\mathcal{F}(a)$ is a pseudo-topological subhyperspace of V with respect to the topologies induced by τ and $\tau_{P^*(V)}$ for all $a \in A$. Therefore, $(\mathcal{F}, A, \tau)$ is a pseudo-topological hypervector space over V. ■

***Example* 2.12** In Example 2.5, the subhypersaces of V are $\{(0, 0)\}$ and all lines passing the origin. Let $A = \mathbb{R}^2$. We define the soft set $\mathcal{F}$:

$$\forall\ (x, y) \in \mathbb{R}^2,\ \mathcal{F}(x, y) = \begin{cases} \text{line pass origin and point (x,y)} & \text{if } (x, y) \neq (0, 0), \\ \{(0, 0)\} & \text{if } (x, y) = (0, 0). \end{cases}$$

Therefore, $(\mathcal{F}, A, \tau)$ is a soft pseudo-topological hypervector space over V.

Definition 2.13 Let $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau)$ be two soft two pseudo-topological hypervector spaces over V. Then, $(\mathcal{K}, B, \tau)$ is called a soft pseudo-topological subhyperspace of $(\mathcal{F}, A, \tau)$ if the following conditions are satisfied:

1. $B \subseteq A$;
2. $\mathcal{K}(b)$ is a subhyperspace of $\mathcal{F}(b)$ for all $a \in Supp(\mathcal{K}, B)$;
3. The mappings:

$$+ : \mathcal{K}(b) \times \mathcal{K}(b) \to \mathcal{K}(b),\ i : \mathcal{K}(b) \to \mathcal{K}(b) \text{ and } \circ : K \times \mathcal{K}(b) \to P^*(\mathcal{K}(b))$$

are continuous with respect to the topologies induced by $\tau \times \tau$ and $\tau_{P^*(V)}$ for all $b \in Supp(\mathcal{K}, B)$.

***Example* 2.14** Let $(\mathcal{F}, A, \tau)$ be a soft pseudo-topological hypervector space over V. If $B \subseteq A$, then $(\mathcal{F}|_B, B, \tau)$ is a soft pseudo-topological subhyperspace of $(\mathcal{F}, A, \tau)$.

Theorem 2.15 If $(\mathcal{K}, B, \tau)$ is a soft pseudo-topological subhyperspace of $(\mathcal{F}, A, \tau)$ and $(\mathcal{N}, C, \tau)$ is a soft pseudo-topological subhyperspace of $(\mathcal{K}, B, \tau)$, then $(\mathcal{N}, C, \tau)$ is the soft pseudo-topological subhyperspace of $(\mathcal{F}, A, \tau)$.

Proof. Straightforward. ■

Theorem 2.16 Let $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau)$ be two soft pseudo-topological hypervector spaces over V. If $(\mathcal{K}, B)$ is a soft subset of $(\mathcal{F}, A)$, then $(\mathcal{K}, B, \tau)$ is a soft pseudo-topological subhyperspace of $(\mathcal{F}, A, \tau)$.

Proof. Suppose that $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau)$ are two soft pseudo-topological hypervector spaces over V. If $(\mathcal{K}, B)$ is a soft subset of $(\mathcal{F}, A)$, then $B \subseteq A$ and $\mathcal{K}(b) \subseteq \mathcal{F}(b)$ for all $b \in Supp(\mathcal{K}, B)$. Thus, $\mathcal{K}(b)$ is a pseudo-topological subhyperspace of $\mathcal{F}(b)$ with respect to the topology induced by τ. Therefore, $(\mathcal{K}, B, \tau)$ is a soft pseudo-topological subhyperspace of $(\mathcal{F}, A, \tau)$. ■

Let $(V_,+_1, \circ_1, K)$ and $(V_2, +_2, \circ_2, K)$ be two strongly distributive hypervector spaces over the field K. Setting $V = V_1 \times V_2$, we define the following two mappings:

$$\oplus : V \times V \to (P^*(V_1), P^*(V_2)),\ \oplus : ((x_1, x_2), (y_1, y_2)) \mapsto (x_1 +_1 y_1, x_2 +_2 y_2)$$

and

$$\odot : K \times V \to V,\ \odot : (a, (x_1, x_2)) \mapsto (a \circ_1 x_1, a \circ_2 x_2)$$

Theorem 2.17 $(V, \oplus, \odot, K)$ is a strongly distributive hypervector space over the field K.

Proof. Let $a, b \in K$, $(x_1, x_2), (y_1, y_2) \in V$ and $0_V = (0_{V_1}, 0_{V_2})$. Then,

(1) It is clear that $V = V_1 \times V_2$ is an abelian group;
(2) $a \odot ((x_1, x_2) \oplus (y_1, y_2)) = a \odot (x_1, x_2) \oplus a \odot (y_1, y_2)$;
(3) $(a + b) \odot (x_1, x_2) = a \odot (x_1, x_2) \oplus b \odot (x_1, x_2)$;
(4) $a \odot (b \odot (x_1, x_2)) = (a \cdot b) \odot (x_1, x_2)$;
(5) $(-a) \odot (x_1, x_2) = a \odot (-(x_1, x_2)) = -(a \odot (x_1, x_2))$;
(6) $1 \odot (x_1, x_2) = (1 \circ_1 x_1, 1 \circ_2 x_2)$, and so $(x_1, x_2) \in 1 \odot (x_1, x_2)$.

Therefore, $(V, \oplus, \odot, K)$ is a strongly distributively hypervector space over K. ■

Theorem 2.18 Let V_1 and V_2 be two hypervector spaces over the field K. Let W_1 and W_2 be subhyperspaces of V_1 and V_2, respectively. Then $W_1 \times W_2$ is a subhyperspace of $V = V_1 \times V_2$.

Proof. Let $a \in K$ and $(x_1, x_2), (y_1, y_2) \in W_1 \times W_2$. We have

$$(x_1, x_2) - (y_1, y_2) = (x_1 - y_1, x_2 - y_2) \in W_1 \times W_2.$$

Also we have $a \odot (x_1, x_2) = (a \circ_1 x_1, a \circ_2 x_2) \subseteq W_1 \times W_2$. Therefore, $W_1 \times W_2$ is a subhyperspace of $V_1 \times V_2$. ■

Theorem 2.19 If $(V_1, +_1, \circ_1, K, \tau_{V_1}, \tau_K)$ and $(V_2, +_2, \circ_2, K, \tau_{V_2}, \tau_K)$ are two pseudo-topological strongly distributively hypervector spaces over the topological field K, then $(V_1 \times V_2, \oplus, \odot, K, \tau_{V_1} \times \tau_{V_2}, \tau_K)$ is a pseudo-topological strongly distributive hypervector space over K.

Proof. We denote the mappings "$+_1$", "$+_2$" and "$\oplus$" by h_1, h_2 and h, respectively. The mappings h_1 and h_2 are continuous. So the mapping

$$h_1 \times h_2 : (V_1 \times V_1, \tau_{V_1} \times \tau_{V_1}) \times (V_2 \times V_2, \tau_{V_2} \times \tau_{V_2}) \to (V_1 \times V_2, \tau_{V_1} \times \tau_{V_2}),$$
$$(h_1 \times h_2)((x_1, y_1), (x_2, y_2)) = (h_1(x_1, y_1), h_2(x_2, y_2))$$

is continuous. Also, the mapping

$$\alpha : (V_1 \times V_2, \tau_{V_1} \times \tau_{V_2})) \times (V_1 \times V_2, \tau_{V_1} \times \tau_{V_2}) \to (V_1 \times V_1, \tau_{V_1} \times \tau_{V_1}) \times (V_2 \times V_2, \tau_{V_2} \times \tau_{V_2}),$$
$$\alpha((x_1, x_2), (y_1, y_2)) = ((x_1, y_1), (x_2, y_2))$$

is continuous. Therefore, the mapping $h = (h_1 \times h_2) \circ \alpha$ is continuous. Let

$$\begin{cases} i_1 : V_1 \to V_1 \\ i_1(x) = -x \end{cases}, \begin{cases} i_2 : V_2 \to V_2 \\ i_2(x) = -x. \end{cases} \text{ and } i = i_1 \times i_2.$$

The mappings i_1 and i_2 are continuous, so the mapping

$$i : V_1 \times V_2 \to V_1 \times V_2, \ i(x_1, x_2) = (-x_1, -x_2)$$

is continuous. We denote the mappings "$\circ_1$", "$\circ_2$" and "$\odot$" by g_1, g_2 and g, respectively. It is clear that, the mappings g_1 and g_2 are continuous, so the mapping

$$g_1 \times g_2 : (K \times V_1) \times (K \times V_2) \to V_1 \times V_2,$$
$$(g_1 \times g_2)((a, x_1), (a, x_2)) = (g_1(a, x_1), g_2(a, x_2))$$

is P-continuous. Also, the mapping

$$\beta : K \times (V_1 \times V_2) \to (K \times V_1) \times (K \times V_2),$$
$$\beta(a, (x_1, x_2)) = ((a, x_1), (a, x_2))$$

is continuous, so $g = (g_1 \times g_2) \circ \beta$ is continuous. Therefore, $(V_1 \times V_2, \oplus, \odot, K, \tau_{V_1} \times \tau_{V_2}, \tau_K)$ is a pseudo-topological strongly distributive hypervector space over K. ■

Definition 2.20 Let $(\mathcal{F}_1, A_1, \tau_1)$ and $(\mathcal{F}_2, A_2, \tau_2)$ be two soft pseudo-topological strongly distributive hypervector spaces over V_1 and V_2, respectively. Then the Cartesian product of $(\mathcal{F}_1, A_1, \tau_1)$ and $(\mathcal{F}_2, A_2, \tau_2)$ over $V_1 \times V_2$ is denoted by $(\mathcal{F}, A, \tau) = (\mathcal{F}_1, A_1, \tau_1) \times (\mathcal{F}_2, A_2, \tau_2)$, where $A = A_1 \times A_2, \tau = \tau_1 \times \tau_2, \mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ and $\mathcal{F}(a, b) = \mathcal{F}_1(a) \times \mathcal{F}_2(b)$ for all $(a, b) \in A$.

Theorem 2.21 The Cartesian product of two soft pseudo-topological strongly distributive hypervector spaces is a soft pseudo-topological strongly distributive hypervector space.

Proof. Let $(\mathcal{F}_1, A_1, \tau_1)$ and $(\mathcal{F}_2, A_2, \tau_2)$ be two soft pseudo-topological strongly distributive hypervector spaces over V_1 and V_2, respectively. Then, $\mathcal{F}_1(a) \neq \emptyset, \mathcal{F}_2(b) \neq \emptyset$ and $\mathcal{F}_1(a), \mathcal{F}_2(b)$ are pseudo-topological subhyperspaces of V_1 and V_2, respectively for all $a \in Supp(\mathcal{F}_1, A_1)$, $B \in Supp(\mathcal{F}_2, A_2)$. Thus, $\mathcal{F}_1(a) \times \mathcal{F}_2(b) \neq \emptyset$ and by Theorem 2.19, $\mathcal{F}_1(a) \times \mathcal{F}_2(b)$ is a pseudo-topological subhyperspace of $V_1 \times V_2$ with the product topology

$\tau_1 \times \tau_2$. Therefore, $(\mathcal{F}, A, \tau_1 \times \tau_2) = (\mathcal{F}_1, A_1, \tau_1) \times (\mathcal{F}_2, A_2, \tau_2)$ is a soft pseudo-topological strongly distributive hypervector space over $V_1 \times V_2$. ■

3. Conclusion

In this paper, we considered the (Tallini) hypervectopr space over a field. We introduced the notions of soft hypervector spaces, soft pseudo-topological hypervector spaces, soft pseudo-topological subhyperspaces, and we presented some related important properties. Indeed, we generalized some notions and properties of soft hyperstructures and soft topological hyperstructures to soft hypervector spaces and soft pseudo-topological hypervector spaces respectively, and we obtained the relation between soft hypervector spaces and pseudo-topological hypervector spaces. Also, we show that the product of two soft pseudo-topological hypervector spaces is a soft pseudo-topological hypervector space. We hope the idea presented in this work can be applied to Krasner hypervector spaces, fuzzy hypervector spaces and other hyperstructures.

References

[1] R. Ameri, Topological (transposition) hypergroups, Ital. J. Pure Appl. Math. (2003), 171-176.
[2] R. Ameri, O. R. Dehghan, On dimension of hypervector spaces, European J. Pure. Appl. Math. 1 (2) (2008), 32-50.
[3] R. Ameri, M. Hamidi, A. Samadifam, Topological vector space derived from (Tallini) topological space, J. Linear. Topological. Algebra. 13 (2) (2024), 71-80.
[4] A. O. Atagun, A. Sezgin, Soft substructures of rings, fields and modules, Comput. Math. Appl. 61 (2011), 592-601.
[5] P. Corsini, V. Leoreanu, Applications of Hyperstructure Theory, Advances in Math-ematics, 5, Kluwer Academic Publishers Boston, Mass, 2003.
[6] F. Feng, Y. B. Jun, X. Zhao, Soft semirings, Comput. Math. Appl. 56 (2008), 2621-2628.
[7] S. Hoskova-Mayerova, Topological Hypergroupoids, Comput. Math. Appl. 64 (9) (2012), 2845-2849.
[8] P. K. Maji, R. Biswas, A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (4-5) (2003), 555-562.
[9] F. Marty, Sur une Generalisation de la Notion de Groupe, 8th Congress Mathematiciens Scandinaves, Stockholm, (1934), 45-49.
[10] W. K. Min, A Note on Soft Topological Spaces, Comput. Math. Appl. 62 (2011), 3524-3528.
[11] D. A. Molodtsov, Soft set theory-First results, Comput. Math. Appl. 37(4-5) (1999), 19-31.
[12] G. Oguz, B. Davvaz, Soft topological hyperstructure, J. Intell. Fuzzy Sys. 40 (2021), 8755-8764
[13] Z. Pawlak, Rough sets, Int. J. Infor. Comp. Sci. 11 (1982), 341-356.
[14] T. Shah, S. Shaheen, Soft topological groups and rings, Ann. Fuzzy Math. Inform. 7 (5) (2014), 725-743.
[15] M. S. Tallini, Hypervector Spaces, Proceedings of the Fourth International Congress on Algebraic Hyperstructures and Applications, Xanthi, Greece, 1990.
[16] M. S. Tallini, Weak Hypervector Space and Norms in such Spaces, Algebraic Hyper Structures and Applications, Jast, Rumania, Hadronic Press, 1994.
[17] J. Wanga, M. Yin, W. Gua, Soft polygroups, Comput. Math. Appl. 62 (2011), 3529-3537.
[18] S. Yamak, O. Kazanci, B. Davvaz, Soft hyperstructure, Comput. Math. Appl. 62 (2) (2011), 797-803.
[19] L. A. Zadeh, Fuzzy sets, Infor. Contr. 8 (1965), 338-353.