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Weaving K-frames in Hilbert C^*-modules

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Abstract. In 2016, Bemrose et al. introduced the weaving frames in a Hilbert space which is influenced by a problem in distributed signal processing. Ghobadzadeh et al. proposed the idea of woven frames in Hilbert C^*-modules in 2018. The authors studied and investigated numerous elementary properties of weaving frames in Hilbert C^*-modules. As K-frames and standard frames deviate in several perspectives, we acquaint the notion of weaving K-frames and an atomic system for weaving K-frames in Hilbert C^*-modules. Inside this script, we explore weaving K-frames from an operator theoretic point of view. We provide an identical interpretation for weaving K-frames and characterize weaving K-frames in terms of bounded linear operators. We also inspect the invariance of woven Bessel sequences under an adjointable operator.

Keywords: Hilbert C^*-module, frame, K-frame, woven frame, K-woven frame, adjoint operator.

2010 AMS Subject Classification: 42C15, 46B15.

1. Introduction

Duffin and Schaeffer [9, 1952] first proposed the notion of frames in Hilbert spaces while studying the nonharmonic Fourier series that can be looked at as more flexible substitutes of bases in Hilbert spaces. In 1986, the theory of frames was reintroduced and advanced by Daubechies et al. [7]. As a result of their exceptional framework, the subject got recognition among many mathematicians, physicists, and engineers. Additionally, we can see its applications in discrete well-known fields like signal processing [13], image

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processing [4], coding and communications [20], sampling [10, 11], numerical analysis, filter theory [3]. It originated as a significant tool in compressive sensing, data analysis, and other areas. The notion of K-frames in Hilbert space was presented by Găvruta [15] to examine the atomic systems with regard to a bounded linear operator K. The concept of standard frames in finitely or countably generated Hilbert C^*-modules over an unital C^*-algebra studied by Frank and Larson [14]. It was established in [1] that every Hilbert module over a commutative C^*-algebra $\mathcal{A}$ admits an algebra-valued G-frame if and only if $\mathcal{A}$ is a C^*-algebra of compact operators and the notion of algebra-valued G-frame treated as a notable case of G-frame in a Hilbert C^*-module. Lately, the n-centered operator is introduced for adjointable operators on Hilbert C^*-modules [18], for any natural number n. Furthermore, it is proved that for an adjointable operator that is MoorePenrose invertible and is n-centered, its MoorePenrose inverse is also n-centered. In 2019, the notion of continuous $*$-K-$g-$frame in Hilbert C^*-modules was introduced and some properties were discussed [22].

The concept of weaving frames in Hilbert space was established in [2] and further investigated in [5, 6]. The idea of weaving frames is somewhat induced by the preprocessing of Gabor frames. It has potential utilization in wireless sensor networks that require distributed processing under different kinds of frames, as well as pre-processing of signals using Gabor frames. In 2018, Deepshikha et al. [8] studied the weaving properties of K-frames in Hilbert space. They presented necessary and sufficient conditions for weaving K-frames in Hilbert spaces and sufficient conditions for PaleyWiener type perturbation of weaving K-frames. Also, it is shown that woven K-frames and weakly woven K-frames are equivalent. Woven frames for finitely or countably generated Hilbert C^*-module were introduced and studied in [16]. The authors have investigated some properties of woven frames and obtained some conditions on a perturbed family of sequences.

2. Preliminaries

In this section, we offer some elementary definitions related to frames, K-frames and weaving frames in Hilbert space and Hilbert C^*-modules which we quote from the literature.

Definition 2.1 [17] A sequence $\{f_n\}_{n=1}^{\infty}$ of elements in Hilbert space H is a frame for H if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leqslant B\|f\|^2, \ \forall\, f \in H. \tag{1}$$

In 2012, Găvruta introduced the notion of K-frames in Hilbert space to study the atomic systems with respect to a bounded linear operator K.

Definition 2.2 [15] A sequence $\{f_n\}_{n=1}^{\infty} \subset H$ is called a K-frames for H, if there exist constants $A, B > 0$ such that

$$A\|K^*f\|^2 \leqslant \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leqslant B\|f\|^2, \ \forall\, f \in H. \tag{2}$$

The concept of weaving frames in Hilbert space is motivated by a problem in distributed signal processing. In [2], Bemrose et al. introduced weaving frames in Hilbert space, and elementary properties of woven frames were developed and discussed.

Definition 2.3 [2] Let I be a countable indexing set. A family of frames $\{\{\phi_{ij}\}_{j\in I} : i \in [m]\}$ for H is said to be woven, if there are universal constants A and B such that for every partition $\{\sigma_i\}_{i\in[m]}$ of I, the family $\bigcup\limits_{i\in[m]} \{\phi_{ij}\}_{j\in\sigma_i}$ is a frame for H with frame bounds A and B.

In [8], Deepshikha et al. studied weaving properties of K-frames in Hilbert space and presented necessary and sufficient conditions for weaving K-frames in Hilbert space. They have also shown that woven K-frames and weakly woven K-frames are correlative.

Definition 2.4 [8] A family of K-frames $\{\{\phi_{ij}\}_{j\in I} : i \in [m]\}$ for H is said to be K-woven if there exist universal positive constants A and B such that for any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{N}$, the family $\bigcup\limits_{i\in[m]} \{\phi_{ij}\}_{j\in\sigma_i}$ is a K-frame for H with lower and upper K-frame bounds A and B, respectively. Each family $\bigcup\limits_{i\in[m]} \{\phi_{ij}\}_{j\in\sigma_i}$ is called a weaving.

Hilbert C^*-modules are generalizations of Hilbert spaces by permitting the inner product to take values in a C^*-algebra rather than in the field of real or complex numbers.

Definition 2.5 [17] Let $\mathcal{A}$ be a unital C^*-algebra and $j \in \mathbb{J}$ be a finite or countable index set. A sequence $\{\psi_j\}_{j\in\mathbb{J}}$ of elements in a Hilbert $\mathcal{A}$ -module $\mathcal{H}$ is said to be a frame if there exist two constants $C, D > 0$ such that

$$C\langle f, f\rangle \leqslant \sum_{j\in\mathbb{J}} \langle f, \psi_j\rangle\langle\psi_j, f\rangle \leqslant D\langle f, f\rangle, \ \forall\ f \in \mathcal{H}. \tag{3}$$

Definition 2.6 [19] A sequence $\{\psi_j\}_{j\in\mathbb{J}}$ of elements in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ is said to be a K-frame ($K \in L(\mathcal{H})$) if there exist constants $C, D > 0$ such that

$$C\langle K^*f, K^*f\rangle \leqslant \sum_{j\in\mathbb{J}} \langle f, \psi_j\rangle\langle\psi_j, f\rangle \leqslant D\langle f, f\rangle, \ \forall\ f \in \mathcal{H}. \tag{4}$$

The notion of weaving frame in Hilbert C^*-module were introduced in [16]. The authors investigated and discussed some properties of woven frames in the Hilbert C^*-module.

Definition 2.7 [16] A family $\{\{\phi_{ij}\}_{i\in I}\}_{j\in J}$ of frames for U is called woven if there exist universal constants $0 < A < B < \infty$ such that for every partition $\{\sigma_j\}_{j\in J}$ of I, the family $\{\{\phi_{ij}\}_{i\in I}\}_{j\in J}$ is a frame for U with lower and upper frame bounds A and B, respectively. Each family $\{\{\phi_{ij}\}_{i\in\sigma_i}\}_{j\in J}$ is called a weaving.

3. Weaving K-frames

We introduce the concept of weaving K-frames in Hilbert C^*-module which is encouraged by the above discussed work in the literature. Here, we investigate weaving K-frames from an operator theoretic point of view as well as establish the atomic system for weaving K-frames in Hilbert C^*-module. We also present an equivalent definition of weaving K-frames and characterization theorems of weaving K-frames in terms of operator theory in Hilbert C^*-module. For the rest of the paper, we assume that $\mathcal{H}$ is a Hilbert C^*-module over unital C^*-algebra $\mathcal{A}$ with $\mathcal{A}$-valued inner product $\langle .,. \rangle$, norm $\|.\|$ and $L(\mathcal{H})$ denotes the set of all adjointable operators on Hilbert C^*-module $\mathcal{H}$.

Definition 3.1 Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module over a unital C^*-algebra. A family of K-frames $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ for $\mathcal{H}$ is said to be K-woven if there exist universal positive constants A and B such that for any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$, the family $\bigcup\limits_{i\in I}\{f_{ij}\}_{j\in\sigma_i}$ is a K-frame for $\mathcal{H}$ with lower and upper K-frame bounds A and B, respectively. Each family $\bigcup\limits_{i\in I}\{f_{ij}\}_{j\in\sigma_i}$ is called a weaving.

The woven frame is called a tight woven frame if $A = B$ and it is called a normalized woven tight frame if $A = B = 1$. For any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$, we define the space as

$$\bigoplus_{i\in I} l^2(\sigma_i) = \big\{\{c_{ij}\}_{j\in\sigma_i, i\in I} \mid c_{ij} \in \mathcal{A}, \sum_{i\in I}\sum_{j\in\sigma_i} c_{ij}c_{ij}^* \text{ converges in } \|\cdot\|_{\mathcal{A}}\big\}$$

with the inner product $\langle \{c_{ij}\}_{j\in\sigma_i,i\in I}, \{d_{ij}\}_{j\in\sigma_i,i\in I}\rangle = \sum\limits_{i\in I}\sum\limits_{j\in\sigma_i} c_{ij}d_{ij}^*$. Let the family of K-frames $\{F_i = \{f_{ij}\}_{j\in J} : i \in I\}$ be woven for $\mathcal{H}$, for any partition $\{\sigma_i\}_{i\in I}$ of J and $W = \{f_{ij}\}_{j\in\sigma_i,i\in I}$ be a K-frame for $\mathcal{H}$, then we have the corresponding synthesis operator, analysis operator, and frame operator as follows:

The operator $T_W : \bigoplus\limits_{i\in I} l^2(\sigma_i) \to \mathcal{H}$ defined by

$$T_W(\{c_{ij}\}) = \sum_{i\in I} T_{F_i} D_{\sigma_i}(\{c_{ij}\}) = \sum_{i\in I}\sum_{j\in\sigma_i} c_{ij}f_{ij} \tag{5}$$

is called the synthesis or pre-frame operator, where T_{F_i} is the synthesis operator of F_i and D_{σ_i} is a $|J| \times |J|$ diagonal matrix with $d_{jj} = 1$ for $j \in \sigma_i$ and otherwise 0. The adjoint of T_W is

$$\langle f, T_W\{c_{ij}\}\rangle = \langle f, \sum_{i\in I}\sum_{j\in\sigma_i} c_{ij}f_{ij}\rangle = \sum_{i\in I}\sum_{j\in\sigma_i} c_{ij}^*\langle f, f_{ij}\rangle \tag{6}$$

$\implies \langle f, T_W\{c_{ij}\}\rangle = \langle \{\langle f, f_{ij}\rangle\}, \{c_{ij}\}\rangle \implies T_W^*(f) = \{\langle f, f_{ij}\rangle\}_{j\in\sigma_i,i\in I}$.

The adjoint operator $T_W^* : \mathcal{H} \to \bigoplus\limits_{i\in I} l^2(\sigma_i)$ is given by

$$T_W^*(f) = \sum_{i\in I} D_{\sigma_i} T_{F_i}^{\sigma_i *}(f) = \{\langle f, f_{ij}\rangle\}_{j\in\sigma_i,i\in I} \tag{7}$$

and is called the analysis operator. By composing T_W and T_W^*, we obtain the frame operator $S_W : \mathcal{H} \to \mathcal{H}$

$$S_W(f) = T_W T_W^*(f) = (\sum_{i\in I} T_{F_i} D_{\sigma_i})(\sum_{i\in I} T_{F_i} D_{\sigma_i})^* = \sum_{i\in I}\sum_{j\in\sigma_i} \langle f, f_{ij}\rangle f_{ij}. \tag{8}$$

We now state some of the important properties of the synthesis, analysis, and frame operators of weaving K-frames in the Hilbert C^*-module.

Lemma 3.2 Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a woven Bessel sequence then the synthesis operator T_W is linear and bounded.

Proof. Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a woven Bessel sequence with universal Bessel bound B. Now,

$$\begin{aligned} T_W(\{\lambda c_{ij} + d_{ij}\}) &= \sum_{i\in I}\sum_{j\in\sigma_i}(\lambda c_{ij} + d_{ij})f_{ij} \\ &= \sum_{i\in I}\sum_{j\in\sigma_i}\lambda c_{ij}f_{ij} + \sum_{i\in I}\sum_{j\in\sigma_i} d_{ij}f_{ij} \\ &= \lambda T_W(\{c_{ij}\}) + T_W(\{d_{ij}\}) \end{aligned} \tag{9}$$

and

$$\|T_W f\|^2 = \|\langle T_W f, T_W f\rangle\| = \|\langle T_W T_W^* f, f\rangle\| = \|\langle S_W f, f\rangle\| \leqslant B\|f\|^2, \tag{10}$$

$\implies \|T_W f\| \leqslant \sqrt{B}\|f\|$. Hence, the synthesis operator T_W is linear and bounded. ■

Lemma 3.3 Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be woven frame for $\mathcal{H}$ with universal bounds A and B. Then the frame operator S_W is self-adjoint, positive, bounded, and invertible on $\mathcal{H}$.

Proof. Since $S_W^* = (T_W T_W^*)^* = T_W T_W^* = S_W$, the frame operator S_W is self adjoint. Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be woven frame for $\mathcal{H}$ with universal bounds A and B. Let $f \in \mathcal{H}$ and $S_W(f) = \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle f_{ij}$. Then

$$\langle S_W f, f\rangle = \Big\langle \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle f_{ij}, f\Big\rangle = \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle$$

$\implies A\langle f, f\rangle \leqslant \langle S_W f, f\rangle \leqslant B\langle f, f\rangle \implies AI \leqslant S_W \leqslant BI$.

Therefore, the frame operator S_W is positive, bounded, and invertible. ■

We now give an example of woven frames and woven K-frames in Hilbert C^*-module.

***Example* 3.4** Let $\mathcal{H} = C_0$ be the set of all sequences converging to zero and $\{e_j\}_{j=1}^{\infty}$ be the standard orthonormal basis for $\mathcal{H}$. For any $u = \{u_j\}_{j=1}^{\infty} \in \mathcal{H}$ and $v = \{v_j\}_{j=1}^{\infty} \in \mathcal{H}$, $\langle u, v\rangle = uv^* = \{u_j v_j^*\}_{j=1}^{\infty}$. Let $\phi = \{\phi_{1j}\}_{j=1}^{\infty}$ and $\psi = \{\phi_{2j}\}_{j=1}^{\infty}$ be defined as follows:

$$\begin{aligned} \{\phi_{1j}\}_{j=1}^{\infty} &= \{e_1, e_2, 0, e_3, 0, e_4, 0, e_5, ...\} \\ \{\phi_{2j}\}_{j=1}^{\infty} &= \{0, e_2, e_2, e_3, e_3, e_4, e_4, e_5, e_5, ...\} \end{aligned}$$

Let $f = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, ...\} \in \mathcal{H}$. Then $\langle f, f\rangle = \alpha_1\alpha_1^* + \alpha_2\alpha_2^* + \alpha_3\alpha_3^* + \alpha_4\alpha_4^* + \cdots$. For any subset σ of $\mathbb{N}$, we have

$$\sum_{j\in\sigma}\langle f, \phi_{1j}\rangle\langle\phi_{1j}, f\rangle + \sum_{j\in\sigma^c}\langle f, \phi_{2j}\rangle\langle\phi_{2j}, f\rangle \leqslant 2\sum_{j=1}^{\infty}\langle f, e_j\rangle\langle e_j, f\rangle = 2\langle f, f\rangle.$$

On the other hand, let $f \in \mathcal{H}$. Then we have

$$\langle f, f\rangle = \sum_{j=1}^{\infty}\langle f, e_j\rangle\langle e_j, f\rangle \leqslant \sum_{j\in\sigma}\langle f, \phi_{1j}\rangle\langle \phi_{1j}, f\rangle + \sum_{j\in\sigma^c}\langle f, \phi_{2j}\rangle\langle \phi_{2j}, f\rangle.$$

Hence, ϕ and ψ are woven frames with universal lower and upper frame bounds 1 and 2, respectively.

***Example* 3.5** Let $\mathcal{H} = C_0$ be the set of all sequences converging to zero and K be the orthogonal projection of $\mathcal{H}$ onto span$\{e_j\}_{j=3}^{\infty}$. For any $u = \{u_j\}_{j=1}^{\infty} \in \mathcal{H}$ and $v = \{v_j\}_{j=1}^{\infty} \in \mathcal{H}$, $\langle u, v\rangle = uv^* = \{u_j v_j^*\}_{j=1}^{\infty}$. Let $\phi = \{\phi_{1j}\}_{j=1}^{\infty} \in \mathcal{H}$ and $\psi = \{\phi_{2j}\}_{j=1}^{\infty} \in \mathcal{H}$ be defined as follows:

$$\{\phi_{1j}\}_{j=1}^{\infty} = \{0, e_3, 0, e_4, 0, e_5, 0, e_6, ...\}$$
$$\{\phi_{2j}\}_{j=1}^{\infty} = \{0, e_3, e_3, e_4, e_4, e_5, e_5, e_6, e_6, ...\}$$

where $\{e_j\}_{j=1}^{\infty}$ be the standard orthonormal basis for $\mathcal{H}$. Let $f = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, ...\} \in \mathcal{H}$. Then $\langle f, f\rangle = \alpha_1\alpha_1^* + \alpha_2\alpha_2^* + \alpha_3\alpha_3^* + \alpha_4\alpha_4^* + \cdots$. For any subset σ of $\mathbb{N}$, we have

$$\sum_{j\in\sigma}\langle f, \phi_{1j}\rangle\langle \phi_{1j}, f\rangle + \sum_{j\in\sigma^c}\langle f, \phi_{2j}\rangle\langle \phi_{2j}, f\rangle \leqslant 2\sum_{j=1}^{\infty}\langle f, e_j\rangle\langle e_j, f\rangle = 2\langle f, f\rangle.$$

On the other hand, let $f \in \mathcal{H}$. Then $f = \sum_{j=1}^{\infty}\alpha_j e_j$. Thus, we have

$$\begin{aligned}\langle K^*f, K^*f\rangle &= \Big\langle K^*\Big(\sum_{j=1}^{\infty}\alpha_j e_j\Big), K^*\Big(\sum_{j=1}^{\infty}\alpha_j e_j\Big)\Big\rangle \\ &= \Big\langle \sum_{j=3}^{\infty}\alpha_j e_j, \sum_{j=3}^{\infty}\alpha_j e_j\Big\rangle \\ &= \sum_{j=3}^{\infty}\langle f, e_j\rangle\langle e_j, f\rangle \\ &\leqslant \sum_{j\in\sigma}\langle f, \phi_{1j}\rangle\langle \phi_{1j}, f\rangle + \sum_{j\in\sigma^c}\langle f, \phi_{2j}\rangle\langle \phi_{2j}, f\rangle\end{aligned}$$

Hence ϕ and ψ are K-woven frames with universal lower and upper frame bounds 1 and 2, respectively.

We now introduce a woven atomic system for weaving K-frames in Hilbert C^*-module.

Definition 3.6 The sequence $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ of $\mathcal{H}$ is said to be a woven atomic system for $K \in L(\mathcal{H})$, if for any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$, the family $\bigcup_{i\in I}\{f_{ij}\}_{j\in\sigma_i}$ is a woven atomic system for K, i.e. the following statements hold:

(i) The series $\sum_{i\in I}\sum_{j\in\sigma_i} c_{ij} f_{ij}$ converges for all $\{c_{ij}\}_{j\in\sigma_i, i\in I} \in l^2(\mathcal{A})$.

(ii) There exist $C > 0$ such that for every $f \in \mathcal{H}$, there exists $\{a_{ij,f}\}_{j\in\sigma_i, i\in I} \in l^2(\mathcal{A})$ such that $\sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,f}a^*_{ij,f} \leqslant C\langle f, f\rangle$ and $Kf = \sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,f} f_{ij}$.

Theorem 3.7 If $K \in L(\mathcal{H})$, then there exists a woven atomic system for K.

Proof. Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a standard normalized woven tight frame for $\mathcal{H}$ with universal frame bound $A = B = 1$. Since $f = \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle f_{ij}$, we have $Kf = \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle Kf_{ij}$. For $f \in \mathcal{H}$, $a_{ij,f} = \langle f, f_{ij}\rangle$ and $g_{ij} = Kf_{ij}$,

$$\begin{aligned}\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, g_{ij}\rangle\langle g_{ij}, f\rangle &= \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, Kf_{ij}\rangle\langle Kf_{ij}, f\rangle \\ &= \sum_{i\in I}\sum_{j\in\sigma_i}\langle K^*f, f_{ij}\rangle\langle f_{ij}, K^*f\rangle \\ &= \langle K^*f, K^*f\rangle \\ &\leqslant \|K^*\|^2\langle f, f\rangle.\end{aligned}$$

Therefore, $\{\{g_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven Bessel sequence for $\mathcal{H}$ with Bessel bound $\|K^*\|^2$ and we conclude that the series $\sum_{i\in I}\sum_{j\in\sigma_i} c_{ij}g_{ij}$ converges for all $\{c_{ij}\}_{j\in\sigma_i,i\in I} \in l^2(\mathcal{A})$. We also have

$$\sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,f}a^*_{ij,f} = \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle = \langle f, f\rangle,$$

which completes the proof. ■

Since it is more convenient to work with an equivalent definition of weaving K- frames in Hilbert C^*-modules, we would like to introduce an equivalent definition in the following result. First, we requisite the successive result to validate an equivalent definition of weaving K-frames in Hilbert C^*-modules.

Theorem 3.8 [12] Let $\mathcal{F}, \mathcal{H}, \mathcal{K}$ be Hilbert C^*-modules over a C^*-algebra $\mathcal{A}$. Also, let $S \in L(\mathcal{K}, \mathcal{H})$ and $T \in L(\mathcal{F}, \mathcal{H})$ with $\overline{R(T^*)}$ orthogonally complemented. The following statements are equivalent:

(i) $SS^* \leqslant \lambda TT^*$ for some $\lambda > 0$;

(ii) there exists $\mu > 0$ such that $\|S^*z\| \leqslant \|T^*z\|$ for all $z \in \mathcal{H}$;

(iii) there exists $D \in L(\mathcal{K}, \mathcal{F})$ such that $S = TD$, i.e., $TX = S$ has a solution;

(iv) $R(S) \subseteq R(T)$.

Theorem 3.9 For any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$, let the family $\bigcup_{i\in I}\{f_{ij}\}_{j\in\sigma_i}$ be a woven Bessel sequence for $\mathcal{H}$ and $K \in L(\mathcal{H})$. Suppose that $T^* \in L(\mathcal{H}, l^2(\mathcal{A}))$ given by $T^*(f) = \{\langle f, f_{ij}\rangle\}_{j\in\sigma_i,i\in I}$ and $\overline{R(T^*)}$ is orthogonally complemented then the following statements are equivalent:

(i) The sequence $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ of $\mathcal{H}$ is a woven atomic system for K.

(ii) There exist $A, B > 0$ such that

$$A\|K^*f\|^2 \leqslant \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle\| \leqslant B\|f\|^2$$

(iii) There exists $D \in L(\mathcal{H}, l^2(\mathcal{A}))$ such that $K = TD$.

Proof. (i) $\implies$ (ii) For every $f \in \mathcal{H}$, we have

$$\|K^*f\| = \sup_{\|g\|=1}\|\langle g, K^*f\rangle\| = \sup_{\|g\|=1}\|\langle Kg, f\rangle\|.$$

Since $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven atomic system for K, there exist $C > 0$ such that for every $g \in \mathcal{H}$, there exist $a_g = \{a_{ij,g}\}_{j\in\sigma_i, i\in I} \in l^2(\mathcal{A})$ for which $\sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,g}a_{ij,g}^* \leqslant C\langle g, g\rangle$ and $Kg = \sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,g}f_{ij}$. Therefore,

$$\begin{aligned}
\|K^*f\|^2 &= \sup_{\|g\|=1}\|\langle Kg, f\rangle\|^2 \\
&= \sup_{\|g\|=1}\|\langle\sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,g}f_{ij}, f\rangle\|^2 \\
&= \sup_{\|g\|=1}\|\sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,g}\langle f_{ij}, f\rangle\|^2 \\
&\leqslant \sup_{\|g\|=1}\|\sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,g}\|^2\|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f_{ij}, f\rangle\|^2 \\
&= \sup_{\|g\|=1}\|\sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,g}a_{ij,g}^*\|\|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\ \langle f_{ij}, f\rangle\| \\
&\leqslant \sup_{\|g\|=1} C\|\langle g, g\rangle\|\|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\ \langle f_{ij}, f\rangle\| \\
&= \sup_{\|g\|=1} C\|g\|^2\|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\ \langle f_{ij}, f\rangle\| \\
&= C\|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\ \langle f_{ij}, f\rangle\|,
\end{aligned}$$

which implies $\frac{1}{C}\|K^*f\|^2 \leqslant \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\ \langle f_{ij}, f\rangle\|$. Moreover, $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven Bessel sequence for $\mathcal{H}$. Hence (ii) holds.
(ii) $\implies$ (iii) Since $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven Bessel sequence for $\mathcal{H}$, we get

$$T(\{e_{ij}\}) = \sum_{i\in I}\sum_{j\in\sigma_i} e_{ij}f_{ij} = f_{ij}.$$

where $\{e_{ij}\}_{j\in\sigma_i, i\in I}$ is the standard orthonormal basis for $l^2(\mathcal{A})$. Therefore, for every $f \in \mathcal{H}$

$$\begin{aligned}
A\|K^*f\|^2 &\leqslant \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle \ \langle f_{ij}, f\rangle\| \\
&= \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, T\{e_{ij}\}\rangle \ \langle T\{e_{ij}\}, f\rangle\| \\
&= \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle T^*f, \{e_{ij}\}\rangle \ \langle \{e_{ij}\}, T^*f\rangle\| = \|T^*f\|^2.
\end{aligned}$$

By using Theorem 3.8, there exist an operator $D \in L(\mathcal{H}, l^2(\mathcal{A}))$ such that $K = TD$.
(iii) $\implies$ (i) For every $f \in \mathcal{H}$, we have

$$Df = \sum_{i\in I}\sum_{j\in\sigma_i}\langle Df, e_{ij}\rangle e_{ij} \implies TDf = \sum_{i\in I}\sum_{j\in\sigma_i}\langle Df, e_{ij}\rangle Te_{ij}.$$

Let $a_{ij,f} = \langle Df, e_{ij}\rangle$, so for all $f \in \mathcal{H}$, we get

$$\sum_{i\in I}\sum_{j\in\sigma_i} a_{ij,f}a_{ij,f}^* = \sum_{i\in I}\sum_{j\in\sigma_i}\langle Df, e_{ij}\rangle\langle e_{ij}, Df\rangle = \langle Df, Df\rangle \leqslant \|D\|^2\langle f, f\rangle.$$

Since the sequence $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven Bessel sequence for $\mathcal{H}$, we conclude that $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven atomic system for K. ■

Corollary 3.10 Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a woven frame for $\mathcal{H}$ with universal frame bounds $A, B > 0$ and $K \in L(\mathcal{H})$. Then $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven atomic system for K with bounds lower and upper frame bounds $\dfrac{1}{A^{-1}\|K\|^2}$ and B, respectively.

Proof. Let S be the frame operator of $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$. Since $\{\{S^{-1}f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven frame for $\mathcal{H}$ with bounds $B^{-1}, A^{-1} > 0$ and
$f = \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle S^{-1}f_{ij}$, for all $f \in \mathcal{H}$.

$$\begin{aligned}
\|K^*f\|^2 &= \sup_{\|g\|=1} \|\langle K^*f, g\rangle\|^2 \\
&= \sup_{\|g\|=1} \|\langle \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle K^*S^{-1}f_{ij}, g\rangle\|^2 = \sup_{\|g\|=1} \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle K^*S^{-1}f_{ij}, g\rangle\|^2 \\
&\leqslant \sup_{\|g\|=1} \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle\|\|\sum_{i\in I}\sum_{j\in\sigma_i}\langle Kg, S^{-1}f_{ij}\rangle\langle S^{-1}f_{ij}, Kg\rangle\| \\
&\leqslant \sup_{\|g\|=1} A^{-1}\|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle\|\|Kg\|^2 \\
&\leqslant \sup_{\|g\|=1} A^{-1}\|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle\|\|K\|^2\|g\|^2 \\
&= A^{-1}\|K\|^2\|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle\|.
\end{aligned}$$

which implies

$$\frac{1}{A^{-1}\|K\|^2}\|K^*f\|^2 \leqslant \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle\| \leqslant B\|f\|^2$$

and shows that the condition (ii) of Theorem 3.9 hold. Therefore, $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven atomic system for K with lower and upper frame bounds $\frac{1}{A^{-1}\|K\|^2}$ and B, respectively. ■

Corollary 3.11 Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a woven atomic system for K. If $K \in L(\mathcal{H})$ is onto, then $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven frame for $\mathcal{H}$.

Proof. As we know, $K \in L(\mathcal{H})$ is surjective if and only if there exists $M > 0$ such that

$$M\|f\| \leqslant \|K^*f\|, \forall f \in \mathcal{H}. \tag{11}$$

Since $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven atomic system for K, so there exists $A, B > 0$ such that

$$A\|K^*f\|^2 \leqslant \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle\| \leqslant B\|f\|^2 \tag{12}$$

for any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$. By using (11) and (12), we get

$$AM^2\|f\|^2 \leqslant \|\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle\| \leqslant B\|f\|^2,$$

which completes the proof. ■

Proposition 3.12 Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a K-frame for $\mathcal{H}$ with K-frame bounds A_i and B_i. Then, for any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$, the family $\bigcup_{i\in I}\{f_{ij}\}_{j\in\sigma_i}$ is a woven Bessel sequence with Bessel bound $\sum_{i\in I} B_i$.

Proof. Let $\{\sigma_i\}_{i\in I}$ be any partition of $\mathbb{N}$. Then, for any $f \in \mathcal{H}$,

$$\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle \leqslant \sum_{i\in I}\sum_{j\in\mathbb{N}}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle \leqslant \sum_{i\in I} B_i\langle f, f\rangle.$$

■

The following theorem gives a characterization of weaving K-frames in terms of a bounded linear operator in Hibert C^*-module.

Theorem 3.13 For each $i \in I$, suppose $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a K-frame for $\mathcal{H}$ with bounds A_i and B_i. Then the following conditions are equivalent:

(i) The family $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is K-woven.

(ii) There exist $A > 0$ such that for any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$, there exist a bounded linear operator $M_\sigma : l^2(\mathcal{A}) \to \mathcal{H}$ such that

$$M_\sigma(e_j) = \begin{cases} f_{1j}, & j \in \sigma_1 \\ f_{2j}, & j \in \sigma_2 \\ \vdots \\ f_{mj}, & j \in \sigma_m \end{cases}$$

and $AKK^* \leqslant M_\sigma M_\sigma^*$, where $\{e_j\}_{j=1}^\infty$ is the canonical orthonormal basis.

Proof. (i) $\implies$ (ii): Suppose A is a universal lower K-frame bound for the family $\{\{f_{ij}\}_{j=1}^\infty : i \in I\}$. For any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$, let T_σ be the synthesis operator associated with the Bessel sequence $\bigcup_{i\in I}\{f_{ij}\}_{j\in\sigma_i}$. Choose $M_\sigma = T_\sigma$. Then $M_\sigma(e_j) = T_\sigma(e_j) = f_{ij}$, $\forall j \in \sigma_i, i \in I$. Now,

$$\begin{aligned} A\langle K^*f, K^*f\rangle &= A\langle KK^*f, f\rangle \\ &\leqslant \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle \\ &= \sum_{j\in\mathbb{N}}\langle f, M_\sigma(e_j)\rangle\langle M_\sigma(e_j), f\rangle \\ &= \sum_{j\in\mathbb{N}}\langle M_\sigma^*f, e_j\rangle\langle e_j, M_\sigma^*f\rangle \\ &= \Big\langle \sum_{j\in\mathbb{N}}\langle M_\sigma^*f, e_j\rangle e_j, M_\sigma^*f\Big\rangle \\ &= \langle M_\sigma^*f, M_\sigma^*f\rangle \\ &= \langle M_\sigma M_\sigma^*f, f\rangle. \end{aligned}$$

This implies $AKK^* \leqslant M_\sigma M_\sigma^*$.
(ii) $\implies$ (i): Let $\{\sigma_i\}_{i\in I}$ be any partition of $\mathbb{N}$. Now,

$$A\langle KK^*f, f\rangle \leqslant \langle M_\sigma M_\sigma^*f, f\rangle = \langle M_\sigma^*f, M_\sigma^*f\rangle = \sum_{j\in\mathbb{N}}\langle M_\sigma^*f, e_j\rangle\langle e_j, M_\sigma^*f\rangle = \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle.$$

This gives the universal lower K-frame bound A. And by Proposition 3.12, $\sum_{i\in I} B_i$ is one of the choices of a universal upper K-frame bound. ■

In the following lemma, we inspect the invariance of woven Bessel sequence under an adjointable operator.

Lemma 3.14 Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and $\{\{f_{ij}\}_{j=1}^\infty : i \in I\}$ be a woven Bessel sequence with universal Bessel bound D. Then $\{\{Mf_{ij}\}_{j=1}^\infty : i \in I\}$ is a woven Bessel sequence with universal Bessel bound with $D\|M^*\|^2$ for every $M \in L(\mathcal{H})$.

Proof. Suppose $\{\{f_{ij}\}_{j=1}^\infty : i \in I\}$ is a woven Bessel sequence with universal Bessel bound D. Then we have

$$\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle \leqslant D\langle f, f\rangle$$

for any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$. Then, for any $f\in\mathcal{H}$, we have

$$\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, Mf_{ij}\rangle\langle Mf_{ij}, f\rangle = \sum_{i\in I}\sum_{j\in\sigma_i}\langle M^*f, f_{ij}\rangle\langle f_{ij}, M^*f\rangle \leqslant D\langle M^*f, M^*f\rangle \leqslant D\|M^*\|^2\langle f, f\rangle.$$

■

In the following result, we study the action of an operator on a K-woven frame.

Proposition 3.15 Let $\{\{f_{ij}\}_{j=1}^{\infty} : i\in I\}$ be a family of K-frames for $\mathcal{H}$. Then the following statements are equivalent:
(i) $\{\{f_{ij}\}_{j=1}^{\infty} : i\in I\}$ is K-woven.
(ii) $\{\{Uf_{ij}\}_{j=1}^{\infty} : i\in I\}$ is UK-woven for all $U\in L(\mathcal{H})$.

Proof. (i) $\Longrightarrow$ (ii): Let $\{\{f_{ij}\}_{j=1}^{\infty} : i\in I\}$ be a family of K-frames for $\mathcal{H}$ with universal frame bounds A and B. Let $\{\sigma_i\}_{i\in I}$ be any partition of $\mathbb{N}$. Then for any $f\in\mathcal{H}$, we have

$$\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, Uf_{ij}\rangle\langle Uf_{ij}, f\rangle = \sum_{i\in I}\sum_{j\in\sigma_i}\langle U^*f, f_{ij}\rangle\langle f_{ij}, U^*f\rangle \leqslant B\langle U^*f, U^*f\rangle \leqslant B\|U^*\|^2\langle f, f\rangle.$$

Similarly, for any $f\in\mathcal{H}$, we have

$$\begin{aligned}\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, Uf_{ij}\rangle\langle Uf_{ij}, f\rangle &= \sum_{i\in I}\sum_{j\in\sigma_i}\langle U^*f, f_{ij}\rangle\langle f_{ij}, U^*f\rangle\\ &\geqslant A\langle K^*U^*f, K^*U^*f\rangle\\ &\geqslant A\langle (UK)^*f, (UK)^*f\rangle.\end{aligned}$$

Hence, the family $\{\{Uf_{ij}\}_{j=1}^{\infty} : i\in I\}$ is UK-woven with universal frame bounds A and $B\|U^*\|^2$.
(ii) $\Longrightarrow$ (i): The family $\{\{f_{ij}\}_{j=1}^{\infty} : i\in I\}$ is K-woven if we choose $U = I$, the identity operator on $\mathcal{H}$. ■

In the following example, we show that if ϕ and ψ be K-frames for $\mathcal{H}$ such that $U\phi$ and $U\psi$ are UK-woven for some $U\in L(\mathcal{H})$. Then, in general ϕ and ψ are not K-woven.

***Example* 3.16** Let $\mathcal{H} = C_0$ be the set of all sequences converging to zero and let K be the orthogonal projection of H onto $\text{span}\{e_j\}_{j=2}^{\infty}$. Let $\phi = \{\phi_{1j}\}_{j=1}^{\infty}$ and $\psi = \{\phi_{2j}\}_{j=1}^{\infty}$ be defined as follows:

$$\begin{aligned}\phi &\equiv \{\phi_{1j}\}_{j=1}^{\infty} = \{0, e_1, 0, e_2, 0, e_3, 0, e_4, 0, e_5, ...\}\\ \psi &\equiv \{\phi_{2j}\}_{j=1}^{\infty} = \{e_1, 0, e_2, 0, e_3, e_3, e_4, e_4, e_5, e_5, ...\}\end{aligned}$$

where $\{e_j\}_{j=1}^{\infty}$ is the standard orthonormal basis for $\mathcal{H}$. Then, ϕ is K-frame for $\mathcal{H}$ with lower and upper frame bound 1. One can easily verify ψ is also K-frame for $\mathcal{H}$. Let $f = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, ...\}\in\mathcal{H}$. Then $\langle f, f\rangle = \alpha_1\alpha_1^* + \alpha_2\alpha_2^* + \alpha_3\alpha_3^* + \alpha_4\alpha_4^* + \cdots$. Let

U be the orthogonal projection of $\mathcal{H}$ onto span$\{e_j\}_{j=3}^{\infty}$. To show that $U\phi$ and $U\psi$ are UK-woven frames for $\mathcal{H}$, first we note that

$$U\phi \equiv \{U(\phi_{1j})\}_{j=1}^{\infty} = \{0,0,0,0,0,e_3,0,e_4,0,e_5,0,...\}$$
$$U\psi \equiv \{U(\phi_{2j})\}_{j=1}^{\infty} = \{0,0,0,0,e_3,e_3,e_4,e_4,e_5,e_5,...\}$$

For any subset σ of $\mathbb{N}$ and $f \in \mathcal{H}$, we have

$$\sum_{j\in\sigma}\langle f, U\phi_{1j}\rangle\langle U\phi_{1j}, f\rangle + \sum_{j\in\sigma^c}\langle f, U\phi_{2j}\rangle\langle U\phi_{2j}, f\rangle \leqslant 2\sum_{j=1}^{\infty}\langle f, e_j\rangle\langle e_j, f\rangle = 2\langle f, f\rangle.$$

On the other hand, let $f \in \mathcal{H}$ and we can represent it as $f = \sum_{j=1}^{\infty}\alpha_j e_j$. Thus, we have

$$\begin{aligned}
\langle (UK)^*f, (UK)^*f\rangle &= \langle U^*f, U^*f\rangle \\
&= \langle U^*(\sum_{j=1}^{\infty}\alpha_j e_j), U^*(\sum_{j=1}^{\infty}\alpha_j e_j)\rangle \\
&= \langle \sum_{j=1}^{\infty}\alpha_j U^*e_j, \sum_{j=1}^{\infty}\alpha_j U^*e_j\rangle \\
&= \langle \sum_{j=3}^{\infty}\alpha_j e_j, \sum_{j=3}^{\infty}\alpha_j e_j\rangle \\
&= \sum_{j=3}^{\infty}\langle f, e_j\rangle\langle e_j, f\rangle \\
&\leqslant \sum_{j\in\sigma}\langle f, U\phi_{1j}\rangle\langle U\phi_{1j}, f\rangle + \sum_{j\in\sigma^c}\langle f, U\phi_{2j}\rangle\langle U\phi_{2j}, f\rangle.
\end{aligned}$$

Hence, $U\phi$ and $U\psi$ are UK-woven frames with universal lower and upper frame bounds 1 and 2, respectively. Now, to show that ϕ and ψ are not K-woven, we choose $\sigma = \mathbb{N}\backslash\{2,4\}$. Then the family $\{\phi_{1j}\}_{j\in\sigma}\bigcup\{\phi_{2j}\}_{j\in\sigma^c} = \{0,0,0,0,e_3,0,e_4,0,e_5,...\}$ is not a K-frame for $\mathcal{H}$, since for any $A > 0$, we have

$$\sum_{j\in\sigma}\langle e_2, \phi_{1j}\rangle\langle \phi_{1j}, e_2\rangle + \sum_{j\in\sigma^c}\langle e_2, \phi_{2j}\rangle\langle \phi_{2j}, e_2\rangle = \sum_{j\geqslant 3}\langle e_2, e_j\rangle\langle e_j, e_2\rangle = 0$$

So, there exist no $A > 0$ such that

$$\sum_{j\in\sigma}\langle e_2, \phi_{1j}\rangle\langle \phi_{1j}, e_2\rangle + \sum_{j\in\sigma^c}\langle e_2, \phi_{2j}\rangle\langle \phi_{2j}, e_2\rangle \geqslant A\langle K^*e_2, K^*e_2\rangle$$

holds. Thus, ϕ and ψ are not K-woven.

Theorem 3.17 Let $K \in L(\mathcal{H})$ and $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ be K-woven for $\mathcal{H}$. If $T \in L(\mathcal{H})$ with closed range such that $\overline{R(TK)}$ is orthogonally complemented and K, T commutes with each other. Then $\{\{Tf_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a K-woven frame for $R(T)$.

Proof. Since T has closed range then T has Moore-Penrose inverse operator $T^{\dagger}$ such that $TT^{\dagger}T = T$ and $T^{\dagger}TT^{\dagger} = T^{\dagger}$. So $TT^{\dagger}|_{R(T)} = I_{R(T)}$ and $(TT^{\dagger})^* = I^* = I = TT^{\dagger}$. For every $f \in R(T)$, we have

$$\begin{aligned}\langle K^*f, K^*f\rangle &= \big\langle (TT^{\dagger})^*K^*f, (TT^{\dagger})^*K^*f\big\rangle \\ &= \big\langle T^{\dagger *}T^*K^*f, T^{\dagger *}T^*K^*f\big\rangle \\ &\leqslant \|(T^{\dagger})^*\|^2\big\langle T^*K^*f, T^*K^*f\big\rangle\end{aligned}$$

This implies that

$$\|(T^{\dagger})^*\|^{-2}\big\langle K^*f, K^*f\big\rangle \leqslant \big\langle T^*K^*f, T^*K^*f\big\rangle. \tag{13}$$

As $R(T^*K^*) \subset R(K^*T^*)$, by using Theorem 3.8, there exists some $\lambda' > 0$ such that

$$\big\langle T^*K^*f, T^*K^*f\big\rangle \leqslant \lambda'\big\langle K^*T^*f, K^*T^*f\big\rangle. \tag{14}$$

Since $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ is K-woven with universal bound A and B, we have

$$\begin{aligned}\sum_{i\in I}\sum_{j\in\sigma_i}\langle f, Tf_{ij}\rangle\langle Tf_{ij}, f\rangle &= \sum_{i\in I}\sum_{j\in\sigma_i}\langle T^*f, f_{ij}\rangle\langle f_{ij}, T^*f\rangle \\ &\geqslant A\big\langle K^*T^*f, K^*T^*f\big\rangle \\ &\geqslant \frac{A}{\lambda'}\langle T^*K^*f, T^*K^*f\rangle \quad \text{(using (14))} \\ &\geqslant \frac{A}{\lambda'}\|(T^{\dagger})^*\|^{-2}\langle K^*f, K^*f\rangle \quad \text{(using (13))}\end{aligned}$$

On the other hand by Lemma 3.2, $\{\{Tf_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven Bessel sequence. Hence, $\{\{Tf_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a K-woven frame for $R(T)$. ■

We need the following Theorem to prove our next result.

Theorem 3.18 [21] Let E be a Hilbert module, $A, B_1, B_2 \in L(E)$ and $R(B_1) + R(B_2)$ is closed. The following statements are equivalent.
(1) $R(A) \subset R(B_1) + R(B_2)$;
(2) $AA^* \leqslant \lambda(B_1B_1^* + B_2B_2^*)$;
(3) There exist $X, Y \in L(E)$ such that $A = B_1X + B_2Y$.

Theorem 3.19 Let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ and $\{\{g_{ij}\}_{j=1}^{\infty} : i \in I\}$ be two K-woven frame for $\mathcal{H}$. Let L_1 and L_2 be defined as $L_1, L_2 \colon l^2(\mathcal{A}) \to \mathcal{H}$, $L_1e_{ij} = f_{ij}$ and $L_2e_{ij} = g_{ij}$ and $R(K) \subseteq R(L_1)$, $R(K) \subseteq R(L_2)$, where $\{e_{ij}\}_{j\in\sigma_i, i\in I}$ is the canonical orthonormal basis for $l^2(\mathcal{A})$ and $\{\sigma_i\}_{i\in I}$ be any partition of $\mathbb{N}$. If $L_1L_2^*$ and $L_2L_1^*$ are positive operators and $R(L_1) + R(L_2)$ is closed, then $\{\{f_{ij} + g_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a K-woven for $\mathcal{H}$.

Proof. By the hypothesis we have $L_1e_{ij} = f_{ij}$, $L_2e_{ij} = g_{ij}$, $R(K) \subseteq R(L_1)$ and $R(K) \subseteq R(L_2)$. So $R(K) \subseteq R(L_1)+R(L_2)$, and by Theorem 3.18, we have $KK^* \leqslant \lambda(L_1L_1^*+L_2L_2^*)$ for some $\lambda > 0$.

Now, let $\{\sigma_i\}_{i\in I}$ be any partition of $\mathbb{N}$.

$$\begin{aligned}
\sum_{i\in I}\sum_{j\in\sigma_i}\big\langle f, f_{ij}+g_{ij}\big\rangle\big\langle f_{ij}+g_{ij}, f\big\rangle &= \sum_{i\in I}\sum_{j\in\sigma_i}\big\langle f, L_1e_{ij}+L_2e_{ij}\big\rangle\big\langle L_1e_{ij}+L_2e_{ij}, f\big\rangle\\
&= \sum_{i\in I}\sum_{j\in\sigma_i}\big\langle f, (L_1+L_2)e_{ij}\big\rangle\big\langle (L_1+L_2)e_{ij}, f\big\rangle\\
&= \sum_{i\in I}\sum_{j\in\sigma_i}\big\langle (L_1+L_2)^*f, e_{ij}\big\rangle\big\langle e_{ij}, (L_1+L_2)^*f\big\rangle\\
&= \big\langle (L_1+L_2)^*f, (L_1+L_2)^*f\big\rangle\\
&= \big\langle (L_1+L_2)(L_1+L_2)^*f, f\big\rangle\\
&= \big\langle (L_1+L_2)(L_1^*+L_2^*)f, f\big\rangle\\
&= \big\langle (L_1L_1^*+L_1L_2^*+L_2L_1^*+L_2L_2^*)f, f\big\rangle\\
&\geqslant \big\langle L_1L_1^*+L_2L_2^*f, f\big\rangle\\
&\quad \text{(since } L_1L_2^* \text{ and } L_2L_1^* \text{ are positive operators)}\\
&\geqslant \frac{1}{\lambda}\big\langle KK^*f, f\big\rangle\\
&= \frac{1}{\lambda}\big\langle K^*f, K^*f\big\rangle.
\end{aligned}$$

For the upper bound, let $\{\{f_{ij}\}_{j=1}^{\infty} : i \in I\}$ and $\{\{g_{ij}\}_{j=1}^{\infty} : i \in I\}$ be two woven Bessel sequence with Bessel bound B_1 and B_2. Then it is easy to see that $\{\{f_{ij}+g_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a woven Bessel sequence with bound B_1+B_2. And hence, $\{\{f_{ij}+g_{ij}\}_{j=1}^{\infty} : i \in I\}$ is a K-woven for $\mathcal{H}$. ■

Theorem 3.20 For $i \in I$, let $F_i = \{f_{ij}\}_{j=1}^{\infty}$ be a K-frame for $\mathcal{H}$ with bounds A_i and B_i. For any $\sigma \subset \mathbb{N}$ and a fix $t \in I$, let $P_i^{\sigma}(f) = \sum_{j\in\sigma}\langle f, f_{ij}\rangle f_{ij} - \sum_{j\in\sigma}\langle f, f_{tj}\rangle f_{tj}$ for $i \neq t$. If P_i^{σ} is a positive linear operator, then the family of K-frames $\{F_i\}_{i\in I}$ is K-woven.

Proof. Let $\{\sigma_i\}_{i\in I}$ be any partition of $\mathbb{N}$. Then, for every $f \in \mathcal{H}$, a fix $t \in I$ and $j \in \sigma_i$, we have

$$\begin{aligned}
\sum_{j\in\sigma_i}\langle f, f_{tj}\rangle\langle f_{tj}, f\rangle &= \Big\langle \sum_{j\in\sigma_i}\langle f, f_{tj}\rangle f_{tj}, f\Big\rangle\\
&= \Big\langle \sum_{j\in\sigma_i}\langle f, f_{ij}\rangle f_{ij} - P_i^{\sigma}(f), f\Big\rangle\\
&\leqslant \Big\langle \sum_{j\in\sigma_i}\langle f, f_{ij}\rangle f_{ij}, f\Big\rangle \quad \text{(As } P_i^{\sigma} \text{ is a positive linear operator)}\\
&= \sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle. \qquad (15)
\end{aligned}$$

Now, using (15), we have

$$
\begin{aligned}
A_t\langle K^*f, K^*f\rangle &\leqslant \sum_{j\in J}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle \\
&= \sum_{j\in\sigma_1}\langle f, f_{tj}\rangle\langle f_{tj}, f\rangle + \cdots + \sum_{j\in\sigma_i}\langle f, f_{tj}\rangle\langle f_{tj}, f\rangle + \cdots + \sum_{j\in\sigma_m}\langle f, f_{tj}\rangle\langle f_{tj}, f\rangle \\
&\leqslant \sum_{j\in\sigma_1}\langle f, f_{1j}\rangle\langle f_{1j}, f\rangle + \cdots + \sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle + \cdots + \sum_{j\in\sigma_m}\langle f, f_{mj}\rangle\langle f_{mj}, f\rangle \\
&\leqslant (B_1 + \cdots + B_i + \ldots + B_m)\langle f, f\rangle = \sum_{i\in I} B_i\langle f, f\rangle,
\end{aligned}
$$

which implies

$$
A_t\langle K^*f, K^*f\rangle \leqslant \sum_{i\in I}\sum_{j\in\sigma_i}\langle f, f_{ij}\rangle\langle f_{ij}, f\rangle \leqslant \sum_{i\in I} B_i\langle f, f\rangle.
$$

■

References

[1] M. B. Asadi, Z. Hassanpour-Yakhdani, Algebra-valued G-Frames in Hilbert C^*-modules, Operators and Matrices. 13 (1) (2019), 197-201.

[2] T. Bemrose, P. G. Casazza, Karlheinz Gröchenig, C. Mark Lammers, R. G. Lynch, Weaving frames, Operator and Matrices. 10 (4) (2016), 1093-1116.

[3] H. Bolcskei, F. Hlawatsch, H. G. Feichtinger, Frame-theoretic analysis of oversampled filter banks, IEEE Trans. Signal Process. 46 (12) (1998), 3256-3268.

[4] E. J. Candes, D. L. Donoho, New tight frames of curvelets and optimal representations of objects with piecewise C^2 singularities, Commun. Pure. Appl. Math. 57 (2) (2004), 219-266.

[5] P. G. Casazza, D. Freeman, R. G. Lynch, Weaving schauder frames, J. Approximation Theory. 211 (2016), 42-60.

[6] P. G. Casazza, G. Richard Lynch, Weaving Properties of Hilbert Space Frames, International Conference on Sampling Theory and Applications, IEEE, (2015), 110-114.

[7] I. Daubechies, A. Grossmann, Y. Meyer, Painless nonorthogonal expansions, J. Math. Phys. 27 (1986), 1271-1283.

[8] -. Deepshikha, L. K. Vashisht, Weaving K-frames in Hilbert spaces, Results Math. 73 (2018), 1-20.

[9] R. J. Duffin, A. C. Schaeffer, A class of nonharmonic Fourier series, Trans. Math. Soc. 72 (1952), 341-366.

[10] Y. C. Eldar, Sampling with arbitrary sampling and reconstruction spaces and oblique dual frame vectors, J. Fourier. Anal. Appl. 9 (1) (2003), 77-96.

[11] Y. C. Eldar, T. Werther, General framework for consistent sampling in Hilbert spaces, Int. J. Walvelets Multi. Inf. Process. 3 (4) (2005), 497-509.

[12] X. Fang, J. Yu, H. Yao, Solutions to operator equations on Hilbert C^*-modules, Linear Algebra. Appl. 431 (11) (2009), 2142-2153.

[13] P. J. S. G. Ferreira, Mathematics for Multimedia Signal Processing II: Discrete Finite Frames and Signal Reconstruction, Signals Processing for Multimedia, 1999.

[14] M. Frank, D. R. Larson, Frames in Hilbert C^*-modules and C^*-algebras, J. Operator Theory, 48 (2002), 273-314.

[15] L. Găvruta, Frames for operators, Appl. Comput. Harmon. Anal. 32 (2012), 139-144.

[16] F. Ghobadzadeh, A. Najati, G. A. Anastassiou, C. Park, Woven frames in Hilbert C^*-module spaces, J. Comput. Anal. Appl. 25 (2018), 1220-1232.

[17] W. Jing, Frames in Hilbert C^*-Modules, Ph.D. Thesis, University of Central Florida, 2006.

[18] N. Liu, W. Luo, Q. Xu, The polar decomposition for adjointable operators on Hilbert C^*-modules and n-centered operators, Banach J. Math. Anal. 13 (3) (2019), 627-646.

[19] A. Najati, M. M. Saem, P. Găvruta, Frames and operators in Hilbert C^*-modules, Operators and Matrices. 10 (1) (2016), 73-81.

[20] T. Strohmer, R. Jr. Heath, Grassmanian frames with applications to coding and communications, Appl. Comput. Harmon. Anal. 14 (2003), 257-275.

[21] G. A. Tabadkan, A. A. Arefijamaal, M. Mahmoudieh, K-frames in Hilbert C^*-modules, arXiv: 1802.01947.

[22] A. Touri, M. Rossafi, H. Labrigui, A. Akhlidj, Continuous $*$-Kg-Frame in Hilbert C^*-Modules, arXiv:1902.05970.