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Hyperbolic partial differential equations from the point of view of shearlet transforms

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Abstract. It is shown that a nonlinear partial differential equation with complex coefficients can be obtained through continuous shearlet transforms. We introduce a shearlet for constructing a nonlinear partial differential equation with complex coefficients and show that the initial value obtained from the continuous shearlet transform coincides with solutions of this equation.

Keywords: Continuous shearlet system, hyperbolic partial differential equation, complex coefficients.

2010 AMS Subject Classification: 42C15, 42C40.

1. Introduction and Preliminaries

Partial differential equations (PDE's) with complex coefficients have been established as a powerful tool in several areas of mathematics, from harmonic analysis to geometry, physics, electromagnetic waves and quantum mechanical systems. In recent years, numerical methods have been applied to solve a class of nonlinear PDE's. Nonlinear PDE's play an important role in the study of many problems in large varieties of physical, chemical, geometrical and biological phenomena [9].

Hyperbolic partial differential equations are widely applied in many fields of science and engineering such as fluid dynamics and aerodynamics, the theory of elasticity, optics, electromagnetic waves, direct and inverse scattering and the general theory of relativity [12]. One of the noteworthy topics in which hyperbolic partial differential equations are

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used, is the global optimal scheduling of unmanned aerial vehicle navigation channel. Wave equations as a well-known type of hyperbolic partial differential equations have been investigated by many authors, e.g. [1, 13], whereas in this paper, we present another type of hyperbolic partial differential equations with its solutions.

One of the important issues in PDEs, which is used in image processing, is to find an equation via its solution. In particular, an effective technique for processing irregular signals and images is the smoothing them via convolution with a suitable filter. Most practical devices for image analysis assume a preliminary step in the processing of the pictures. This step consists in passing from the original picture to smoothed versions via convolution, which still contain significant information. More obviously, we define a "multiscale analysis" to be a family of transforms $(T_t)_{t\geqslant 0}$ which, when applied to the original picture $f(x)$, yield a sequence of pictures $u(t,x) = (T_t f)(x)$. $T_t f$ is a semi-local version of f where a neighborhood of t around x has been scanned for determining the value of $T_t f(x)$. In accordance with the foregoing assumptions on a multiscale analysis T_t, all sequences of pictures $u(t,x) = (T_t f)(x)$ are solutions of a partial differential equation of second order $\frac{\partial u}{\partial t} = F(D^2 u, Du, t)$ with $u(0,x) = f(x)$.

In the past, the continuous wavelet transform was used for solving various PDE's like wave equations [8]. In recent times, several new representation systems, including the complex wavelets [4], the ridgelets, the curvelets [7] and etclets [2, 6], are proposed to solve the PDE's. These representation systems often give rise to numerical solutions of PDE's. Shearlets are almost new representation systems that are equipped with a rich mathematical structure similar to wavelets. In fact, theory and algorithms of shearlets can be carried over the continuous wavelet transform. The continuous shearlet transform is based on special affine systems generated by one single function $\psi \in L^2(\mathbb{R}^2)$. Moreover, compared with wavelets, the continuous shearlet transform has a coherent matrix structure for n-dimensions so that it is useful for solving the higher dimensional PDE's [3, 5]. In general, it is not easy to achieve initial values for the equation like (8) through analytical methods and they often are computed numerically. In [11], Postnikov with the aid of wavelets, obtained an initial value for the simpler kind of PDE which is well-studied. Using a proper shearlet, we gain an initial value for more complicated PDE's with complex coefficients (e.g. equation (8)). Changing the function in the shearlet transform, we obtain various initial values for the equation defined in (8), (see example 2.1 below).

The paper is organized as follows. In the rest of this section, we recall required notation and definitions about shearlets and linear partial differential equation of order 2. In Section 2, we present a new shearlet and attain a PDE via this shearlet. We represent two examples to obtain various initial values of this PDE in Section 3, to illustrate our approach.

We commence recalling some notation and definitions on shearlets. For $\psi \in L^2(\mathbb{R}^n)$, we take the shearlet family $\{\psi_{a,s,b}(\cdot)\}_{a,s,b}$ by $\psi_{a,s,b}(\cdot) = |\det A_a|^{-\frac{1}{2}}\psi(A_a^{-1}S_s^{-1}(\cdot - b))$, where $a \in \mathbb{R}\setminus\{0\}, s \in \mathbb{R}^{n-1}, b \in \mathbb{R}^n$ and $A_a = \begin{bmatrix} a & 0_{n-1}^T \\ 0_{n-1} & \operatorname{sgn}(a)|a|^{(1/n)} I_{n-1}\end{bmatrix}$, $S_s = \begin{bmatrix} 1 & s^T \\ 0_{n-1} & I_{n-1}\end{bmatrix}$. Especially, let $\psi \in L^2(\mathbb{R}^2)$. Shearlets in 2-dimensions are given by

$$\psi_{a,s,t,t'}(\cdot) = a^{-\frac{3}{4}}\psi(A_a^{-1}S_s^{-1}(\cdot - (t,t'))), \tag{1}$$

where $a \in \mathbb{R}^+, s \in \mathbb{R}, (t,t') \in \mathbb{R}^2$ and $A_a = \begin{bmatrix} a & 0 \\ 0 & \sqrt{a}\end{bmatrix}$, $S_s = \begin{bmatrix} 1 & s \\ 0 & 1\end{bmatrix}$, and ψ is admissible in the sense that $\int_{\mathbb{R}^2} \frac{|\hat{\psi}(\xi_1,\xi_2)|^2}{|\xi_1|^2} d\xi_2 d\xi_1 < \infty$.

The continuous shearlet transform is the mapping

$$f \to SH_\psi f(a,s,t,t') = \langle \psi_{a,s,t,t'}, f \rangle. \tag{2}$$

For more details about shearlets, see [5]. Furthermore, the general linear partial differential equation of order 2 in two independent variables has the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G, \tag{3}$$

where A, B, C, D, E, F and G may depend on x and y (but not u) [10]. If a second-order equation with independent variables x and y does not have the form of (3), it is nonlinear. If $G = 0$, the equation is homogeneous; otherwise it is nonhomogeneous. The equation (3) is

(i) elliptic if $B^2 - 4AC < 0$,
(ii) hyperbolic if $B^2 - 4AC > 0$,
(iii) parabolic if $B^2 - 4AC = 0$.

The hyperbolic partial differential equations are usually introduced by the equations which have the coefficients $A, C \neq 0$ and $B = 0$, for example the wave equation. Here, we consider an equation with $A, C = 0$ and $B \neq 0$. It may be claimed that we have created a different sample of the hyperbolic partial differential equations with its solution.

2. Main result: PDE's based on shearlet transforms

To construct suitable shearlets, we introduce a continuous function $\psi \in L^2(\mathbb{R}^2)$ which is admissible. Consider

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2\pi i}} \cdot \frac{e^{-ix_1} - e^{-2ix_1}}{x_1} \cdot e^{-x_2^2},$$

(see fig. 1). Then according to (1), the shearlets will be given by

$$\begin{aligned}
\psi_{a,s,t,t'}(x_1,x_2) &= a^{-\frac{3}{4}}\psi(A_a^{-1}S_s^{-1}((x_1,x_2)-(t,t'))) \\
&= a^{-\frac{3}{4}}\psi(\frac{(x_1-t)-s(x_2-t')}{a}, \frac{x_2-t'}{\sqrt{a}}) \\
&= \frac{a^{-\frac{3}{4}}}{\sqrt{2\pi i}} \cdot \frac{e^{-i(\frac{(x_1-t)-s(x_2-t')}{a})} - e^{-2i(\frac{(x_1-t)-s(x_2-t')}{a})}}{(\frac{(x_1-t)-s(x_2-t')}{a})} \cdot e^{-(\frac{x_2-t'}{\sqrt{a}})^2} \\
&= \frac{a^{\frac{1}{4}}}{\sqrt{2\pi i}} \cdot \frac{e^{-i(\frac{(x_1-t)-s(x_2-t')}{a})} - e^{-2i(\frac{(x_1-t)-s(x_2-t')}{a})}}{(x_1-t)-s(x_2-t')} \cdot e^{-(\frac{x_2-t'}{\sqrt{a}})^2} \\
&= \frac{a^{\frac{1}{4}}}{\sqrt{2\pi i}} \left[\frac{e^{-i(\frac{(x_1-t)-s(x_2-t')}{a})}}{(x_1-t)-s(x_2-t')} - \frac{e^{-2i(\frac{(x_1-t)-s(x_2-t')}{a})}}{(x_1-t)-s(x_2-t')} \right] e^{-(\frac{x_2-t'}{\sqrt{a}})^2} \\
&= \psi_1(y_1,y_2) - \psi_2(y_1,y_2),
\end{aligned} \tag{4}$$

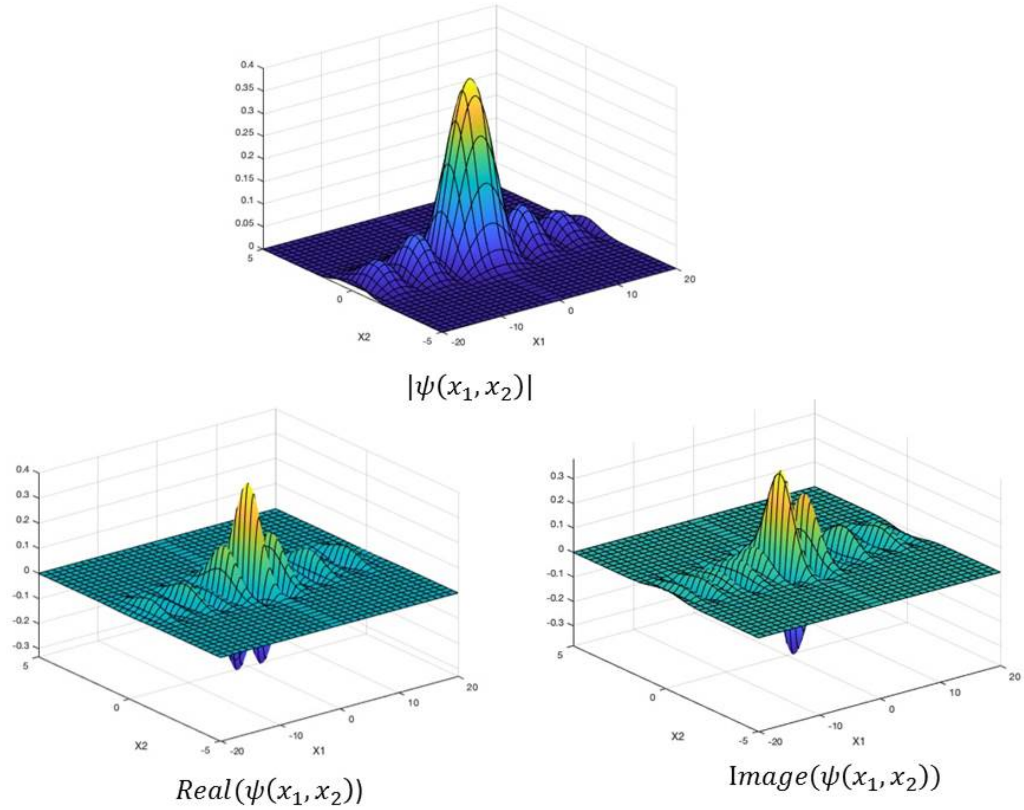

Figure 1.

where $y_1 = \frac{x_1 - t}{a}$, $y_2 = \frac{x_2 - t'}{a}$ and

$$\psi_1(y_1, y_2) = \frac{a^{-\frac{3}{4}}}{\sqrt{2\pi}i} \frac{e^{-i(y_1 - sy_2)}}{y_1 - sy_2} e^{-ay_2^2}, \quad \psi_2(y_1, y_2) = \frac{a^{-\frac{3}{4}}}{\sqrt{2\pi}i} \frac{e^{-2i(y_1 - sy_2)}}{y_1 - sy_2} e^{-ay_2^2}.$$

By the definition of continuous shearlet transforms (2) and (4), we have

$$\begin{aligned} SH_\psi f(a, s, t, t') &= \int_{\mathbb{R}^2} \overline{f}(x_1, x_2) \psi_{a,s,t,t'}(x_1, x_2) dx_1 dx_2 \\ &= \int_{\mathbb{R}^2} \overline{f}(x_1, x_2)(\psi_1(y_1, y_2) - \psi_2(y_1, y_2)) dx_1 dx_2 \\ &= W_1 - W_2, \end{aligned} \tag{5}$$

in which

$$W_1 = \int_{\mathbb{R}^2} \overline{f}(x_1, x_2) \psi_1(y_1, y_2) dx_1 dx_2, \tag{6}$$

$$W_2 = \int_{\mathbb{R}^2} \overline{f}(x_1, x_2) \psi_2(y_1, y_2) dx_1 dx_2. \tag{7}$$

First, we consider $s, t' = 0$. Differentiating W_1 with respect to t and a yields

$$\frac{\partial W_1}{\partial t} = \int_{\mathbb{R}^2} \overline{f}(x_1,x_2)\psi_1(y_1,y_2)\left(\frac{i}{a}+\frac{1}{(x_1-t)}\right)dx_1dx_2,$$

$$\frac{\partial W_1}{\partial a} = \int_{\mathbb{R}^2} \overline{f}(x_1,x_2)\psi_1(y_1,y_2)\left(\frac{1}{4a}+\frac{i(x_1-t)}{a^2}+\frac{x_2^2}{a^2}\right)dx_1dx_2,$$

$$\frac{\partial^2 W_1}{\partial t\partial a} = \int_{\mathbb{R}^2} \overline{f}(x_1,x_2)\psi_1(y_1,y_2)\left(\frac{i}{4a^2}+\frac{1}{4a(x_1-t)}+\frac{ix_2^2}{a^3}-\frac{(x_1-t)}{a^3}+\frac{x_2^2}{a^2(x_1-t)}\right)dx_1dx_2.$$

Then we have

$$\begin{aligned}&\frac{\partial^2 W_1}{\partial t\partial a}-\frac{5}{4a}\frac{\partial W_1}{\partial t}-\frac{i}{a}\frac{\partial W_1}{\partial a}+\frac{5i}{4a^2}W_1\\ &\qquad= \int_{\mathbb{R}^2} \overline{f}(x_1,x_2)\psi_1(y_1,y_2)(\frac{-1}{a(x_1-t)}+\frac{x_2^2}{a^2(x_1-t)})dx_1dx_2.\end{aligned} \tag{8}$$

Put $f(x_1,x_2) = f_1(x_1).f_2(x_2)$ for an arbitrary $f_1 \in L^2(\mathbb{R}), f_2(x_2) = 2x_2$ for $x_2 > 0$. Hence

$$\int_{\mathbb{R}^2} \overline{f}(x_1,x_2)\psi_1(y_1,y_2)(\frac{-1}{a(x_1-t)}+\frac{x_2^2}{a^2(x_1-t)})dx_1dx_2$$

$$\begin{aligned}&= \int_{\mathbb{R}} \overline{f_1}(x_1)\frac{a^{\frac14}}{\sqrt{2\pi}i}\frac{e^{-i(x_1-t)}}{x_1-t}\left(\frac{-1}{a(x_1-t)}\right)dx_1\int_0^\infty 2x_2e^{-\frac{x_2^2}{a}}dx_2\\ &+\int_{\mathbb{R}} \overline{f_1}(x_1)\frac{a^{\frac14}}{\sqrt{2\pi}i}\frac{e^{-i(x_1-t)}}{x_1-t}\left(\frac{1}{a^2(x_1-t)}\right)dx_1\int_0^\infty 2x_2^3e^{-\frac{x_2^2}{a}}dx_2\\ &= \int_{\mathbb{R}} \overline{f_1}(x_1)\frac{a^{\frac14}}{\sqrt{2\pi}i}\frac{e^{-i(x_1-t)}}{x_1-t}\left(\frac{-1}{a(x_1-t)}\right)(a)dx_1\\ &+\int_{\mathbb{R}} \overline{f_1}(x_1)\frac{a^{\frac14}}{\sqrt{2\pi}i}\frac{e^{-i(x_1-t)}}{x_1-t}\left(\frac{1}{a^2(x_1-t)}\right)(a^2)dx_1 = 0.\end{aligned}$$

By the way, (8) is a hyperbolic partial differential equation with complex coefficients.

In the following example, we calculate two initial values for the equation (8).

***Example* 2.1** First of all, we consider the function f in (6) as

$$f(x_1,x_2) = e^{ix_1}.2x_2, \qquad x_2 \geqslant 0,$$

in which $a > 1$ (see fig. 2).

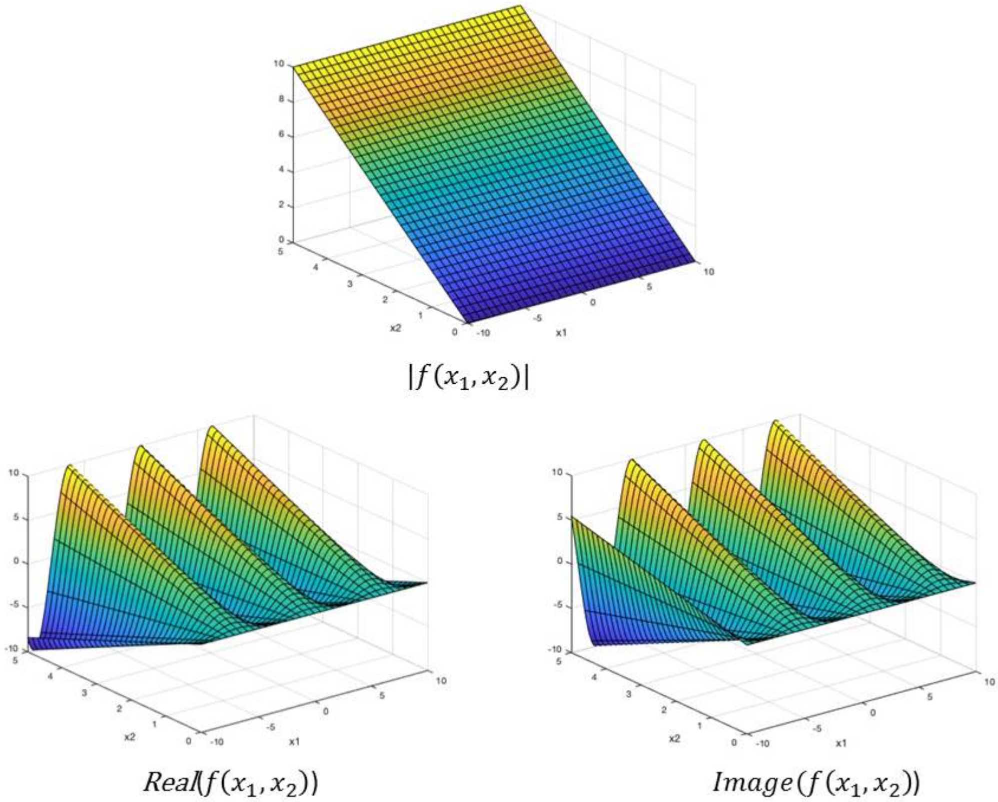

Figure 2.

Using the Cauchy principal value sense, we have

$$
\begin{aligned}
W_1(t,a) &= \int_{\mathbb{R}^2} f(x_1,x_2).\psi_1(y_1,y_2)dx_1dx_2 \\
&= \int_{\mathbb{R}^2} f(x_1,x_2).\psi_1(\frac{x_1-t}{a},\frac{x_2}{a})dx_1dx_2 \\
&= \frac{1}{\sqrt{2\pi}ia^{\frac{3}{4}}} \int_{\mathbb{R}}\int_{\mathbb{R}} f(x_1,x_2).\frac{e^{-i(\frac{x_1-t}{a})}}{\frac{x_1-t}{a}}.e^{-\frac{x_2^2}{a}}dx_1dx_2 \qquad (9) \\
&= \frac{a^{\frac{1}{4}}}{\sqrt{2\pi}i}\left(\int_0^\infty 2x_2e^{-\frac{x_2^2}{a}}dx_2\right)\left(\text{P.V.}\int_{-\infty}^{+\infty} e^{ix_1}\frac{e^{-i(\frac{x_1-t}{a})}}{x_1-t}dx_1\right) \\
&= a^{\frac{5}{4}}\sqrt{\frac{\pi}{2}}\ \ e^{it}.
\end{aligned}
$$

According to (9), we have the initial value of W_1, as

$$W_1(t_0,a_0) = a_0^{\frac{5}{4}}\sqrt{\frac{\pi}{2}}\ \ e^{it_0}. \qquad (10)$$

We obtain the partial derivatives of $W_1(t,a)$ with respect to a and t, as follows

$$\frac{\partial W_1}{\partial t} = a^{\frac{5}{4}}\sqrt{\frac{\pi}{2}}ie^{it}, \qquad \frac{\partial W_1}{\partial a} = \frac{5}{4}a^{\frac{1}{4}}\sqrt{\frac{\pi}{2}}e^{it}, \qquad \frac{\partial^2 W_1}{\partial a \partial t} = \frac{5}{4}a^{\frac{1}{4}}\sqrt{\frac{\pi}{2}}ie^{it}. \quad (11)$$

By substituting (11) in (8), we have

$$\begin{aligned}\frac{\partial^2 W_1}{\partial t \partial a} &- \frac{5}{4a}\frac{\partial W_1}{\partial t} - \frac{i}{a}\frac{\partial W_1}{\partial a} + \frac{5i}{4a^2}W_1 \\ &= \tfrac{5}{4}a^{\frac{1}{4}}\sqrt{\tfrac{\pi}{2}}ie^{it} - \tfrac{5}{4a}a^{\frac{5}{4}}\sqrt{\tfrac{\pi}{2}}ie^{it} - \tfrac{i}{a}\tfrac{5}{4}a^{\frac{1}{4}}\sqrt{\tfrac{\pi}{2}}e^{it} + \tfrac{5i}{4a^2}a^{\frac{5}{4}}\sqrt{\tfrac{\pi}{2}}\ e^{it} \\ &= 0.\end{aligned} \quad (12)$$

Hence, (10) is one of the solutions of (8).

To find another initial value for (8), we put f in (6) as $f(x_1, x_2) = \chi_{[0,1]}(x_1).2x_2$ with $x_2 \geqslant 0$, (see fig. 3). Then

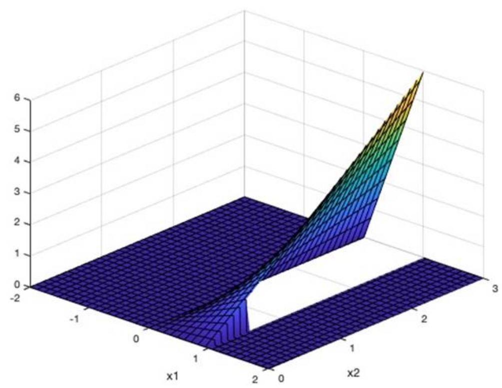

Figure 3.

$$\begin{aligned}W_1(t,a) &= \int_{\mathbb{R}^2} f(x_1,x_2).\psi_1(\frac{x_1 - t}{a}, \frac{x_2}{a})dx_1dx_2 \\ &= \frac{1}{\sqrt{2\pi}ia^{\frac{3}{4}}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x_1,x_2).\frac{e^{-i(\frac{x_1-t}{a})}}{\frac{x_1-t}{a}}.e^{-\frac{x_2^2}{a}}dx_1dx_2 \\ &= \frac{a^{\frac{1}{4}}}{\sqrt{2\pi}i}\int_0^\infty 2x_2e^{-\frac{x_2^2}{a}}dx_2\int_0^1 \frac{e^{-i(\frac{x_1-t}{a})}}{x_1 - t}dx_1 \\ &= \frac{a^{\frac{5}{4}}}{\sqrt{2\pi}i}[\Gamma(0, \frac{ti}{a}) + \Gamma(0, \frac{(1-t)i}{a})],\end{aligned} \quad (13)$$

where Γ is the gamma function. According to (13), we have the initial value of W_1, as

$$W_1(t_0, a_0) = \frac{a_0^{\frac{5}{4}}}{\sqrt{2\pi i}}[\Gamma(0, \frac{t_0 i}{a_0}) + \Gamma(0, \frac{(1-t_0)i}{a_0})]. \tag{14}$$

Differentiating W_1 in (13) with respect to a, t, we obtain

$$\begin{aligned}
\frac{\partial W_1}{\partial t} &= \frac{a^{\frac{5}{4}}}{\sqrt{2\pi i}}\left(\frac{e^{\frac{it}{a}}}{t} + \frac{e^{\frac{-(1-t)i}{a}}}{1-t}\right), \\
\frac{\partial^2 W_1}{\partial a \partial t} &= \frac{1}{\sqrt{2\pi i}}\left[\frac{5}{4}a^{\frac{1}{4}}\left(\frac{e^{\frac{it}{a}}}{t} + \frac{e^{\frac{-(1-t)i}{a}}}{1-t}\right) + a^{\frac{5}{4}}\left(-\frac{it}{a^2}\frac{e^{\frac{it}{a}}}{t} + \frac{(1-t)i}{a^2}\frac{e^{\frac{-(1-t)i}{a}}}{1-t}\right)\right], \\
\frac{\partial^2 W_1}{\partial t^2} &= \frac{a^{\frac{5}{4}}}{\sqrt{2\pi i}}\left(\frac{ie^{\frac{it}{a}}}{at} - \frac{e^{\frac{it}{a}}}{t^2} + \frac{ie^{\frac{-(1-t)i}{a}}}{a(1-t)} + \frac{e^{\frac{-(1-t)i}{a}}}{(1-t)^2}\right), \\
\frac{\partial^3 W_1}{\partial a \partial t^2} &= \frac{a^{\frac{1}{4}}}{\sqrt{2\pi i}}\left[e^{\frac{it}{a}}\left(-\frac{5}{4}\frac{1}{t^2} + \frac{5}{4}\frac{i}{at} + \frac{1}{a^2}\right)\right. \\
&\qquad \left. + e^{\frac{-(1-t)i}{a}}\left(\frac{5}{4}\frac{1}{(1-t)^2} + \frac{5}{4}\frac{i}{a(1-t)} - \frac{1}{a^2}\right)\right].
\end{aligned} \tag{15}$$

By substituting (15) in (8), we have

$$\frac{\partial^3 W_1}{\partial t^2 \partial a} - \frac{5}{4a}\frac{\partial^2 W_1}{\partial t^2} - \frac{i}{a}\frac{\partial^2 W_1}{\partial t \partial a} + \frac{5i}{4a^2}\frac{\partial W_1}{\partial t} = 0. \tag{16}$$

Hence, (14) is one of the solutions of (8).

Remark 1 *In the same way as in the above example, we obtain an equation for W_2 in (7), then choosing suitable $f_1(x), f_2(x)$, we get*

$$\frac{\partial^3 W_2}{\partial t^2 \partial a} - \frac{5}{4}\frac{1}{a}\frac{\partial^2 W_2}{\partial t^2} - \frac{i}{a}\frac{\partial^2 W_2}{\partial t \partial a} + \frac{5}{4}\frac{i}{a^2}\frac{\partial W_2}{\partial t} = 0. \tag{17}$$

3. Conclusion

In this paper, we constructed a proper shearlet and its continuous shearlet transform to obtain a solution of PDE's like (16). Meanwhile, Changing f in (6), one may obtain various amounts for the right hand side of the equation (8), whereby some novel PDE's with their solutions can be acquired. One might also consider other shearlets, to develop several higher dimensional PDE's. Besides that, by derivation (6) with respect to the four parameters t, t', s, a, one can reach different PDE's.

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