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New estimates for the Berezin number and Berezin norm

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Abstract. This paper intends to establish several inequalities employing the Cartesian decomposition of the operator. We used the results to determine the Berezin number inequalities. Our results extend and improve some earlier inequalities.

Keywords: Berezin number, Berezin norm, inner product, Cartesian decomposition.

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1. Introduction

In a complex Hilbert space $\mathscr{H}$ with the inner product $\langle\cdot,\cdot\rangle$, we denote the C^*-algebra of all bounded linear operators on $\mathscr{H}$ as $\mathscr{B}(\mathscr{H})$. In the case when $\dim\mathscr{H}=n$, we identify $\mathscr{B}(\mathscr{H})$ with the matrix algebra $\mathscr{M}_n$ of all $n\times n$ matrices with entries in the complex field $\mathbb{C}$. For any $T\in\mathscr{B}(\mathscr{H})$, we can write $T=A+\mathrm{i}B$ in which $A=\Re T=\frac{T+T^*}{2}$ and $B=\Im T=\frac{T-T^*}{2\mathrm{i}}$ are self-adjoint operators. This is the so-called Cartesian decomposition of T. For any $T\in\mathscr{B}(\mathscr{H})$, we can determine its numerical radius and the operator norm, respectively illustrated by $\omega(T)=\sup_{\|x\|=1}|\langle Tx,x\rangle|$ and $\|T\|=\sup_{\|x\|=1}\|Tx\|$. Two meaningful inequalities for the usual operator norm and numerical radius are that

$$\|T^n\|\le\|T\|^n \text{ and } \omega\left(T^n\right)\le\omega^n\left(T\right);\ n=1,2,\ldots.$$

If T is normal, indicating $T^*T=TT^*$, it is widely known that $\omega(T)=\|T\|$. However, this equality fails for non-normal operators. Instead, we can establish the following inequality

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for any $T \in \mathscr{B}(\mathscr{H})$:

$$\frac{1}{2}\|T\| \leqslant \omega(T) \leqslant \|T\|. \tag{1}$$

This inequality is essential because it approximates the numerical radius $\omega(T)$ in terms of the more computationally manageable quantity $\|T\|$.

As a result, researchers have been concentrating on sharpening this and other inequalities for the numerical radius, as found in [1, 14, 16–18, 21, 23]. Below, we list some results concerning the inequality (1).

Kittaneh [20, Theorem 1] proposed an improvement of (1) in the following manner:

$$\frac{1}{4}\left\||T|^2+|T^*|^2\right\| \le \omega^2(T) \le \frac{1}{2}\left\||T|^2+|T^*|^2\right\|.$$

In [18, Corollary 3.4], the previouse inequality was improved as follows:

$$\omega(T) \le \frac{1}{2}\sqrt{\left\||T|^2+|T^*|^2\right\| + 2\omega(|T|\,|T^*|)}. \tag{2}$$

After that, in [22, Corollary 2.8], inequality (2) was refined:

$$\omega(T) \le \frac{1}{2}\sqrt{\left\||T|^2+|T^*|^2\right\| + \|\,|T|\,|T^*| + |T^*|\,|T|\,\|}. \tag{3}$$

Inequality (3) can be written in the following setup:

$$\omega(T) \le \frac{1}{2}\sqrt{\left\||T|^2+|T^*|^2\right\| + 2\,\|\Re(|T|\,|T^*|)\|}.$$

Here, we point out that inequalities (2) and (3) have been established and generalized individually in [7] and [8].

A functional Hilbert space $\mathbb{H} = \mathbb{H}(\Omega)$ is a Hilbert space of complex-valued functions on a (nonempty) set Ω, which has the property that point evaluations are continuous, i.e., for each $\lambda \in \Omega$ the map $f \mapsto f(\lambda)$ is a continuous linear functional on $\mathbb{H}$. The Riesz representation theorem ensure that for each $\lambda \in \Omega$ there is a unique element $k_\lambda \in \mathbb{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathbb{H}$. The collection $\{k_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of $\mathbb{H}$. If $\{e_n\}$ is an orthonormal basis for a functional Hilbert space $\mathbb{H}$, then the reproducing kernel of $\mathbb{H}$ is given by $k_\lambda(z) = \sum_n \overline{e_n(\lambda)} e_n(z)$; (see [15, problem 37]). For $\lambda \in \Omega$, let $\hat{k_\lambda} = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathbb{H}$. For a bounded linear operator T on $\mathbb{H}$, the function $\widetilde{T}$ defined on Ω by $\widetilde{T}(\lambda) = \langle T\hat{k_\lambda}, \hat{k_\lambda} \rangle$ is the Berezin symbol of T, which firstly have been presented by Berezin [4, 5]. Berezin set and Berezin number of the operator, T, are determined by

$$\mathbf{Ber}(T) := \{\widetilde{T}(\lambda) : \lambda \in \Omega\} \qquad \text{and} \qquad \mathbf{ber}(T) := \sup\{|\widetilde{T}(\lambda)| : \lambda \in \Omega\},$$

respectively, (see [19]). Of course, the Berezin norm of T can also be defined as follows:

$$\|T\|_{\mathbf{ber}} = \sup\left\{\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right| : \ \lambda, \mu \in \Omega\right\}.$$

We understand that $\mathbf{ber}(T) \leq \omega(T)$. Moreover, the Berezin number and the Berezing norm of an operator T fulfills the following properties:

(P1) $\|\alpha T\|_{\mathbf{ber}} = |\alpha| \, \|T\|_{\mathbf{ber}}$ and $\mathbf{ber}(\alpha T) = |\alpha| \mathbf{ber}(T)$ for all $\alpha \in \mathbb{C}$.
(P2) $\mathbf{ber}(S+T) \leqslant \mathbf{ber}(S) + \mathbf{ber}(T)$.
(P3) $\|S+T\|_{\mathbf{ber}} \leq \|S\|_{\mathbf{ber}} + \|T\|_{\mathbf{ber}}$.
(P4) $\|T\|_{\mathbf{ber}} = \|T^*\|_{\mathbf{ber}}$ and $\mathbf{ber}\,(T) = \mathbf{ber}\,(T^*)$.
(P5) [9, Proposition 2.11] $\|T\|_{\mathbf{ber}} = \mathbf{ber}\,(T)$, whenever T is positive.

The Berezin symbol has been thoroughly examined for the Toeplitz and Hankel operators on the Hardy and Bergman spaces; it is broadly utilized in various analytical inquiries and exclusively characterizes the operator (i.e., for all $\lambda \in \Omega, \widetilde{T}(\lambda) = \widetilde{S}(\lambda)$ implies $T = S$).

The Berezin number inequalities have been investigated by many mathematicians over the years, the curious readers can see [2, 3, 9].

This paper desires to show considerable inequalities for inner products through the operator's Cartesian decomposition. The results are then used to determine the inequalities in the Berezin number. Furthermore, our research improves and generalizes earlier established inequalities.

In order to achieve these purposes, we will need the following facts:

(I) (Mixed Schwarz inequality [15, pp. 75–76]) For any $T \in \mathscr{B}(\mathscr{H})$ and $x, y \in \mathscr{H}$,

$$|\langle Tx, y\rangle|^2 \leq \left\langle |T|^{2\nu} x, x \right\rangle \left\langle |T^*|^{2(1-\nu)} y, y \right\rangle; \;\; (\nu \in [0,1]) \,. \tag{4}$$

(II) [12, (2.26)] For any $x, y, z \in \mathscr{H}$,

$$|\langle z, x\rangle|^2 + |\langle z, y\rangle|^2 \leq \|z\|^2 \max\left(\|x\|^2, \|y\|^2\right) + |\langle x, y\rangle| \,. \tag{5}$$

(III) (Buzano inequality [10]) For any $x, y, z \in \mathscr{H}$,

$$|\langle z, x\rangle| \, |\langle z, y\rangle| \leq \frac{\|z\|^2}{2} \left(|\langle x, y\rangle| + \|x\| \, \|y\|\right). \tag{6}$$

(IV) (Arithmetic-geometric mean inequality for the usual operator norm [6]) For any $S, T \in \mathscr{B}(\mathscr{H})$,

$$\|ST\| \leq \frac{1}{2} \left\| |S|^2 + |T^*|^2 \right\|. \tag{7}$$

2. Inner Product Inequalities

The following theorem suggests an upper bound for $\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right|$ using polar decomposition.

Theorem 2.1 Let $S, T \in \mathscr{B}\,(\mathscr{H})$. Then

$$\left|\left\langle (S + \mathrm{i}T)\,\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right|^2 \leq \max\left(\left\|S^*\widehat{k}_\mu\right\|^2, \left\|T^*\widehat{k}_\mu\right\|^2\right) + \left|\left\langle TS^*\widehat{k}_\mu, \widehat{k}_\mu \right\rangle\right| + 2\left|\left\langle S\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right| \left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right|,$$

for any vectors $\widehat{k}_\lambda, \widehat{k}_\mu \in \mathscr{H}$ with $\left\|\widehat{k}_\lambda\right\| = \left\|\widehat{k}_\mu\right\| = 1$. If $T \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $T = A + \mathrm{i}B$, then

$$\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|^2 \leq \max\left(\left\|A\widehat{k}_\mu\right\|^2, \left\|B\widehat{k}_\mu\right\|^2\right) + \left|\left\langle BA\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right| + 2\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|\left|\left\langle B\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|. \tag{8}$$

Proof. Letting $x = S^*\widehat{k}_\mu$, $y = T^*\widehat{k}_\mu$ and $z = \widehat{k}_\lambda$ with $\left\|\widehat{k}_\lambda\right\| = \left\|\widehat{k}_\mu\right\| = 1$, in (5), we obtain

$$\begin{aligned}\left|\left\langle S\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|^2 + \left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|^2 &= \left|\left\langle \widehat{k}_\lambda, S^*\widehat{k}_\mu\right\rangle\right|^2 + \left|\left\langle \widehat{k}_\lambda, T^*\widehat{k}_\mu\right\rangle\right|^2 \\ &\leq \max\left(\left\|S^*\widehat{k}_\mu\right\|^2, \left\|T^*\widehat{k}_\mu\right\|^2\right) + \left|\left\langle S^*\widehat{k}_\mu, T^*\widehat{k}_\mu\right\rangle\right|.\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\left\langle (S+T)\,\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|^2 \\ &= \left|\left\langle S\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle + \left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|^2 \\ &\leq \left(\left|\left\langle S\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right| + \left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|\right)^2 \quad \text{(by the triangle inequality)} \\ &= \left|\left\langle S\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|^2 + \left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|^2 + 2\left|\left\langle S\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right| \\ &\leq \max\left(\left\|S^*\widehat{k}_\mu\right\|^2, \left\|T^*\widehat{k}_\mu\right\|^2\right) + \left|\left\langle TS^*\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right| + 2\left|\left\langle S\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|,\end{aligned}$$

i.e.,

$$\left|\left\langle (S+T)\,\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|^2 \leq \max\left(\left\|S^*\widehat{k}_\mu\right\|^2, \left\|T^*\widehat{k}_\mu\right\|^2\right) + \left|\left\langle TS^*\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right| + 2\left|\left\langle S\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|. \tag{9}$$

We reach the desired inequality by substituting T by $\mathrm{i}T$ in the inequality (9). ∎

Inequality (8) can be stated in the following arrangement:

Corollary 2.2 Let $T \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $T = A + \mathrm{i}B$. Then

$$\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|^2 \leq \frac{1}{2}\left(\left\langle\left(|A|^2 + |B|^2\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle + \left|\left\langle\left(|A|^2 - |B|^2\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right|\right) + \left|\left\langle BA\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right| + 2\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|\left|\left\langle B\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|,$$

for any vectors $\widehat{k}_\lambda, \widehat{k}_\mu \in \mathscr{H}$ with $\left\|\widehat{k}_\lambda\right\| = \left\|\widehat{k}_\mu\right\| = 1$.

Proof. We have

$$\begin{aligned}
&\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|^2 \\
&\leq \max\left(\left\|A\widehat{k}_\mu\right\|^2, \left\|B\widehat{k}_\mu\right\|^2\right) + \left|\left\langle BA\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right| + 2\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|\left|\left\langle B\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right| \\
&= \frac{1}{2}\left(\left\|A\widehat{k}_\mu\right\|^2 + \left\|B\widehat{k}_\mu\right\|^2 + \left|\left\|A\widehat{k}_\mu\right\|^2 - \left\|B\widehat{k}_\mu\right\|^2\right|\right) + \left|\left\langle BA\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right| + 2\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|\left|\left\langle B\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right| \\
&= \frac{1}{2}\left(\left\langle |A|^2\widehat{k}_\mu, \widehat{k}_\mu\right\rangle + \left\langle |B|^2\widehat{k}_\mu, \widehat{k}_\mu\right\rangle + \left|\left\langle |A|^2\widehat{k}_\mu, \widehat{k}_\mu\right\rangle - \left\langle |B|^2\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right|\right) + \left|\left\langle BA\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right| + 2\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|\left|\left\langle B\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right| \\
&= \frac{1}{2}\left(\left\langle\left(|A|^2 + |B|^2\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle + \left|\left\langle\left(|A|^2 - |B|^2\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right|\right) + \left|\left\langle BA\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right| + 2\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|\left|\left\langle B\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|,
\end{aligned}$$

as desired. ∎

The next theorem delivers an upper bound for the product of two operators.

Theorem 2.3 Let $A, B \in \mathscr{B}(\mathscr{H})$. Then

$$\left|\left\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \leq \frac{1}{2}\left(\max\left(\left\||A|^2\widehat{k}_\lambda\right\|^2, \left\||B|^2\widehat{k}_\lambda\right\|^2\right) + \left|\left\langle |B|^2|A|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right),$$

for any vector $\widehat{k}_\lambda \in \mathscr{H}$ with $\left\|\widehat{k}_\lambda\right\| = 1$.

Proof. Taking $x = |A|^2\widehat{k}_\lambda$, $y = |B|^2\widehat{k}_\lambda$, and $z = \widehat{k}_\lambda$, in (5), we have

$$\left|\left\langle \widehat{k}_\lambda, |A|^2\widehat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle \widehat{k}_\lambda, |B|^2\widehat{k}_\lambda\right\rangle\right|^2 \leq \max\left(\left\||A|^2\widehat{k}_\lambda\right\|^2, \left\||B|^2\widehat{k}_\lambda\right\|^2\right) + \left|\left\langle |A|^2\widehat{k}_\lambda, |B|^2\widehat{k}_\lambda\right\rangle\right|. \tag{10}$$

So,

$$\begin{aligned}
2\left|\left\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 &= 2\left|\left\langle A\widehat{k}_\lambda, B\widehat{k}_\lambda\right\rangle\right|^2 \\
&\leq 2\left\|A\widehat{k}_\lambda\right\|^2\left\|B\widehat{k}_\lambda\right\|^2 \quad \text{(by the Cauchy-Schwarz inequality)} \\
&= 2\left\langle A\widehat{k}_\lambda, A\widehat{k}_\lambda\right\rangle\left\langle B\widehat{k}_\lambda, B\widehat{k}_\lambda\right\rangle \\
&= 2\left\langle A^*A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\left\langle B^*B\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\
&= 2\left\langle |A|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\left\langle |B|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\
&\leq \left\langle |A|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^2 + \left\langle |B|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^2 \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&= \left|\left\langle \widehat{k}_\lambda, |A|^2\widehat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle \widehat{k}_\lambda, |B|^2\widehat{k}_\lambda\right\rangle\right|^2 \\
&\leq \max\left(\left\||A|^2\widehat{k}_\lambda\right\|^2, \left\||B|^2\widehat{k}_\lambda\right\|^2\right) + \left|\left\langle |A|^2\widehat{k}_\lambda, |B|^2\widehat{k}_\lambda\right\rangle\right| \quad \text{(by (10))} \\
&= \max\left(\left\||A|^2\widehat{k}_\lambda\right\|^2, \left\||B|^2\widehat{k}_\lambda\right\|^2\right) + \left|\left\langle |B|^2|A|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|.
\end{aligned}$$

Accordingly,

$$\left|\left\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \leq \frac{1}{2}\left(\max\left(\left\||A|^2\widehat{k}_\lambda\right\|^2, \left\||B|^2\widehat{k}_\lambda\right\|^2\right) + \left|\left\langle |B|^2|A|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right),$$

as desired. ■

As a consequence of Theorem 2.3, we have:

Corollary 2.4 Let $T \in \mathscr{B}(\mathscr{H})$ and let $0 \leq \nu \leq 1$. Then

$$\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \leq \frac{1}{2}\left(\max\left(\left\||T|^{2\nu}\widehat{k}_\lambda\right\|^2, \left\||T^*|^{2(1-\nu)}\widehat{k}_\lambda\right\|^2\right) + \left|\left\langle |T^*|^{2(1-\nu)}|T|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right),$$

for any vector $\widehat{k}_\lambda \in \mathscr{H}$ with $\left\|\widehat{k}_\lambda\right\| = 1$.

Proof. Letting $B^* = U|T|^{1-\nu}$ and $A = |T|^\nu$, in Theorem 2.3, we get

$$\begin{aligned}
\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 &\leq \frac{1}{2}\left(\max\left(\left\||T|^{2\nu}\widehat{k}_\lambda\right\|^2, \left\|U|T|^{2(1-\nu)}U^*\widehat{k}_\lambda\right\|^2\right) + \left|\left\langle |T|^{2\nu}, U|T|^{2(1-\nu)}U^*\widehat{k}_\lambda\right\rangle\right|\right) \\
&= \frac{1}{2}\left(\max\left(\left\||T|^{2\nu}\widehat{k}_\lambda\right\|^2, \left\||T^*|^{2(1-\nu)}\widehat{k}_\lambda\right\|^2\right) + \left|\left\langle |T|^{2\nu}\widehat{k}_\lambda, |T^*|^{2(1-\nu)}\widehat{k}_\lambda\right\rangle\right|\right) \\
&\quad \text{(by [13, Theorem 4 (ii), p. 58])} \\
&= \frac{1}{2}\left(\max\left(\left\||T|^{2\nu}\widehat{k}_\lambda\right\|^2, \left\||T^*|^{2(1-\nu)}\widehat{k}_\lambda\right\|^2\right) + \left|\left\langle |T^*|^{2(1-\nu)}|T|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right),
\end{aligned}$$

as needed. ■

Next, we obtain another upper bound for $\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|$ using polar decompostion.

Theorem 2.5 Let $S, T \in \mathscr{B}(\mathscr{H})$. Then for any $0 \leq \nu \leq 1$,

$$\left|\left\langle (S + \mathrm{i}T)\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right| \leq \sqrt{\left\langle \left(|S|^{2\nu} + |T|^{2\nu}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle}\sqrt{\left\langle \left(|S^*|^{2(1-\nu)} + |T^*|^{2(1-\nu)}\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle},$$

for any vectors $\widehat{k}_\lambda, \widehat{k}_\mu \in \mathscr{H}$ with $\left\|\widehat{k}_\lambda\right\| = \left\|\widehat{k}_\mu\right\| = 1$. If $T \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $T = A + \mathrm{i}B$, then

$$\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right| \leq \sqrt{\left\langle \left(|A|^{2\nu} + |B|^{2\nu}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\langle \left(|A|^{2(1-\nu)} + |B|^{2(1-\nu)}\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle}.$$

Proof. Let $\widehat{k}_\lambda, \widehat{k}_\mu \in \mathscr{H}$ with $\left\|\widehat{k}_\lambda\right\| = \left\|\widehat{k}_\mu\right\| = 1$. Then

$$\begin{aligned}
\left|\left\langle (S + \mathrm{i}T)\,\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right| &= \left|\left\langle S\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle + \mathrm{i}\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right| \\
&\le \left|\left\langle S\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right| + \left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right| \quad \text{(by the triangle inequality)} \\
&\le \sqrt{\left\langle |S|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\langle |S^*|^{2(1-\nu)}\widehat{k}_\mu, \widehat{k}_\mu\right\rangle} + \sqrt{\left\langle |T|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\langle |T^*|^{2(1-\nu)}\widehat{k}_\mu, \widehat{k}_\mu\right\rangle} \quad \text{(by (4))} \\
&\le \sqrt{\left\langle |S|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + \left\langle |T|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle}\sqrt{\left\langle |S^*|^{2(1-\nu)}\widehat{k}_\mu, \widehat{k}_\mu\right\rangle + \left\langle |T^*|^{2(1-\nu)}\widehat{k}_\mu, \widehat{k}_\mu\right\rangle} \\
&\quad \text{(by the Cauchy-Schwarz inequality)} \\
&= \sqrt{\left\langle \left(|S|^{2\nu} + |T|^{2\nu}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle}\sqrt{\left\langle \left(|S^*|^{2(1-\nu)} + |T^*|^{2(1-\nu)}\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle},
\end{aligned}$$

i.e.,

$$\left|\left\langle (S + \mathrm{i}T)\,\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right| \le \sqrt{\left\langle \left(|S|^{2\nu} + |T|^{2\nu}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle}\sqrt{\left\langle \left(|S^*|^{2(1-\nu)} + |T^*|^{2(1-\nu)}\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle},$$

as expected. ■

3. Berezin Number Inequalities

This section derives several inequalities for the Berezin number. The first result reads as follows.

Proposition 3.1 Let $S, T \in \mathscr{B}(\mathscr{H})$. Then

$$\begin{aligned}
\|S + T\|_{\mathbf{ber}}^2 \le{}& \frac{1}{2}\min\left(\left\||S|^2 + |T|^2\right\|_{\mathbf{ber}} + \left\||S|^2 - |T|^2\right\|_{\mathbf{ber}}, \left\||S^*|^2 + |T^*|^2\right\|_{\mathbf{ber}} + \left\||S^*|^2 - |T^*|^2\right\|_{\mathbf{ber}}\right) \\
&+ \min\left(\mathbf{ber}\,(T^*S)\,, \mathbf{ber}\,(TS^*)\right) + 2\,\|S\|_{\mathbf{ber}}\,\|T\|_{\mathbf{ber}}\,.
\end{aligned}$$

Proof. It observes from (9) that

$$\begin{aligned}
&\left|\left\langle (S + T)\,\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|^2 \\
&\le \frac{1}{2}\left(\left\langle \left(|S^*|^2 + |T^*|^2\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle + \left|\left\langle \left(|S^*|^2 - |T^*|^2\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right|\right) + \left|\left\langle TS^*\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right| + 2\left|\left\langle S\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right| \\
&\le \frac{1}{2}\left(\left\||S^*|^2 + |T^*|^2\right\|_{\mathbf{ber}} + \left\||S^*|^2 - |T^*|^2\right\|_{\mathbf{ber}}\right) + \mathbf{ber}\,(TS^*) + 2\,\|S\|_{\mathbf{ber}}\,\|T\|_{\mathbf{ber}}\,.
\end{aligned}$$

Now, by taking supremum over all vectors $\widehat{k}_\lambda, \widehat{k}_\mu \in \mathscr{H}$ with $\left\|\widehat{k}_\lambda\right\| = \left\|\widehat{k}_\mu\right\| = 1$, we obtain

$$\|S + T\|_{\mathbf{ber}}^2 \le \frac{1}{2}\left(\left\||S^*|^2 + |T^*|^2\right\|_{\mathbf{ber}} + \left\||S^*|^2 - |T^*|^2\right\|_{\mathbf{ber}}\right) + \mathbf{ber}\,(TS^*) + 2\,\|S\|_{\mathbf{ber}}\,\|T\|_{\mathbf{ber}}\,. \tag{11}$$

If we substitute S and T by S^* and T^*, in (11), we deduce

$$\begin{aligned}
&\|S+T\|_{\mathbf{ber}}^2 \\
&= \|S^*+T^*\|_{\mathbf{ber}}^2 \\
&\leq \frac{1}{2}\left(\left\||S|^2+|T|^2\right\|_{\mathbf{ber}} + \left\||S|^2-|T|^2\right\|_{\mathbf{ber}}\right) + \mathbf{ber}\left(T^*S\right) + 2\left\|S^*\right\|_{\mathbf{ber}}\left\|T^*\right\|_{\mathbf{ber}} \qquad (12)\\
&= \frac{1}{2}\left(\left\||S|^2+|T|^2\right\|_{\mathbf{ber}} + \left\||S|^2-|T|^2\right\|_{\mathbf{ber}}\right) + \mathbf{ber}\left(T^*S\right) + 2\left\|S\right\|_{\mathbf{ber}}\left\|T\right\|_{\mathbf{ber}},
\end{aligned}$$

thanks to (P-4).

We conclude the desired result by combining two inequalities (11) and (12). ■

The second result can be stated as follows.

Proposition 3.2 Let $S, T \in \mathscr{B}(\mathscr{H})$. Then

$$\begin{aligned}
\mathbf{ber}^2\left(S+T\right) \leq \frac{1}{2}\min&\left(\left\||S|^2+|T|^2\right\|_{\mathbf{ber}} + \left\||S|^2-|T|^2\right\|_{\mathbf{ber}}, \left\||S^*|^2+|T^*|^2\right\|_{\mathbf{ber}} + \left\||S^*|^2-|T^*|^2\right\|_{\mathbf{ber}}\right)\\
&+ \min\left(\mathbf{ber}\left(T^*S\right), \mathbf{ber}\left(TS^*\right)\right) + 2\mathbf{ber}\left(S\right)\mathbf{ber}\left(T\right).
\end{aligned}$$

Proof. Letting $\widehat{k}_\mu = \widehat{k}_\lambda$, in (9), we observe that

$$\begin{aligned}
&\left|\left\langle (S+T)\,\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \\
&\leq \frac{1}{2}\left(\left\langle\left(|S^*|^2+|T^*|^2\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + \left|\left\langle\left(|S^*|^2-|T^*|^2\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right) + \left|\left\langle TS^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| + 2\left|\left\langle S\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \\
&\leq \frac{1}{2}\left(\left\||S^*|^2+|T^*|^2\right\|_{\mathbf{ber}} + \left\||S^*|^2-|T^*|^2\right\|_{\mathbf{ber}}\right) + \mathbf{ber}\left(TS^*\right) + 2\mathbf{ber}\left(S\right)\mathbf{ber}\left(T\right),
\end{aligned}$$

which implies

$$\mathbf{ber}^2\left(S+T\right) \leq \frac{1}{2}\left(\left\||S^*|^2+|T^*|^2\right\|_{\mathbf{ber}} + \left\||S^*|^2-|T^*|^2\right\|_{\mathbf{ber}}\right) + \mathbf{ber}\left(TS^*\right) + 2\mathbf{ber}\left(S\right)\mathbf{ber}\left(T\right).$$

If we substitute S and T by S^* and T^*, in the above inequality, we infer

$$\mathbf{ber}^2\left(S+T\right) \leq \frac{1}{2}\left(\left\||S|^2+|T|^2\right\|_{\mathbf{ber}} + \left\||S|^2-|T|^2\right\|_{\mathbf{ber}}\right) + \mathbf{ber}\left(T^*S\right) + 2\mathbf{ber}\left(S\right)\mathbf{ber}\left(T\right),$$

thanks to (P-4).

Now, the result follows by incorporating these two inequalities. ■

The following result is a product of Theorem 2.3.

Corollary 3.3 Let $A, B \in \mathscr{B}(\mathscr{H})$. Then

$$\mathbf{ber}^2\left(B^*A\right) \leq \frac{1}{2}\left(\max\left(\|A\|_{\mathbf{ber}}^4, \|B\|_{\mathbf{ber}}^4\right) + \mathbf{ber}\left(|B|^2|A|^2\right)\right).$$

The following theorem suggests an upper bound for the Berezin number of the product of two operators.

Theorem 3.4 Let $A, B \in \mathscr{B}(\mathscr{H})$. Then for any $r, s \geq 1$,

$$\mathbf{ber}\left(B^*A\right) \leq \sqrt{\left\|\frac{|A|^{2r}+|B|^{2r}}{2}\right\|_{\mathbf{ber}}^{\frac{1}{r}}\left\|\frac{|A|^{2s}+|B|^{2s}}{2}\right\|_{\mathbf{ber}}^{\frac{1}{s}}}.$$

Proof. It has been demonstrated in [11, Corollary 4] that

$$\left\|\frac{B^*A+A^*B}{2}\right\|_{\mathbf{ber}} \leq \sqrt{\left\|\frac{|A|^{2r}+|B|^{2r}}{2}\right\|_{\mathbf{ber}}^{\frac{1}{r}}\left\|\frac{|A|^{2s}+|B|^{2s}}{2}\right\|_{\mathbf{ber}}^{\frac{1}{s}}},$$

which can be written as

$$\left\|\mathfrak{R}\left(B^*A\right)\right\|_{\mathbf{ber}} \leq \sqrt{\left\|\frac{|A|^{2r}+|B|^{2r}}{2}\right\|_{\mathbf{ber}}^{\frac{1}{r}}\left\|\frac{|A|^{2s}+|B|^{2s}}{2}\right\|_{\mathbf{ber}}^{\frac{1}{s}}}.$$

Replacing A by $e^{\mathrm{i}\theta}A$, we receive

$$\left\|\mathfrak{R}\mathrm{e}^{\mathrm{i}\theta}\left(B^*A\right)\right\|_{\mathbf{ber}} \leq \sqrt{\left\|\frac{|A|^{2r}+|B|^{2r}}{2}\right\|_{\mathbf{ber}}^{\frac{1}{r}}\left\|\frac{|A|^{2s}+|B|^{2s}}{2}\right\|_{\mathbf{ber}}^{\frac{1}{s}}}.$$

Now taking supremum over $\theta \in \mathbb{R}$, we infer that

$$\mathbf{ber}\left(B^*A\right) \leq \sqrt{\left\|\frac{|A|^{2r}+|B|^{2r}}{2}\right\|_{\mathbf{ber}}^{\frac{1}{r}}\left\|\frac{|A|^{2s}+|B|^{2s}}{2}\right\|_{\mathbf{ber}}^{\frac{1}{s}}},$$

due to $\sup\limits_{\theta\in\mathbb{R}} \left\|\mathfrak{R}\mathrm{e}^{\mathrm{i}\theta}T\right\|_{\mathbf{ber}} = \mathbf{ber}\,(T)$ [24]. ■

Remark 1 *The case $s = r$, in Theorem 3.4, reduces to* $\mathbf{ber}^r\left(B^*A\right) \leq \frac{1}{2}\left\||A|^{2r}+|B|^{2r}\right\|_{\mathbf{ber}}$.

By using the same technique as in the proof of Corollary 2.4, we can write from Theorem 3.4 that:

Corollary 3.5 Let $T \in \mathscr{B}(\mathscr{H})$. Then

$$\mathbf{ber}\,(T) \leq \sqrt{\left\|\frac{|T|^{2r\nu}+|T^*|^{2r(1-\nu)}}{2}\right\|_{\mathbf{ber}}^{\frac{1}{r}}\left\|\frac{|T|^{2s\nu}+|T^*|^{2s(1-\nu)}}{2}\right\|_{\mathbf{ber}}^{\frac{1}{s}}};\ \ (r, s \geq 1, 0 \leq \nu \leq 1).$$

Remark 2 *The case $s = r$, in Corollary 3.5, reduces to*

$$\mathbf{ber}^r\,(T) \leq \frac{1}{2}\left\||T|^{2r\nu}+|T^*|^{2r(1-\nu)}\right\|_{\mathbf{ber}}.$$

It is easy to see that if $T = A + \mathrm{i}B$ is the Cartesian decomposition of $T \in \mathscr{B}(\mathscr{H})$, then $\|T\|_{\mathbf{ber}} \le \|A\|_{\mathbf{ber}} + \|B\|_{\mathbf{ber}}$. Closely related to this inequality, one may note the following result, a natural consequence of Theorem 2.5.

Corollary 3.6 Let $S, T \in \mathscr{B}(\mathscr{H})$ be two self-adjoint operators. Then for any $0 \le \nu \le 1$,

$$\|S + \mathrm{i}T\|_{\mathbf{ber}} \le \sqrt{\left\||S|^{2\nu} + |T|^{2\nu}\right\|_{\mathbf{ber}} \left\||S|^{2(1-\nu)} + |T|^{2(1-\nu)}\right\|_{\mathbf{ber}}}.$$

If $T \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $T = A + \mathrm{i}B$, then

$$\|T\|_{\mathbf{ber}} \le \sqrt{\left\||A|^{2\nu} + |B|^{2\nu}\right\|_{\mathbf{ber}} \left\||A|^{2(1-\nu)} + |B|^{2(1-\nu)}\right\|_{\mathbf{ber}}}.$$

Remark 3 *Put $\nu = 1$ in Corollary 3.6, we obtain $\|S + \mathrm{i}T\|_{ber}^2 \le 2\left\||S|^2 + |T|^2\right\|_{ber}$.*

Remark 4 *Letting $\nu = \frac{1}{2}$ in Corollary 3.6 to get*

$$\|S + \mathrm{i}T\|_{\mathbf{ber}} \le \|\, |S| + |T| \,\|_{\mathbf{ber}} \le \|S\|_{\mathbf{ber}} + \|T\|_{\mathbf{ber}}$$

where the second inequality is obvious by the triangle inequality. By substituting $S = \Re T$ and $T = \Im T$, we deduce

$$\|T\|_{\mathbf{ber}} \le \frac{1}{2}\left\|\sqrt{TT^* + T^*T + 2\Re T^2} + \sqrt{TT^* + T^*T - 2\Re T^2}\right\|_{\mathbf{ber}} \le \|\Re T\|_{\mathbf{ber}} + \|\Im T\|_{\mathbf{ber}}.$$

Corollary 3.7 Let $T \in \mathscr{B}(\mathscr{H})$ with the Cartesian decomposition $T = A + \mathrm{i}B$. Then for any $0 \le \nu \le 1$,

$$\mathbf{ber}(T) \le \frac{1}{2}\left\||A|^{2\nu} + |A|^{2(1-\nu)} + |B|^{2\nu} + |B|^{2(1-\nu)}\right\|_{\mathbf{ber}}.$$

Proof. Letting $\widehat{k}_\mu = \widehat{k}_\lambda$, in Theorem 2.5, we can write

$$\begin{aligned}
\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| &\le \sqrt{\left\langle\left(|A|^{2\nu} + |B|^{2\nu}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\langle\left(|A|^{2(1-\nu)} + |B|^{2(1-\nu)}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle} \\
&\le \frac{1}{2}\left\langle\left(|A|^{2\nu} + |A|^{2(1-\nu)} + |B|^{2\nu} + |B|^{2(1-\nu)}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\
&\le \frac{1}{2}\left\||A|^{2\nu} + |A|^{2(1-\nu)} + |B|^{2\nu} + |B|^{2(1-\nu)}\right\|_{\mathbf{ber}},
\end{aligned}$$

where the second inequality is observed from the arithmetic-geometric mean inequality. Taking supremum over all vectors $\widehat{k}_\lambda \in \mathscr{H}$ yields the desired result. ■

Another corresponding result can be stated as follows.

Proposition 3.8 Let $T \in \mathscr{M}_n$. Then $\left\||T|^2\right\|_{\mathbf{ber}} \le \left\|T^*T + \mathrm{i}\Im T^2\right\|_{\mathbf{ber}}$.

Proof. For any $A, B \in \mathscr{M}_n$

$$\begin{aligned}\left\| |A+B|^2 \right\|_{\mathbf{ber}} &= \left\| (A+B)^* (A+B) \right\|_{\mathbf{ber}} \\ &= \left\| A^*A + B^*B + A^*B + B^*A \right\|_{\mathbf{ber}} \\ &= \left\| A^*A + B^*B + 2\Re\left(A^*B\right) \right\|_{\mathbf{ber}} \\ &= \left\| \Re\left(A^*A + B^*B + 2A^*B\right) \right\|_{\mathbf{ber}} \\ &\le \left\| A^*A + B^*B + 2A^*B \right\|_{\mathbf{ber}}.\end{aligned}$$

Thus, $\left\| |A+B|^2 \right\|_{\mathbf{ber}} \le \left\| A^*A + B^*B + 2A^*B \right\|_{\mathbf{ber}}$. If we replace B by $\mathrm{i}B$, we obtain $\left\| |A+\mathrm{i}B|^2 \right\|_{\mathbf{ber}} \le \left\| A^*A + B^*B + 2\mathrm{i}A^*B \right\|_{\mathbf{ber}}$. Now, if $T = A + \mathrm{i}B$ is the Cartesian decomposition of $T \in \mathscr{M}_n$ (noticing that A and B are self-adjoint now), then

$$\begin{aligned}\left\| |T|^2 \right\|_{\mathbf{ber}} &= \left\| |A+\mathrm{i}B|^2 \right\|_{\mathbf{ber}} \\ &\le \left\| A^2 + B^2 + 2\mathrm{i}AB \right\|_{\mathbf{ber}} \\ &\le \left\| (\Re T)^2 + (\Im T)^2 + 2\mathrm{i}\,(\Re T)\,(\Im T) \right\|_{\mathbf{ber}} \\ &= \left\| T^*T + \mathrm{i}\Im T^2 \right\|_{\mathbf{ber}}\end{aligned}$$

i.e., $\left\| |T|^2 \right\|_{\mathbf{ber}} \le \left\| T^*T + \mathrm{i}\Im T^2 \right\|_{\mathbf{ber}}$, as required. ■

Remark 5 *Notice that*

$$\begin{aligned}\left\| |T|^2 \right\|_{ber} &= \left\| |A+\mathrm{i}B|^2 \right\|_{ber} \\ &\le \left\| A^2 + B^2 + 2\mathrm{i}AB \right\|_{ber} \\ &= \left\| A\,(A+\mathrm{i}B) + (\mathrm{i}A+B)\,B \right\|_{ber} \\ &= \left\| A\,(A+\mathrm{i}B) + (A+\mathrm{i}B)^*\,(\mathrm{i}B) \right\|_{ber} \\ &\le \|A\|_{ber}\|A+\mathrm{i}B\|_{ber} + \|(A+\mathrm{i}B)^*\|_{ber}\|\mathrm{i}B\|_{ber} \\ &= \|A+\mathrm{i}B\|_{ber}\left(\|A\|_{ber} + \|B\|_{ber}\right) \\ &= \|T\|_{ber}\left(\|\Re T\|_{ber} + \|\Im T\|_{ber}\right).\end{aligned}$$

Therefore, by Proposition 3.8, we conclude that

$$\left\| |T|^2 \right\|_{ber} \le \left\| T^*T + \mathrm{i}\Im T^2 \right\|_{ber} \le \|T\|_{ber}\left(\|\Re T\|_{ber} + \|\Im T\|_{ber}\right).$$

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