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Autocentralizer automorphisms of groups

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Abstract. Let G be a group and $C_G(a)$ be a normal centralizer subgroup of G for some $a \in G$. Assume that, $\Upsilon_2^a(G) = [G, C_G(a)]$ and $Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G)$ is the set of all automorphisms of G that centralizes $\frac{G}{\Upsilon_2^a(G)}$ and $Z(C_G(a))$. In this paper, we focus on the group $Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G)$ and try to characterize its properties.

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1. Introduction

Our notations are standard. Let G be a group, by $\gamma_n(G), Z_n(G),\ C_G(a),\ Aut(G)$ and $Inn(G)$ we denote the n-th term of the lower and upper central series of G, the centralizer of a in G, the group of all automorphisms and inner automorphisms of G, respectively. Also, $Hom(G, H)$ denotes the group of all homomorphisms of G to H for an abelian group H.

An automorphism α of G is called central if $x^{-1}\alpha(x) \in Z(G)$ for each $x \in G$. The set of all central automorphisms of G is a normal subgroup of $Aut(G)$ and is denoted by $Aut_c(G)$. For finite p-group G, it is shown that $Aut_c(G) = Inn(G)$ if and only if $Z(G)$ is cyclic and $G' \leqslant Z(G)$ [3]. Furthermore, let C^* denote the set of all central automorphisms of G fixing $Z(G)$ elementwise. Attar [4] gave a classification of finite p-groups for which

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$C^* \cong Inn(G)$. More precisely, he proved that $C^* = Inn(G)$ if and only if G is an abelian group or a nilpotent group of class 2 with cyclic center.

Also, an automorphism β of group G is called a pointwise inner automorphism if x and $\beta(x)$ are conjugate for each $x \in G$. The set of all pointwise inner automorphisms of G denoted by $Aut_{pwi}(G)$. Yadav [5] showed that for each finite nilpotent group G of class two $Aut_{pwi}(G)$ is isomorphic to $Hom(G/Z(G), G')$. Azhdari and Akhavan-Malayeri [2] generalized the concept of pointwise inner automorphism for finitely generated nilpotent groups of class $k+1$. After then, the existence of a relation between $Aut_{k-pwi}(G)$ and $Hom(\frac{G}{Z_k(G)}, \gamma_{k+1}(G))$ are proved and some sufficient conditions for which these automorphisms are equal to $Inn(G)$, C^* or $Aut_c(G)$ are given.

In this paper, we first define the notion of autocentralizer automorphism for a normal centralizer subgroup of G. Then, it is shown that, the set of all autocentralizer automorphisms forms a normal subgroup of $Aut(G)$ and it is included in $Inn(G)$. Moreover, some necessary and sufficient conditions for which these automorphisms are equal to subgroups of $Inn(G)$ will be provided. At the end, a relation between autocentralizer automorphisms and pointwise inner automorphisms are obtained.

2. Motivation and Preliminaries

The aim of this section is to collect several definitions and basic results that will be applied in the rest of this paper. Let G be a group, H a subgroup of G and $x, y \in G$. Then x^y denotes the conjugate element of x by y. The element $[x, y] = x^{-1}y^{-1}xy$ is called the commutator of x and y. Inductively, $[x_1, \ldots, x_{n-1}, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$ for elements $x_1, \ldots, x_n \in G$. The subgroup generated by the elements $[g, h]$ in which $g \in G$ and $h \in H$, is denoted by $[G, H]$.

Lemma 2.1 Let N be a normal subgroup of a group G and θ be an endomorphism of G such that $\theta(N) \leqslant N$. Denote by $\overline{\theta}$ and θ_0 the endomorphism induced by θ in $\frac{G}{N}$ and N, respectively. If $\overline{\theta}$ and θ_0 are surjective(injective), then so is θ.

Lemma 2.2 [2, Corollary 1.4] Let A, B be two finite abelian groups and $exp(A) | \, exp(B)$. Then $Hom(A, B) \cong A$ if and only if $B \cong C_m \times H$ in which $C_m \cong \prod_{p\in\pi(A)} B_p$ and $H \cong \prod_{p\notin\pi(A)} B_p$. In particular, if $\pi(A) = \pi(B)$ then it is equivalent to B is a cyclic group.

Lemma 2.3 [5, Lemma 2.5] Let A and B be two finite abelian p-group such that $A = C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_s}}$, where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s > 0$ and $B = C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_r}}$, where $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_r > 0$. Let $b_j \geqslant a_j$ for all $j, 1 \leqslant j \leqslant s$, and $b_j > a_j$ for some such j. Let t be the smallest integer between 1 and s such that $a_j = b_j$ for all j such that $t+1 \leqslant j \leqslant s$. Then, for any finite abelian p-group C, $|Hom(A, C)| < |Hom(B, C)|$ if and only if the exponent of C is at least p^{a_t+1}.

Now, we define two series for a normal subgroup of G, in which under some conditions we will arrive to the known series, lower and upper central series of G. Let $C_G(a)$ be a normal subgroup of G for some $a \in G$. Then, we nominate the subgroup $[G, C_G(a)]$ as autocentralizer of $C_G(a)$ and show it by $\Upsilon_2^a(G)$. Inductively, the n-th autocentralizer subgroup of $C_G(a)$ is defined as $\Upsilon_n^a(G) = \langle [x, y] | x \in \Upsilon_{n-1}^a(G), y \in C_G(a) \rangle$. If $\Upsilon_1^a(G) = G$, then we will have the following descending series $G = \Upsilon_1^a(G) \supseteq \Upsilon_2^a(G) \supseteq \ldots \supseteq \Upsilon_n^a(G) \supseteq \ldots$ of G that we call it the lower series of $C_G(a)$. Although, it need not reach to 1 or even terminate. Of course, for every positive integer $n \geqslant 2$, $\Upsilon_n^a(G)$ is a normal subgroup of

$C_G(a)$ and $\Upsilon^a_n(G) \subseteq \gamma_n(G)$. The smallest number k in which $\Upsilon^a_{k+1}(G) = 1$ is called the length of the lower series of $C_G(a)$. Now, let $Z_1(C_G(a)) = Z(C_G(a))$ be the center of $C_G(a)$. For $n \geqslant 2$, the n-th center of $C_G(a)$ is defined as:

$$Z_n(C_G(a)) = \{x \in G | [x, y] \in Z_{n-1}(C_G(a)) \;\; \forall y \in G\}.$$

The ascending series $\{1\} \subseteq Z_1(C_G(a)) \subseteq Z_2(C_G(a)) \subseteq ... \subseteq Z_n(C_G(a)) \subseteq ...$ of G is called the upper series of $C_G(a)$. Although, it need not reach to G or even terminate. Clearly, for every positive integer n, $Z_n(G) \subseteq Z_n(C_G(a))$. The smallest number k in which $Z_k(C_G(a)) = G$ is called the length of the upper series of $C_G(a)$.

In what follows, a relation between central series of a group and two new defined series is explained.

Lemma 2.4 Let $1 = G_0 \leqslant G_1 \leqslant ... \leqslant G_n = G$ be a central series for a nilpotent group G and $C_G(a)$ be a normal subgroup of G. Then

(i) $\Upsilon^a_i(G) \leqslant G_{n-i+1}$, in particular, $\Upsilon^a_{n+1}(G) = 1$.
(ii) $G_i \leqslant Z_i(C_G(a))$, in particular, $Z_n(C_G(a)) = G$.
(iii) Lower and upper series of $C_G(a)$ have the same length.

Proof. Note that $\Upsilon^a_n(G) \subseteq \gamma_n(G)$ and $Z_n(G) \subseteq Z_n(C_G(a))$. Now, one can obtain the result by induction on i. ∎

3. Main results

In this section, we first define a subgroup of automorphisms of G that is a new set of automorphisms on centralizer subgroups. Then, some properties of this subgroup are given.

Definition 3.1 An automorphism α of $Aut(G)$ is defined autocentralizer, if $x^{-1}\alpha(x) \in \Upsilon^a_2(G)$ for each $x \in G$. We denote the set of all autocentralizer automorphisms of G by $Aut^{\Upsilon^a_2(G)}(G)$.

Note that $Aut^{\Upsilon^a_2(G)}(G)$ is a normal subgroup of $Aut(G)$ and it is included in $Aut_{pwi}(G)$. Moreover, by $Aut_{Z(C_G(a))}(G)$, we mean the subgroup of all automorphisms of G fixing $Z(C_G(a))$ elementwise. We denote $Aut^{\Upsilon^a_2(G)}(G) \cap Aut_{Z(C_G(a))}(G)$ by $Aut^{\Upsilon^a_2(G)}_{Z(C_G(a))}(G)$. It is easy to see that $Aut^{\Upsilon^a_2(G)}_{Z(C_G(a))}(G)$ is equal to $Aut^{G'}_{Z(G)}(G)$ whenever $C_G(a) = G$. Also, $Aut^{\Upsilon^a_2(G)}_{Z(C_G(a))}(G)$ is trivial if and only if $C_G(a) = Z(G)$ and $Aut^{\Upsilon^a_2(G)}_{Z(C_G(a))}(G) \leqslant Aut_c(G)$ if and only if G be a nilpotent group of class 2. Let G be a finitely generated nilpotent group of class 2 with torsion subgroup $T(G)$. Azhdari and Akhavan-Malayeri [1] proved that $exp(T(G/Z(G)))$ divides $exp(T(G'))$. Moreover $exp(G/Z(G)) = exp(G')$ whenever $G/Z(G)$ is torsion. In what follows, a similar result as above is given for each autocentralizer subgroup and a quotient of $G/Z(G)$.

Lemma 3.2 Let G be a finitely generated nilpotent group of class 2 and $C_G(a)$ be a normal subgroup of G. If $G/Z(C_G(a))$ is minimally generated by $\overline{x_i}$'s for $1 \leqslant i \leqslant d$, then

(i) $\Upsilon^a_2(G) = \langle [x_i, a_j] | 1 \leqslant i \leqslant d, a_j \in C_G(a)) \rangle$.
(ii) If $G/Z(C_G(a))$ is torsion, then $\Upsilon^a_2(G)$ is torsion and $exp(G/Z(C_G(a))) = exp(\Upsilon^a_2(G))$.
(iii) $exp(T(G/Z(C_G(a)))) | exp(T(Z(C_G(a))))$.

Proof. (i) The proof is straightforward.

(ii) Let $exp(G/Z(C_G(a)) = m$. Then $x_i^m \in Z(C_G(a))$ for $1 \leqslant i \leqslant d$. Therefore, $[x_i, a_j]^m = [x_i^m, a_j] = 1$ and hence, $exp(\Upsilon_2^a(G))|m$ and $\Upsilon_2^a(G)$ is torsion subgroup. Moreover, if $exp(\Upsilon_2^a(G)) = e$, then $[x_i^e, a_j] = [x_i, a_j]^e = 1$ for each $a_j \in C_G(a))$ and $1 \leqslant i \leqslant d$. Therefore $x_i^e \in Z(C_G(a))$ and so $m|e$. It follows that $exp(G/Z(C_G(a)) = exp(\Upsilon_2^a(G))$.
(iii) Let $exp(T(G/Z(C_G(a))) = m$ and $exp(T(Z(C_G(a))) = n$. Suppose that $\{\overline{x_1}, \ldots, \overline{x_s}\}$ is a minimal generator set for $T(G/Z(C_G(a)))$, where $s \leqslant d$. One can obtain that $[x_i, a_j]^m = [x_i^m, a_j] = 1$ for each $a_j \in C_G(a)$ and $i = 1, \ldots, s$. Therefore, $[x_i, a_j] \in \Upsilon_2^a(G) \leqslant Z(C_G(a))$ and hence, $[x_i^n, a_j] = [x_i, a_j]^n = 1$ for each $a_j \in C_G(a)$. This implies that $x_i^n \in Z(C_G(a))$ and $m|n$. ■

The next result gives a relationship between $Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G)$ and some homomorphisms.

Proposition 3.3 Let G be a finitely generated nilpotent group of class 2 and $C_G(a)$ be a normal subgroup of G. Then

(i) $Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G) \cong Hom(\frac{G}{Z(C_G(a))}, \Upsilon_2^a(G))$.
(ii) $Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G) \cong \frac{K \cap Z(C_G(a))}{Z(G)}$ where $Z(\frac{G}{\Upsilon_2^a(G)}) = \frac{K}{\Upsilon_2^a(G)}$.

Proof. (i) Let $\theta : Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G) \to Hom(\frac{G}{Z(C_G(a))}, \Upsilon_2^a(G))$ be defined by $\theta(\alpha) = \alpha^*$, where $\alpha^*(xZ(C_G(a))) = x^{-1}\alpha(x)$ for each $\alpha \in Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G)$. Since α is an automorphism fixing $Z(C_G(a))$ elementwise α^* is a well-defined homomorphism of $\frac{G}{Z(C_G(a))}$ to $\Upsilon_2^a(G)$. Therefore θ is a well-defined map. Clearly, θ is one-to-one.

In the first place, θ is a homomorphism: let $\alpha_1, \alpha_2 \in Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G)$ and $x \in G$. Since G is nilpotent group of class 2, we have $x^{-1}\alpha_1(x) \in \Upsilon_2^a(G) \leqslant G' \leqslant Z(C_G(a))$, $\alpha_1(x^{-1}\alpha_2(x)) = x^{-1}\alpha_2(x)$. This implies that

$$\begin{aligned}(\alpha_1\alpha_2)^*(xZ(C_G(a))) &= x^{-1}\alpha_1\alpha_2(x) = x^{-1}\alpha_1(\alpha_2(x)) \\ &= x^{-1}\alpha_1(xx^{-1}\alpha_2(x)) = x^{-1}\alpha_1(x).\alpha_1(x^{-1}\alpha_2(x)) \\ &= x^{-1}\alpha_1(x).x^{-1}\alpha_2(x) = \alpha_1^*(xZ(C_G(a)))\alpha_2^*(xZ(C_G(a))).\end{aligned}$$

Now, we show θ is surjective. For this, let $\beta \in Hom(\frac{G}{Z(C_G(a))}, \Upsilon_2^a(G))$ and define the map

$$\begin{aligned}&\alpha : G \longrightarrow G, \\ &x \longmapsto x\beta(xZ(C_G(a))).\end{aligned}$$

Evidently, α is a well-defined homomorphism. By Lemma 2.1, α is an isomorphism. Furthermore α centralizes $\frac{G}{\Upsilon_2^a(G)}$ and $Z(C_G(a))$ and consequently $\alpha \in Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G)$. Also, by the definition of θ, we have $\alpha^* = \beta$ and it follows that θ is an isomorphism of $Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G)$ to $Hom(\frac{G}{Z(C_G(a))}, \Upsilon_2^a(G))$.
(ii) It is straightforward to see that $g \in K \cap Z(C_G(a))$ iff $I_g \in Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G)$. Now, a quick calculation shows that the map $\phi : K \cap Z(C_G(a)) \longrightarrow Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}$ defined by $\phi(x) = I_{x^{-1}}$ for all $x \in K \cap Z(C_G(a))$ is an epimorphism with the kernel equal to $Z(G)$. ■

Let G be a finitely generated nilpotent group of class 2 with cyclic commutator subgroup. Using Proposition 3.3, one can see $Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G)$ is isomorphic to a subgroup of

$G/Z(G)$, equivalently, there exists a monomorphism from $Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G)$ into $Inn(G)$.

Lemma 3.4 Let G be a finitely generated nilpotent group of class 2 and $C_G(a)$ be a normal subgroup of G. Then

$$Hom(\frac{G}{Z(C_G(a))}, \Upsilon_2^a(G)) \cong \frac{G}{Z(C_G(a))} \text{ if and only if } \Upsilon_2^a(G) \text{ is cyclic.}$$

Proof. By Lemma 3.2 and Proposition 3.3(i), one can get the result. ■

***Example* 3.5** Let $G = D_8 = \langle x, y | x^4 = y^2 = 1, [x, y] = x^2 \rangle$. We have $Z(G) = G' = \langle x^2 \rangle \cong \mathbb{Z}_2$ and $Inn(G) = \{\alpha_1 = id, \alpha_2, \alpha_3, \alpha_4\}$. In fact, $\alpha_i : G \to G$ is defined by $\alpha_1 = id(x \to x, y \to y), \alpha_2(x \to x^3, y \to y), \alpha_3(x \to x, y \to yx^2), \alpha_4(x \to x^3, y \to yx^2)$. Let $C_G(x) = \langle x \rangle$ be a normal centralizer subgroup of G. It is clear that $Z(G) < C_G(x)$ and $\Upsilon_2^x(G) = \langle x^2 \rangle \cong \mathbb{Z}_2$. We have $G/(Z(C_G(x)) \cong \{\alpha_1, \alpha_3\}$ and $Hom(\frac{G}{Z(C_G(x))}, \Upsilon_2^x(G)) \cong \mathbb{Z}_2$.

The following results are immediate consequences of Lemma 3.4 and Proposition 3.3(i).

Corollary 3.6 Let G be a finite nilpotent group of class 2 and $C_G(a)$ be a normal subgroup of G. Then $Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G) = Inn_G(C_G(a))$.

Corollary 3.7 Let G be a finitely generated nilpotent group of class 2 and $C_G(a)$ be a normal subgroup of G. Then

(i) $Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G)$ is abelian.

(ii) If G is a p-group then so is $Aut_{Z(C_G(a))}^{\Upsilon_2^a(G)}(G)$.

One can easily generalized the Definition 3.1 and introduced the notion of $(k+1)$-autocentralizer as follows:

$$Aut_{Z_k(C_G(a))}^{\Upsilon_{k+1}^a(G)}(G) = \{\alpha | [x, \alpha] \in \Upsilon_{k+1}^a(G), \forall x \in G \ \ and \ \alpha(t) = t, \ \forall t \in Z_k(C_G(a))\}.$$

Clearly, if G is a nilpotent group of class $k+1$, then $Aut_{Z_k(C_G(a))}^{\Upsilon_{k+1}^a(G)}(G)$ is a non-trivial subgroup of $Aut_c(G)$. By a similar argument used in the proof of Proposition 3.3(i), we will have the following result.

Corollary 3.8 Let G be a finitely generated nilpotent group of class $k+1$ and $C_G(a)$ be a normal subgroup of G. Then

$$Aut_{Z_k(C_G(a))}^{\Upsilon_{k+1(G)}^a}(G) \cong Hom(G/Z_k(C_G(a)), \Upsilon_{k+1}^a(G)).$$

Corollary 3.9 Let G be a finite p-group of nilpotency class $k+1$ and $\Upsilon_{k+1}^a(G) = \gamma_{k+1}(G) = [x, \gamma_k(G)]$ for all $x \in G \setminus C_G(\gamma_k(G))$. Moreover,

$$G/Z_k(C_G(a)) \cong C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_s}}, G/Z_k(G) \cong C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_r}},$$

in which $a_1 \geqslant a_2 \geqslant ... \geqslant a_s > 0$ and $b_1 \geqslant b_2 \geqslant ... \geqslant b_r > 0$. Then $Aut_{Z_k(C_G(a))}^{\Upsilon_{k+1}^a(G)}(G) = Aut_{k-pwi}(G)$ if and only if $s = r$ and $exp(\gamma_{k+1}(G)) \leqslant p^{a_t}$, where t is the smallest integer between 1 and s such that $a_j = b_j$ for all $t+1 \leqslant j \leqslant s$.

Proof. First, assume that $r = s$ and $\gamma_{k+1}(G) = \Upsilon_{k+1}^a(G)$. Hence, by Lemma 2.3, we will have $Hom(\frac{G}{Z_k(G)}, \gamma_{k+1}(G)) \cong Hom(\frac{G}{Z_k(C_G(a))}, \Upsilon_{k+1}^a(G))$. The existence of an one-to-one correspondence between $Aut_{k-pwi}(G)$ and $Hom(\frac{G}{Z_k(G)}, \gamma_{k+1}(G))$ is shown in [2, Proposition 1.7]. Now, one can deduce the result by Corollary 3.8.

Conversely, let $Aut^{\Upsilon^a_{k+1}(G)}_{Z_k(C_G(a))}(G) = Aut_{k-pwi}(G)$. We first claim that $r = s$. By the way of contradiction, suppose that $s < r$. Since $b_j \geqslant a_j > 0$ for all j such that $1 \leqslant j \leqslant s$ and $b_j > 0$, for $s+1 \leqslant j \leqslant r$, we have

$$\begin{aligned}|Hom(\frac{G}{Z_k(C_G(a))}, \gamma_{k+1}(G))| &= |Hom(C_{p^{a_1}} \times \cdots \times C_{p^{a_s}}, \gamma_{k+1}(G))| \\ &\leqslant |Hom(C_{p^{b_1}} \times \cdots \times C_{p^{b_s}}, \gamma_{k+1}(G))| \\ &< |Hom(C_{p^{b_1}} \times \cdots \times C_{p^{b_s}}, \gamma_{k+1}(G))| \\ &\quad \times |Hom(C_{p^{b_{s+1}}} \times \cdots \times C_{p^{b_r}}, \gamma_{k+1}(G))| \\ &= |Hom(C_{p^{b_1}} \times \cdots \times C_{p^{b_r}}, \gamma_{k+1}(G))|.\end{aligned}$$

This yields $Aut^{\Upsilon^a_{k+1}(G)}_{Z_k(C_G(a))}(G) < Aut_{k-pwi}(G)$, which is a contradiction and the claim is approved. Furthermore, $Hom(\frac{G}{Z_k(G)}, \gamma_{k+1}(G)) \cong Hom(\frac{G}{Z_k(C_G(a))}, \Upsilon^a_{k+1}(G))$ and by Lemma 2.3, one can obtain $exp(\gamma_{k+1}(G)) \leqslant p^{a_t}$. ■

***Example* 3.10** Let G be the direct product of two copies of D_8 that is $G = D_8 \times D_8$. Then G is 16-centralizer group and its centralizer subgroups are not abelian. It is easy to see that $C_G(x, y)$ is a normal centralizer subgroup of G and $(D_8 \times D_8)/Z(C_G(x, y)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. We also have $G' = \Upsilon^{(x,y)}_2(G)$ and $exp(G') = 2$. Therefore,

$$\begin{aligned}&Hom((D_8 \times D_8)/Z(C_G(x, y)), \Upsilon^{(x,y)}_2(G)) \cong \\ &\qquad Hom((D_8 \times D_8)/Z(D_8 \times D_8), G') \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.\end{aligned}$$

Then, by [2, Proposition 1.7] and Proposition 3.3(i), $Aut^{\Upsilon^{(x,y)}_2(G)}_{Z(C_G(x,y))}(G) = Aut_{pwi}(G)$.

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