

Numerical solution of second-order stochastic differential equations with Gaussian random parameters

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Abstract. In this paper, we present the numerical solution of ordinary differential equations (or SDEs), from each order especially second-order with time-varying and Gaussian random coefficients. We indicate a complete analysis for second-order equations in special case of scalar linear second-order equations (damped harmonic oscillators with additive or multiplicative noises). Making stochastic differential equations system from this equation, it could be approximated or solved numerically by different numerical methods. In the case of linear stochastic differential equations system by Computing fundamental matrix of this system, it could be calculated based on the exact solution of this system. Finally, this stochastic equation is solved by numerically method like Euler-Maruyama and Milstein. Also its Asymptotic stability and statistical concepts like expectation and variance of solutions are discussed.

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1. Introduction

One of the most important and applicable concepts in various sciences is Newtons second law of motion which relates force and acceleration together. Therefore, second-order differential equations are most common in various scientific applications, As we can see in some articles, the famous and well-known differential equations of second-order such as OrnsteinUhlenbeck process and Random harmonic oscillator have been solved by different methods like as Monte Carlo and other numerical methods [1, 6, 14]. The study

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of numerical methods of second-order ordinary differential equations is one of the most applicable branches in numerical analysis issues[9]. With attention to stochastic essence of almost all physical phenomena, the most interesting advances in recent years are the development and extension of previous methods to stochastic systems [12]. Numerical methods have to replace discrete-time dynamics in place of continuous-time with generating values at times t_0, t_1, \dots, t_n . In this process, t_0 and t_n , are fixed points. Some criterions for specification a good numerical method are order of convergency, comparison with exact solution and produced error of a used method and determining its discrete-time dynamics which has an appropriate stationary density as close as possible to that of the corresponding continuous-time system that Wiener chaos expansion (WCE), is one of these methods[4],[17]. The differential equation which describe second-order systems contains some parameters known as damping. The stationary density is completely independent of damping, but dynamical quantities, and the specially the final numerical algorithms, are strongly dependent on it. With some change of variables, the system becomes first order one.

In this paper, we intend to extend our previous work about solving second-order linear stochastic differential equation [16], to solve second-order stochastic differential equation By first-order stochastic linear system equation which has been mentioned in various books like [10], [11], [13] and [15].

That is, we consider $X_t \in \mathbb{R}^n$ and $X_t \in L_n^2(0, T)$, as a stochastic process and unique solution of following S.D.E:

$$\begin{cases} dX_t = (A(t).X_t + B(t))dt + (C(t).X_t + D(t))dW \\ X(0) = X_0. \quad (0 \leq t \leq T) \end{cases} \quad (1)$$

such that $W(\cdot)$, is a m-dimensional Brownian motion. Afterwards, By construction fundamental or Hamiltonian matrix for this system, we solve it with numerical methods as EulerMaruyama(E.M.), Milstein and Rung-Kutta method. finally, in the end of our conclusions, we investigate some statistical properties like expectation and variance by numerical simulation like predictor-corrector E.M.[8], and will have a comparison between exact solution and its solution and find least square error for this equations.

This paper is organized as follow. In section 2, we consider the stochastic linear equation system of stochastic linear second-order equation which has been mentioned in various books like [13] and [11] and latest articles[3] and [5]. Afterwards, by construction fundamental matrix for this system we solve it numerically by Rung-Kutta method. In this section we consider second order S.D.E. examples solve them by this mentioned method and stochastic numerical simulation like predictor-corrector EulerMaruyama and Milstein'method [8]. Also, we take a discussion about expectation, variance of these equations solutions . In final section, the conclusion of this paper has been said again.

2. Making Stochastic Differential Equation System

Let the general form of a second-order stochastic differential equation is defined such this equation:

$$\begin{cases} \ddot{X}_t = (f(X_t, t) + \hat{f}(X_t, t)\xi(t))\dot{X}_t + (g(X_t, t) + \hat{g}(X_t, t)\xi(t))X_t + \bar{h}(X_t, t), \\ X_t(0) = X_0, \dot{X}_0 = X_1. \end{cases} \quad (2)$$

such that we define $\bar{h}(X_t, t) = (h(X_t, t) + \hat{h}(X_t, t)\xi(t))$ and X_t is a 1-dimensional stochastic process defined on closed time interval $[0, T]$ and the real functions $f(X_t, t), g(X_t, t), h(X_t, t)$ and also $\hat{f}(X_t, t), \hat{g}(X_t, t), \hat{h}(X_t, t)$, are stochastic integrable functions. In special case, It shall be considered equation of the following form which has been discussed in[3]:

$$\begin{cases} \ddot{X}_t = f(X_t) - \eta \hat{f}^2(X_t, t) \dot{X}_t + \varepsilon \hat{f}(X_t, t) \xi(t), \\ X_t(0) = X_0, \dot{X}_0 = X_1. \end{cases} \tag{3}$$

The damping parameter and The amplitude of the random forcing are denoted by η and ε respectively such that related to the temperature T and damping coefficient η by the fluctuation-dissipation relation[7], We have $\varepsilon^2 = 2\eta KT$. $\xi(t)$, is defined as White noise that has derivative relation with Wiener process

$$\dot{W}_t = \frac{dW}{dt} = \xi(t),$$

Definition 2.1 Consider the m -dimensional vector $W(t)$ of real stochastic processes $W_i(t), (i = 1, 2, \dots, m)$. It is named Wiener process or Brownian motion if;

- (a) $W(0) = 0$ a.s. (almost sure with probability one),
- (b) $W(t) - W(s)$ is normal distribution (*i.e.* $W \sim N(0, t - s)$), for all $0 \leq s \leq t$
- (c) The random variables $W(t_1), W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1})$, for times $0 \leq t_1 \leq t_2 \leq \dots \leq t_n$, are independent increments.

From this definition, about expectation and variance related to $W_i(t)$, it could be concluded that

$$E[W_i(t)] = 0, E[W_i^2(t)] = t \text{ for } i = 1, 2, \dots, m. \tag{4}$$

The expectation integral for these functions has these properties

$$E[\xi(t), \xi(s)] = \delta_0(s - t), E[W(s), W(t)] = \min(s, t).$$

We can write (2), as a pair of first-order equations for X_t and V_t , the position and velocity stochastic processes:

$$\begin{cases} dX_t = V_t dt, \\ dV_t = (f(X_t, t)V_t + g(X_t, t)X_t + h(X_t, t))dt + (\hat{f}(X_t, t)V_t + \hat{g}(X_t, t)X_t + \hat{h}(X_t, t))dW(t). \end{cases} \tag{5}$$

With initial conditions $X_t(0) = X_0, V_0 = X_1$. Now the second-order SDE(2), could be written again as a first-order system by matrix notation.

$$d \begin{pmatrix} X_t \\ V_t \end{pmatrix} = \left(\Gamma(X_t, t) \begin{pmatrix} X_t \\ V_t \end{pmatrix} + \begin{pmatrix} 0 \\ h(X_t, t) \end{pmatrix} \right) dt + \left(\Sigma(X_t, t) \begin{pmatrix} X_t \\ V_t \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{h}(X_t, t) \end{pmatrix} \right) dW_t. \tag{6}$$

Where we have

$$\Gamma(X_t, t) = \begin{pmatrix} 0 & 1 \\ g(X_t, t) & f(X_t, t) \end{pmatrix}, \quad \Sigma(X_t, t) = \begin{pmatrix} 0 & 0 \\ \hat{g}(X_t, t) & \hat{f}(X_t, t) \end{pmatrix}.$$

If $\Gamma(X_t, t) = \Gamma(t)$ and $\Sigma(X_t, t) = \Sigma(t)$, we reach to a stochastic linear equation system:

$$d \begin{pmatrix} X_t \\ V_t \end{pmatrix} = \left(\Gamma(t) \begin{pmatrix} X_t \\ V_t \end{pmatrix} + \begin{pmatrix} 0 \\ h(t) \end{pmatrix} \right) dt + \left(\Sigma(t) \begin{pmatrix} X_t \\ V_t \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{h}(t) \end{pmatrix} \right) dW_t. \quad (7)$$

If in special case, we have

$$d \begin{pmatrix} X_t \\ V_t \end{pmatrix} = \Gamma(t) \begin{pmatrix} X_t \\ V_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ \hat{h}(t) \end{pmatrix} dW_t, \quad (8)$$

This special case is a better model of *Brownian movement* which is provided by the *Ornstein-Uhlenbeck* equation. For solving this matrix form (8), it could be done by different numerical methods like Euler, Milstein or even Rung-kutta that in [16], it was done that is named linear second-order SDEs in narrow sense case. As another special cases, if we have

$$d \begin{pmatrix} X_t \\ V_t \end{pmatrix} = \Gamma(t) \begin{pmatrix} X_t \\ V_t \end{pmatrix} dt + \Sigma(t) \begin{pmatrix} X_t \\ V_t \end{pmatrix} dW_t. \quad (9)$$

This stochastic differential equation system is well-known to *Geometric Brownian Motion* But as we will observe later, this system just in special case has an exact solution like Black-Scholes model in 1-dimentional case.

Now we perform the same work in extended case on interval $[0, T]$, to solve the system of equations(27) explicitly and numerically. Suppose time interval has been separated to equal subintervals $[t_i, t_{i+1}]$, $(i = 1, 2, \dots, n)$ and we define the following one step recursive equation even for general case(6).

$$\begin{pmatrix} X_{i+1} \\ V_{i+1} \end{pmatrix} = \quad (10)$$

$$\begin{pmatrix} X_i \\ V_i \end{pmatrix} + \left(\Gamma(X_i, t_i) \begin{pmatrix} X_i \\ V_i \end{pmatrix} + \begin{pmatrix} 0 \\ h(X_i, t_i) \end{pmatrix} \right) \Delta t_i + \left(\Sigma(X_i, t_i) \begin{pmatrix} X_i \\ V_i \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{h}(X_i, t_i) \end{pmatrix} \right) \Delta W_i. \quad (11)$$

such that $\Delta t_i = t_{i+1} - t_i = \frac{T}{n}$, and according to Wiener process definition:

$$\Delta W_i = W_{i+1} - W_i \sim N(0, \Delta t_i).$$

With the next theorem, we want to express the existence and uniqueness of second-order stochastic differential equations based on some conditions.

Theorem 2.2 Suppose that $A(t).X_t + B(t) : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^n$ and $C(t).X_t + D(t) : \mathbb{R}^n \times [0, T] \rightarrow \mathcal{M}^{m \times n}$ are continuous and satisfy the following properties:

$$\begin{aligned}
 (1) \quad & \|A(t).(X(t) - \hat{X}_t(t))\| \leq L.\|X_t(t) - \hat{X}_t(t)\|, \\
 & \|C(t).(X_t(t) - \hat{X}_t(t))\| \leq L.\|X_t(t) - \hat{X}_t(t)\|. \quad \text{for all } 0 \leq t \leq T, X_t, \hat{X}_t \in \mathbb{R}^n \\
 (2) \quad & \|A(t).X_t + B(t)\| \leq L|1 + \|X_t\||, \\
 & \|C(t).X_t + D(t)\| \leq L|1 + \|X_t\||. \quad \text{for all } 0 \leq t \leq T, X_t \in \mathbb{R}^n,
 \end{aligned}
 \tag{12}$$

for some suitable $L \in \mathbb{R}$.

Let $X_0 \in \mathbb{R}^n$ is a random variable such that: $E[X_0^2] < \infty$ so, There exist a unique solution $X_t \in L_n^2(0, T)$ of following S.D.E:

$$\begin{cases}
 dX_t = (A(t).X_t + B(t))dt + (C(t).X_t + D(t))dW \\
 X(0) = X_0. \quad (0 \leq t \leq T)
 \end{cases}
 \tag{13}$$

where $W(\cdot)$, is a m-dimensional Brownian motion[6].

Remark 1 Moreover, in special case if each one of matrixes gets its supreme value in a closed interval, that is;

$$\text{Sup}_{0 \leq t \leq T} \{ \|B(t)\|, \|D(t)\|, \|A(t)\|, \|C(t)\| \} < \infty$$

Therefore, $A(t)X_t + B(t)$ and $C(t)X_t + D(t)$ satisfy the hypotheses of uniqueness and Existence theorem [2] for Linear S.D.E. provided that $E[X_0^2] < \infty$. Finally, this condition is necessary that for almost each $W(t)$, the random trajectories of linear S.D.E.

$$\begin{cases}
 dX_t = (A(t).X_t + B(t))dt + (C(t).X_t + D(t))dW, \\
 X(0) = X_0 + \xi
 \end{cases}$$

when with attention to:

$$\lim_{\xi \rightarrow 0} C(t).X_t + D(t) = 0,$$

Converge uniformly on interval $[0, T]$ to the trajectory of its corresponding ordinary differential equation. $\begin{cases} \dot{X}_t = A(t).X_t + B(t) \\ X(0) = X_0. \end{cases}$

In general case, the following theorem that is named (*Dependence on parameters*), indicates the concept of asymptotic stability for Linear stochastic systems.

Theorem 2.3 Suppose for $k = 1, 2, \dots$ that $A^k(t)X_t + B^k(t)$ and $C^k(t)X_t + D^k(t)$ satisfy the hypotheses of existence and uniqueness theorem, with the same constant L which said as a real bond in theorem. Assume further that

$$\lim_{k \rightarrow \infty} E(\|X_0^k - X_0\|) = 0$$

and for each $M > 0$, such that $\|X_t\| \leq M$,

$$\lim_{k \rightarrow \infty} \max_{0 \leq t \leq T} \left(\|B^k(t) - B(t)\| + \|A^k(t) - A(t)\| + \|D^k(t) - D(t)\| + \|C^k(t) - C(t)\| \right) = 0$$

Finally suppose that $X^k(0)$ solves:

$$\begin{cases} dX_t^k = (A^k(t)X_t + B^k(t))dt + (D^k(t)X_t + C^k(t))dW \\ X^k(0) = X_0^k \end{cases}$$

Then

$$\lim E \left(\max_{0 \leq t \leq T} \|X^k(t) - X(t)\|^2 \right) = 0,$$

where X_t is the unique solution of

$$\begin{cases} dX_t = (A(t).X_t + B(t))dt + (C(t).X_t + D(t))dW, \\ X(0) = X_0. \end{cases}$$

In addition, the analytic solution and least square error of O.D.E. Could be found in virtue of expectation and variance of S.D.E. solution such that in special Case, if A, B, C and D contain continuous elements in $[0, T]$, they get their finite maximum values in this interval. With attention to existence and uniqueness solution of Linear S.D.E., the equation has been brought in (8), has a closed and explicit solution which is found according to following theorem[2]:

Theorem 2.4

$$\begin{cases} dX_t = (A(t).X_t + B(t))dt + D(t).dW \\ X(0) = X_0 \end{cases} \quad (14)$$

$$X(t) = \Psi(t) \left(X_0 + \int_0^t \Psi(s)^{-1} (C(s)ds + E(s)dW) \right), \quad (15)$$

where $\Psi(0)$ is the Fundamental matrix of the following O.D.E. system:

$$\frac{d\Psi}{dt} = A(t).\Psi, \quad \Psi(0) = I.$$

In other words, we have:

$$\begin{pmatrix} \dot{\Psi}_{11} & \dot{\Psi}_{12} \\ \dot{\Psi}_{21} & \dot{\Psi}_{22} \end{pmatrix} = \begin{pmatrix} \Psi_{21} & \Psi_{22} \\ B(t)\Psi_{11} + A(t)\Psi_{21} & B(t)\Psi_{21} + A(t)\Psi_{22} \end{pmatrix}$$

$$\text{and } \Psi_{ij}(0) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

Consequently it turns out two second-order equations by different initial Conditions:

$$\begin{cases} \ddot{\Psi}_{11} = B(t)\Psi_{11} + A(t)\dot{\Psi}_{11} \\ \Psi_{11}(0) = 1, \dot{\Psi}_{11}(0) = 0 \end{cases}, \begin{cases} \ddot{\Psi}_{12} = B(t)\Psi_{12} + A(t)\dot{\Psi}_{12} \\ \Psi_{12}(0) = 0, \dot{\Psi}_{12}(0) = 1 \end{cases} \quad (16)$$

so the explicit solution could be computed as follow

$$\begin{pmatrix} X_1 \\ \dot{X}_1 \end{pmatrix} = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \dot{\Psi}_{11} & \dot{\Psi}_{12} \end{pmatrix} \begin{pmatrix} X(0) \\ X'(0) \end{pmatrix} + \int_0^t \frac{1}{\det \Psi} \begin{pmatrix} \dot{\Psi}_{12} & -\Psi_{12} \\ -\dot{\Psi}_{11} & \Psi_{11} \end{pmatrix} \begin{pmatrix} 0 \\ C(s) \end{pmatrix} ds + \begin{pmatrix} 0 & 0 \\ \lambda\gamma A(t) & (1-\lambda)\gamma(t) \end{pmatrix} \begin{pmatrix} dW_2 \\ dW_1 \end{pmatrix}$$

Hence, we should utilize this equality for equation solution:

$$\begin{aligned} X_1 = X_t = & \Psi_{11}X(0) + \Psi_{12}X'(0) + \Psi_{11} \int_0^t \frac{1}{\det \Psi} (-\Psi_{12})(C(s)ds \\ & + \lambda\gamma(s)dW_2 + (1-\lambda)\gamma(s)dW_1) \\ & + \Psi_{12} \int_0^t \frac{1}{\det \Psi} (\Psi_{11})(C(s)ds + \lambda\gamma(s)dW_2 + (1-\lambda)\gamma(s)dW_1). \end{aligned} \quad (17)$$

the equation (16) are second-order Linear O.D.E. we could solve them by various methods like series solution respect to nonsingular point or *Frobinios* series respect to regular points. Also, we could apply *sinc* method to solve directly this equation or convert it to a Linear system equation and solve it by 4th-order *Rung – kutta* method.

afterwards, we decide to Compute from equality (17) that it could be done by numerical methods like E.M. predictor-corrector E.M. and milstein method.

Also in matrix form which is convenient for *Matlab* software, we could get the following recursive procedure.

$$\begin{aligned} X(t) = & \Psi(t) \left(X_0 + \int_0^t \Psi(s)^{-1} (C(s)ds + E(s)dW_s) \right) \\ \begin{cases} \Psi^{-1}(t_{i+1}).X(t_{i+1}) = X_0 + \int_0^{t_{i+1}} \Psi(s)^{-1} (C(s)ds + E(s)dW_s) \\ \Psi^{-1}(t_i).X(t_i) = X_0 + \int_0^{t_i} \Psi(s)^{-1} (C(s)ds + E(s)dW_s) \end{cases} \end{aligned}$$

Consequently, we could have:

$$\begin{aligned} X(t_{i+1}) = X(i+1) = & \Psi(t_{i+1}) \left(\Psi^{-1}(t_i)X_i + \int_{t_i}^{t_{i+1}} \Psi(s)^{-1} (C(s)ds + E(s)dW_s) \right) \\ X(i+1) = & \Psi(t_{i+1})\Psi(t_i)^{-1} (X_i + C(t_i)\Delta t_i + E(t_i)\Delta W_i) \end{aligned} \quad (18)$$

such that;

$$\delta W_i = W(t_{i+1}) - W(t_i) \cong \sqrt{\delta t_i} \xi_i \quad (\xi_i \sim N(0, 1))$$

The last approximation has been concluded from Independent Increment property of

wiener process (for any $t, s \in [0, T]$; $W(t) - W(s) = W(t - s) \sim N(0, t - s)$).

of course, $W(t)$ could be computed by an infinite series of Haar functions with standard Gaussian random variables [6], or even have been expressed according to fundamental conditions in definition of Wiener process.

Example 1 Consider the stochastic differential equation : $t \in [a, b]$, ($0 < a < b < 1$)

$$\begin{cases} X'' = \frac{1+\beta(t)\xi_2}{t-1} X' + \left(\frac{1+\alpha(t)\xi_1}{t} + \frac{1}{1-t}\right)X + \gamma(t)\xi_3 + \frac{t-2}{t-1}; \\ X(0) = 0, X'(0) = 1. \end{cases}$$

that $\xi_3 = \lambda\xi_1 + (1 - \lambda)\xi_2$, $0 \leq \lambda \leq 1$, and $\xi_i = \frac{dW_i}{dt}$ ($i = 1, 2, 3$) are white noise. $\alpha(t)$ is a continuous function on interval $[a, b]$.

At first, we convert this equation to a linear system,

$$\begin{cases} X'_1 = X_2, \\ X'_2 = \left(\frac{1+\alpha(t)\xi_1}{t} + \frac{1}{1-t}\right)X_1 + \left(\frac{1+\beta(t)\xi_2}{t-1}\right)X_2 + \frac{t-2}{t-1} + \gamma(t)\xi_3. \end{cases}$$

$$dX_t = (A(t).X_t + B(t))dt + (F_1(t).X_t + E_1(t)).dW_1 + (F_2(t).X_t + E_2(t)).dW_2$$

such that, $A(t) = \begin{pmatrix} 0 & 1 \\ \frac{1}{t} + \frac{1}{1-t} & \frac{1}{t-1} \end{pmatrix}$, $F_1(t) = \begin{pmatrix} 0 & 0 \\ \frac{\alpha(t)}{t} & 0 \end{pmatrix}$, $F_2(t) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\beta(t)}{t-1} \end{pmatrix}$,
 $E_1(t) = \begin{pmatrix} 0 \\ \gamma(t)\lambda \end{pmatrix}$, $E_2(t) = \begin{pmatrix} 0 \\ \gamma(t)(\lambda-1) \end{pmatrix}$ and $B(t) = \begin{pmatrix} 0 \\ \frac{t-2}{2-1} \end{pmatrix}$
 corresponding fundamental matrix for system is

$$\dot{\Psi} = A(t)\Psi, \Psi(0) = I$$

$$\begin{pmatrix} \dot{\Psi}_{11} & \dot{\Psi}_{12} \\ \dot{\Psi}_{21} & \dot{\Psi}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{1-t} & \frac{1}{t-1} \end{pmatrix} \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} \text{ and } \Psi_{ij}(0) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \text{ and so we have:}$$

$$\begin{cases} \ddot{\Psi}_{11} = \left(\frac{1}{t} + \frac{1}{1-t}\right)\Psi_{11} + \frac{1}{t-1}\dot{\Psi}_{11}, & \begin{cases} \ddot{\Psi}_{12} = \left(\frac{1}{t} + \frac{1}{1-t}\right)\Psi_{12} + \frac{1}{t-1}\dot{\Psi}_{12}, \\ \Psi_{11}(0) = 1, \dot{\Psi}_{11}(0) = 0 \end{cases} \\ \Psi_{11}(0) = 1, \dot{\Psi}_{11}(0) = 0 & \begin{cases} \Psi_{12}(0) = 0, \dot{\Psi}_{12}(0) = 1. \end{cases} \end{cases}$$

These are special case of *heun* equation that multiplying in $t(t - 1)$, they could be solved by *robinious* Series around $t = 0$ (since $t = 0$ is a regular singular point for them), and their solutions is: $\Psi_{12} = t$, $\Psi_{11} = 1 + t \ln(t)$.

Therefore, the matrix Ψ is indicated as follow:

$$\Psi = \begin{pmatrix} 1 + t \ln(t) & t \\ 1 + \ln(t) & 1 \end{pmatrix}, \Psi^{-1} = \frac{1}{1-t} \begin{pmatrix} 1 & -t \\ -1 - \ln(t) & 1 + t \ln(t) \end{pmatrix}$$

According to (10), we can conclude in narrow sense case:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \Psi \left(\begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix} + \int_0^t \Psi^{-1}(s). \left(\begin{pmatrix} 0 \\ \frac{s-2}{s-1} \end{pmatrix} ds + \begin{pmatrix} 0 \\ \alpha(s)\lambda \end{pmatrix} dW_1 \right) \right)$$

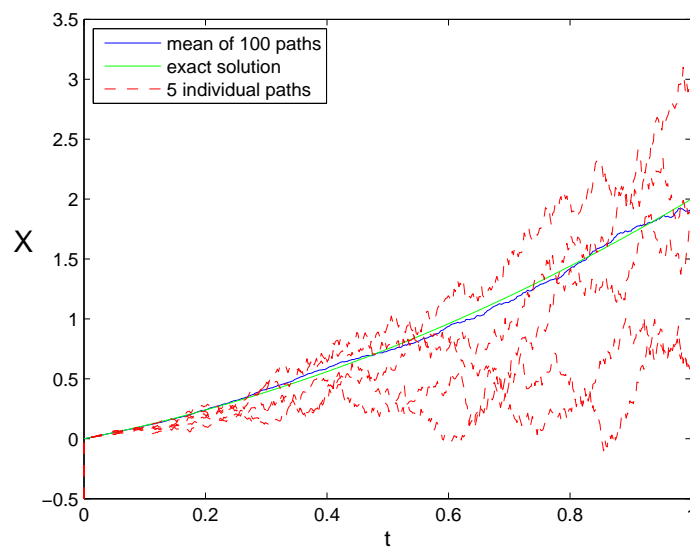


Figure 1. The exact solution with mean of $M = 100$ numerical solution by E.M. Method with $N = 2^8$ subinterval of $[0, 1]$ and also 5 individual numerical solution has been shown. In this case the least square error is 0.02135.

$$+ \left(\begin{matrix} 0 \\ (1 - \lambda)\alpha(s) \end{matrix} dW_2 \right)$$

that the solution of initial S.D.E. is

$$X_1 = X = (\Psi_{11} \ \Psi_{12}) \left(\begin{pmatrix} X(0) \\ X'(0) \end{pmatrix} + \int_0^t \Psi^{-1}(s) \left(\begin{pmatrix} 0 \\ \frac{s-2}{s-1} \end{pmatrix} ds + \begin{pmatrix} 0 \\ \alpha(s)\lambda \end{pmatrix} dW_1 + \begin{pmatrix} 0 \\ (1 - \lambda)\alpha(s) \end{pmatrix} dW_2 \right) \right)$$

with attention to this issue that expectation of itô Integral is zero so we have

$$E[X_t] = X(0)(1 + t \ln(t)) + X'(0)t + (\Psi_{11} \ \Psi_{12}) \int_0^t \Psi^{-1}(s) \cdot \begin{pmatrix} 0 \\ \frac{s-2}{s-1} \end{pmatrix} ds$$

with initial conditions and asymptotic stability of S.D.E., if $\alpha(t) \rightarrow 0$, then implicit solution of it's correspond O.D.E.

$$\begin{cases} \ddot{X} = \frac{1}{t-1} \dot{X} + \left(\frac{1}{t} - \frac{1}{t-1} \right) X + \frac{t-2}{t-1} \\ X(0) = 0, \dot{X}(0) = 1 \end{cases}$$

is the same expectation of S.D.E.:

$$E[X_t] = t + (1 + t \ln(t)) \cdot \int_0^t \frac{s(s-2)}{(s-1)^2} ds + t \int_0^t \frac{2-s}{(s-1)^2} (1 + s \ln s) ds = t + t^2.$$

We have showed the maximum absolute errors in numerical solution of the example in the table for different N , where $\|LE_{EM}^N(h)\|$ are least squares errors for E.M. method

Table 1. Generated errors via Numerical methods *E.M.* and *Milstein*

N	$\ E_{EM}^N(h)\ ^a$	$\ E_M^N(h)\ ^b$
2^7	5.2×10^{-2}	3.24×10^{-2}
2^8	2.36×10^{-2}	2.09×10^{-2}
2^9	2.31×10^{-2}	1.22×10^{-2}
2^{10}	2.12×10^{-2}	4.241×10^{-3}

^aThe value of least squares error generated by *E.M.* method.

^bThe value of least squares error generated by *Milstein* method

and $\|LE_M^N(h)\|$ are least squares errors for Milstein method.

Afterwards, we intend to express an explicitly solution of second-order linear SDEs systems. At first, let express the basic theorem related to general vector linear equation[2].

Theorem 2.5 Suppose $X_t \in \mathbb{R}^n$ is a stochastic process and consider the following S.D.E in $\mathbb{L}^2(0, T)$:

$$\begin{cases} dX_t = (A(t).X_t + B(t))dt + (C(t).X_t + D(t))dW, \\ X(0) = X_0. \quad (0 \leq t \leq T) \end{cases} \quad (19)$$

where $W(\cdot)$, is a m-dimensional Brownian motion, the square matrixes $A, C \in \mathcal{M}^{n \times n}$ of time independent variable, and vector time dependent functions $B, D \in \mathbb{R}^n$. Therefore, the exact solution of this system is

$$X_t = \Psi_t(X_0 + \int_0^t \Psi_s^{-1} dY_s),$$

such that stochastic process Y_t , and the fundamental matrix $\Psi_t \in \mathcal{M}^{n \times n}$ holds in following SDEs:

$$\begin{cases} dY_t = (B(t) - C(t)D(t))dt + (D(t))dW_t, \\ d\Psi_t = A(t)\Psi_t dt + C(t)\Psi_t dW_t, \quad \Psi_0 = I. \end{cases} \quad (20)$$

As we can see easily, with integration by part formula, dY could be represented based on a Lebesgue integral:

$$\begin{aligned} dY_t &= (B(t) - C(t)D(t))dt + (D(t))dW_t \\ &= (B(t) - C(t)D(t))dt + (D(t))W_t - W_t D'(t)dt \\ &= (D(t))W_t + (B(t) - C(t)D(t) + W_t D'(t))dt. \end{aligned}$$

Also, in virtue of zero value of *itô* integral expectation, $E[X_t] = M(t)$, holds in following ODE:

$$\dot{M}(t) = A(t)M(t) + B(t) \quad , \quad M(0) = X_0.$$

Unfortunately, the explicit solution for fundamental matrix of stochastic equations system is represented under some special conditions on related coefficients which don't hold in general case.

Theorem 2.6 Under previous theorem conditions, if $A(t)$ and $C(t)$, are commutative matrixes, then there is an explicit stochastic solution for fundamental matrix is:

$$\Psi(t) = \exp\left(\int_0^t \left(A(s) - \frac{C^2(s)}{2}\right)ds + \int_0^t C(s)dW_s\right). \tag{21}$$

Proof: On account of commutative matrixes, it's enough to do *itô* formula for $\Psi(t) = \exp(Y)$ such that $dY_t = \left(A(t) - \frac{C^2(t)}{2}\right)dt + C(t)dW_t$.

$$d\Psi(t) = d\exp(Y) = \exp(Y)dY + \frac{1}{2}\exp(Y)(dY)^2 \tag{22}$$

$$= \exp(Y)\left(\left(A(t) - \frac{C^2(t)}{2}\right)dt + C(t)dW_t\right) + \frac{1}{2}\exp(Y)C^2(t)dt \tag{23}$$

$$= \exp(Y)\left(A(t)dt + C(t)dW_t\right) \tag{24}$$

and therefore the proof of theorem is completed. □.

Thus we propose to find an explicit solution for fundamental matrix equation which has been produced from linear equations system(27), by appropriate change of variable. Before anything, we pay attention to next theorem

Theorem 2.7 The solution of(28), such that $C(t)$ is an invertible matrix, could be represented as a function of following SDE:

$$dZ_t = P(t)dt + Q(t)dW_t, \quad P(t) = \left(Q(t)C^{-1}(t)\right)\left(A(t) - \frac{1}{2}C(t)^2\right). \tag{25}$$

that $Q(t)$, is considered as a rotation matrix and

$$\Psi(t) = U(Z_t) = U = \exp\left(Z_t C(t) Q^{-1}(t)\right).$$

Proof: Considering $dZ_t = P(t)dt + Q(t)dW_t$, $Q(t) = \begin{pmatrix} \hat{a}(t) & -\hat{b}(t) \\ \hat{b}(t) & \hat{a}(t) \end{pmatrix}$ and applying *itô* formula under change of variable method for $U(Z_t) = \Psi(t)$, on fundamental matrix equation(28), we get:

$$\begin{cases} U'(P(t)) + \frac{1}{2}U''Q^2(t) = UA(t) \\ U'Q(t) = UC(t) \end{cases} \tag{26}$$

Since $Q(t)$ is a commutative and invertible matrix, therefore $U' = UC(t)Q^{-1}(t)$ and consequently $U = \exp\left(Z_t C(t) Q^{-1}(t)\right)$. Afterwards, with substituting this solution in first equation of (26), we get

$$UC(t)Q^{-1}(t)(P(t)) + \frac{1}{2}U(C(t)Q^{-1}(t))^2Q^2(t) = UA(t).$$

in other words, we conclude

$$P(t) = \left(C(t)Q^{-1}(t)\right)^{-1}\left(A(t) - \frac{1}{2}(Q^{-1}(t)C(t))^2Q^2(t)\right)$$

$$= (Q(t)C^{-1}(t))(A(t) - \frac{1}{2}C(t)^2).$$

thus, the proof is completed. \square .

Example 2 Consider the following stochastic model

$$\begin{cases} dX = \frac{3}{4}t^2 X^2 dt + tX^{3/2} dW, \\ X(0) = 0. \end{cases}$$

It could be checked that for this equation the necessary condition holds. according to (26) $u'b(t) = tu^{3/2}$. since u is just a function of Y , so it should be $b(t) = t$, $u = \frac{4}{Y^2}$ and also $\frac{a(t)}{b(t)} = 0$ or $a(t) = 0$. thus $dY = t dW$ and $Y = \int_0^t s dW_s + Y(0)$. Finally $X = u(Y) = 4(\int_0^t s dW_s + Y(0))^{-2}$ In the end, we represent the explicit solution of (27), based on the theorems (2.7) and (2.5).

Theorem 2.8 Suppose $(X_t, V_t)^T \in \mathbb{R}^2$ is a stochastic process and consider the following S.D.E in $\mathbb{L}^2(0, T)$.

$$\begin{cases} d \begin{pmatrix} X_t \\ V_t \end{pmatrix} = \left(\Gamma(t) \begin{pmatrix} X_t \\ V_t \end{pmatrix} + \begin{pmatrix} 0 \\ h(t) \end{pmatrix} \right) dt + \left(\Sigma(t) \begin{pmatrix} X_t \\ V_t \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{h}(t) \end{pmatrix} \right) dW_t, \\ \begin{pmatrix} X_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}. \end{cases} \quad (27)$$

Whence, solution of this system is $\begin{pmatrix} X_t \\ V_t \end{pmatrix} = \Psi_t \left(\begin{pmatrix} X_0 \\ V_0 \end{pmatrix} + \int_0^t \Psi_s^{-1} dY_s \right)$, such that stochastic process $Y_t = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$, and the fundamental matrix $\Psi_t \in \mathcal{M}^{n \times n}$ holds in following SDEs:

$$\begin{cases} dY_t = \left(\begin{pmatrix} 0 \\ h(t) \end{pmatrix} - \Sigma(t) \begin{pmatrix} 0 \\ \hat{h}(t) \end{pmatrix} \right) dt + \begin{pmatrix} 0 \\ \hat{h}(t) \end{pmatrix} dW_t, \\ d\Psi_t = \Gamma(t) \Psi_t dt + \Sigma(t) \Psi_t dW_t, \quad \Psi_0 = I. \end{cases} \quad (28)$$

and also the solution of last equation is $\Psi(t) = U(Z_t) = U = \exp(Z_t C(t) Q^{-1}(t))$, that Z_t , is the solution of following SDE, and $Q(t)$, is considered as a rotation matrix:

$$dZ_t = P(t) dt + Q(t) dW_t, \quad P(t) = (Q(t) \Sigma^{-1}(t)) \left(\Gamma(t) - \frac{1}{2} \Sigma(t)^2 \right). \quad (29)$$

That is, the fundamental matrix is

$$\Psi(t) = \exp \left(\left(\int_0^t P(s) ds + \int_0^t Q(s) dW_s \right) C(t) Q^{-1}(t) \right)$$

3. Conclusion

As it was indicated in this paper, we performed a survey on stochastic ordinary differential equations from second-order with time-varying and Gaussian random coefficients. We indicated a complete analysis for stochastic second-order equations in special case of scalar linear second-order equations (damped harmonic oscillators with additive or multiplicative noises). Afterwards, with making a system of stochastic differential equations from this mentioned equation, In the case of linear stochastic differential equations

system and by computing fundamental matrix of this system, it calculated based on the exact solution of this system. We approximated its solution based on a method with an analytical approach different from other numerical methods. Finally, this stochastic equation was solved by conventional numerically method like E.M. or Milstein. Also its asymptotic stability and statistical concepts like expectation and variance of solutions was discussed.

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