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On the Finsler modules over H^*-algebras

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Abstract. In this paper, applying the concept of generalized A-valued norm on a right H^*-module and also the notion of ϕ-homomorphism of Finsler modules over C^*-algebras we first improve the definition of the Finsler module over H^*-algebra and then define ϕ-morphism of Finsler modules over H^*-algebras. Finally we present some results concerning these new ones.

Keywords: H^*-algebra, full Finsler module, ϕ-morphism, trace class.

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1. Introduction and Preliminaries

Generalized A-valued norm on a right H^*-module has been introduced by [12]Zalar (1995), also Finsler module over a C^*-algebra has been investigated by [7] Phillips and Weaver (1998), then many mathematicians developed these subjects in several directions. The authors of [3] Amyari and Niknam (2003) and [11] Taghavi and Jafarzadeh (2007), studied ϕ-homomorphisms of Finsler modules over C^*-algebras. Taking idea from these notions we are motivated to improve the concept of Finsler module over H^*-algebra (see [1]Ambrose (1945), [4] Balachandran and Swaminathen (1986)) and define ϕ-morphism of Finsler modules over H^*-algebras and investigate some properties for these new ones. A H^*-algebra, introduced by [1]Ambrose (1945) in the associative case, is a Banach algebra A satisfying the following conditions:
(i) A is itself a Hilbert space under an inner product $\langle ., . \rangle$;

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(ii) For each a in A there is an element a^* in A, the so-called adjoint of a, such that we have both $\langle ab, c\rangle = \langle b, a^*c\rangle$ and $\langle ab, c\rangle = \langle a, cb^*\rangle$ for all $b, c \in A$.

***Example* 1.1** The Hilbert space $l^2 = \{\{a_n\}_n : a_n \in \mathbb{C}, \sum_n |a_n|^2 < \infty\}$ is a H^*-algebra, where for each $\{a_n\}_n$ and $\{b_n\}_n$ in l^2, $\{a_n\}_n\{b_n\}_n = \{a_nb_n\}_n$ and $\{a_n\}_n^* = \{\overline{a_n}\}_n$.

***Example* 1.2** Any Hilbert space is a H^*-algebra, where the product each pair of elements to be zero. Of course in this case the adjoint a^* of a need not be unique, in fact every element, is an adjoint of every element.

Recall that $A_0 = \{a \in A : aA = \{0\}\} = \{a \in A : Aa = \{0\}\}$ is called the annihilator ideal of A. A proper H^*-algebra is a H^*-algebra with zero annihilator ideal. [1]Ambrose (1945), proved that a H^*-algebra is proper if and only if every element has a unique adjoint.
The trace class $\tau(A)$ of A is defined by the set $\tau(A) = \{ab : a, b \in A\}$. It is known that $\tau(A)$ is an ideal of A which is a Banach $*$-algebra under a suitable norm $\tau_A(.)$. The norm τ_A is related to the given norm $\|.\|$ on A by $\tau_A(a^*a) = \|a\|^2$ $(a \in A)$ and $\|a\| \leqslant \tau_A(a)$ for each $a \in \tau(A)$ (see [9]Saworotnow (1970)). If A is proper, then $\tau(A)$ is dense in A ([1, Lemma 2.7]). The trace functional tr on $\tau(A)$ is defined by $tr(ab) = \langle a, b^*\rangle = \langle b, a^*\rangle = tr(ba)$ for each $a, b \in A$, in particular $tr(aa^*) = tr(a^*a) = \|a\|^2$ for all $a \in A$. A positive member of A is an element $a \in A$ such that $\langle ax, x\rangle \geqslant 0$ for each $x \in A$. It is known in [9]Saworotnow (1970), that for each $a \in A$ there exists a unique positive member $[a]$ of A such that $[a]^2 = a^*a$. A nonzero element $e \in A$ is called a projection, if it is self adjoint and idempotent. Two idempotents e and e' are doubly orthogonal if $\langle e, e'\rangle = 0$ and $ee' = e'e = 0$. An idempotent is primitive if it can not be expressed as the sum of two doubly orthogonal idempotents. Every proper H^*-algebra contains a maximal family of doubly orthogonal primitive self adjoint idempotents ([1, Theorem 3.3]). If $\{e_i\}_{i\in I}$ is a maximal family of doubly orthogonal primitive self adjoint idempotents in a proper H^*-algebra A, then $A = \sum_{i\in I} e_iA = \sum_{i\in I} Ae_i$ ([1, Theorem 4.1]) and $a = \sum_{i\in I} e_ia = \sum_{i\in I} ae_i$ for each $a \in A$. For, if $a \in A$, then $a = \sum_{i\in I} e_ib_i$ for some $b_i \in A$ and so for each $j \in I$, $e_jb_j = e_j^2b_j = e_j \sum_i e_ib_i = e_ja$. The next part is proved similarly. We recall from [9]Saworotnow (1970), that if a is a nonzero element in A, then there exists a sequence $\{e_n\}_n$ of doubly orthogonal projections and a sequence $\{\lambda_n\}_n$ of positive numbers such that $a^*a = \sum_n \lambda_ne_n$. In this case, $[a] = \sum_n \lambda_n^{\frac{1}{2}}e_n$ and if a is in $\tau(A)$, then $\tau_A(a) = tr([a])$. Throughout this note we mean by a morphism a $*$-homomorphism of proper H^*-algebras.

The notion of Hilbert H^*-module is introduced by [8]Saworotnow (1968) under the name of generalized Hilbert space. It has been studied by Smith, Molnar, Cabrera, Martinez, Rodriguez and others.

Definition 1.3 Let A be a proper H^*-algebra. A Hilbert H^*-module is a left module E over A with a mapping $[\cdot|\cdot] : E \times E \to \tau(A)$, which satisfies the following conditions:
(i) $[\alpha x|y] = \alpha[x|y]$,
(ii) $[x + y|z] = [x|z] + [y|z]$,
(iii) $[ax|y] = a[x|y]$,
(iv) $[x|y]^* = [y|x]$,
(v) For each nonzero element x in E there is a nonzero element a in A such that $[x|x] = a^*a$,
(vi) E is a Hilbert space with the inner product $(x, y) = tr([x|y])$,
for each $\alpha \in \mathbb{C}$, $x, y, z \in E$, $a \in A$. For example every H^*-algebra A is a Hilbert A-module whenever we define $[x|y] = xy^*$. We say Hilbert A-module E is full, if the linear

space generated the set $\{[x|y] : x, y \in E\}$ is τ_A-dense in $\tau(A)$. For the basic facts about Hilbert H^*-modules the reader is referred in [5]Bakic and Guljas (2001), [6]Cabrera, Martinez and Rodriguez (1995), [10]Smith (1974) and references cited therein.

Finsler modules over H^*-algebras are generalization of Hilbert H^*-modules. It first was introduced by [12]Zalar (1995) by defining a generalized A-valued norm on a right H^*-module. It is proved in [12]Zalar (1995), that a generalized A-valued norm ρ on a H^*-module E over a proper H^*-algebra A arises from a $\tau(A)$-valued inner product $[.|.]$ on E, if and only if ρ satisfies the parallelogram low. In this paper, we improve and investigate some facts concerned with this concept. In the sequel, we extend the definition of ϕ-homomorphism of Finsler modules over H^*-algebras by the name of ϕ-morphisms and describe some basic properties of such class of module maps ([11]Taghavi and Jafarzadeh (2007)). This work is a reconstruction of some results appeared in [2]Amyari and Niknam (2003), [3]Amyari and Niknam (2003), [11]Taghavi and Jafarzadeh (2007), to Finsler modules over H^*-algebras and is also interesting in its own.

2. Main Results

Definition 2.1 ([12]Zalar (1995)) Let A be a proper H^*-algebra and E be a complex linear space which is a left A-module (and $\lambda(ax) = (\lambda a)x = a(\lambda x)$ where $\lambda \in \mathbb{C}$, $a \in A$ and $x \in E$) equipped with a map $\rho_A : E \to \{a^*a : a \in A\}$ such that
(i) the map $\|.\|_E : x \mapsto tr(\rho_A(x))^{\frac{1}{2}}$ is a norm on E;
(ii) $\rho_A(ax) = a\rho_A(x)a^*$ for each $a \in A$ and $x \in E$.
Then E is called a pre-Finsler module over H^*-algebra A. If $(E, \|.\|_E)$ is complete, then E is called a Finsler module. For instance, every Hilbert H^*-module E with the map $\rho_A(x) = [x|x]$ $(x \in E)$ is a Finsler module.
E is said to be a full Finsler module, if the linear subspace generated by $\{\rho_A(x) : x \in E\}$ which is denoted by $\langle\rho_A(E)\rangle$ is τ_A-dense in $\tau(A)$, more precisely $\overline{\langle\rho_A(E)\rangle}^{\tau_A} = \tau(A)$.

Example **2.2** The set $A = l^2$, is a proper H^*-algebra and $\tau(A) = A$ (since A is unital). It is easy to verify that $\{e_i\}_{i\in\mathbb{N}}$ (e_i, has 1 as i-th position and 0 elsewhere) is a maximal family of doubly orthogonal projections for A. If $\{a_n\}_n \in A$, then $\{a_n\}_n^*\{a_n\}_n = \{|a_n|^2\}_n = \sum_n |a_n|^2 e_n$, $[\{a_n\}_n] = \sum_n |a_n| e_n$ and $\tau_A(\{a_n\}_n) = tr([\{a_n\}_n]) = tr(\sum_n |a_n|e_n) = \sum_n |a_n| tr(e_n) = \sum_n |a_n|$. Since $tr(e_n) = tr(e_n^2) = 1$.
Let $E = A$ and $\rho_A : E \to \{\{a_n\}_n^*\{a_n\}_n : \{a_n\}_n \in A\}$ be defined by $\rho_A(\{a_n\}_n) = \{|a_n|^2\}_n$. Then E is a full (Hilbert module) Finsler module over A. For fullness of E, let $\epsilon > 0$ be given and $\{a_n\}_n \in \tau(A)$. Then by definition of τ_A, it is easy to find complex numbers λ_i and $a_{i,n}$ $(n \in \mathbb{N},\ i = 1, ..., k)$, in which $\tau_A(\{\sum_{i=1}^{k}\lambda_i|a_{i,n}|^2 - a_n\}_n) < \epsilon$. Now surjectivity of ρ_A gives the desired result, i.e. $\overline{\langle\rho_A(E)\rangle}^{\tau_A} = \tau(A)$.

The following lemmas which are interesting, will be used frequently later.

Lemma 2.3 Let E be a Finsler module over H^*-algebra A. Then it is a Banach A-module.

Proof. By the definition of Finsler module, E is a Banach space. It remains to show that $\|ax\|_E \leqslant \|a\|\|x\|_E$ for all $a \in A$ and $x \in E$. For, let $x \in E$. Then $\rho_A(x) = b^*b$ for some $b \in A$ and $\|x\|_E = tr(\rho_A(x))^{\frac{1}{2}} = tr(b^*b)^{\frac{1}{2}} = \|b\|$. So $\|ax\|_E^2 = tr(\rho_A(ax)) = tr(a\rho_A(x)a^*) = tr(ab^*ba^*) = \|ba^*\|^2 \leqslant \|b\|^2\|a\|^2 = \|x\|_E^2\|a\|^2$. ■

As a consequence of the above lemma we have $\|ax\|_E \leqslant \tau_A(a)\|x\|_E$ for each $a \in \tau(A)$ and $x \in E$.

Lemma 2.4 Let E be a full Finsler module over H^*-algebra A and $a \in A$. Then $ax = 0$ for all $x \in E$ if and only if $a = 0$.

Proof. Firstly, suppose that $a \in \tau(A)$ and also $b \in \tau(A)$ is arbitrary. Since E is full, there exists a sequence $\{u_n\}_n$ in $\langle\rho_A(E)\rangle$ such that $b = \lim_{n\to\infty}{}^{\tau_A} u_n$. Each u_n is of the form $u_n = \sum_{i=1}^{k_n}\lambda_{i,n}\rho_A(x_{i,n})$ in which $\lambda_{i,n} \in \mathbb{C}$, $x_{i,n} \in E$. Hence,

$$aba^* = \lim_{n\to\infty}{}^{\tau_A} au_na^* = \lim_{n\to\infty}{}^{\tau_A}(a\sum_{i=1}^{k_n}\lambda_{i,n}\rho_A(x_{i,n})a^*) = \lim_{n\to\infty}{}^{\tau_A}\sum_{i=1}^{k_n}\lambda_{i,n}\rho_A(ax_{i,n}) = 0. \quad (1)$$

Relation (1) holds since if x is an arbitrary element in E, then $\rho_A(ax) = c^*c$ for some $c \in A$ and by assumption $\|c\|^2 = tr(c^*c) = tr(\rho_A(ax)) = \|ax\|_E^2 = 0$. It implies that $c = 0$ and so $\rho_A(ax) = 0$. Replacing b by a^*a in (1) we get $tr(aba^*) = tr(aa^*aa^*) = \|aa^*\|^2 = 0$. Consequently $aa^* = 0$ and by [1, Lemma 2.2], $a = 0$. Secondly, suppose that $a \in A$ and $ax = 0$ for all $x \in E$. Let $b \in A$ be arbitrary, then by Lemma 2.3. $bax = 0$ for all $x \in E$. By the above discussion and since $ba \in \tau(A)$, so $ba = 0$ for each $b \in A$. It implies that $Aa = 0$. Hence $a = 0$, because A is proper. ■

Remark 1 *If $\phi : A \to B$ is an isometric morphism of H^*-algebras, then for each $a \in A$, $\|\phi(a)\|^2 = \|a\|^2$ and so $\langle\phi(a), \phi(a)\rangle = \langle a, a\rangle$. Whence $tr(\phi(aa^*)) = tr(aa^*)$. If in addition ϕ is an isomorphism, then for each $b \in B$, $tr(\phi^{-1}(bb^*)) = tr(bb^*)$.*

Taking idea from [2]Amyari and Niknam (2003), we have two following theorems.

Theorem 2.5 Let E be a full Finsler module over H^*-algebra B, $\phi : A \to B$ be a morphism of H^*-algebras such that $\phi|_{\tau(A)} : \tau(A) \to \tau(B)$ be a τ-continuous isomorphism and isometric with respect to $\|.\|$. Then by the module action, $ax = \phi(a)x$ and the map $x \mapsto \rho_A(x)$ defined by $\rho_A(x) = \phi^{-1}(\rho_B(x))$, E is a full Finsler A-module.

Proof. It is clear that E is a complex linear space, and by morphism of ϕ, E is a left A-module. Because of isometric isomorphism of $\phi|_{\tau(A)}$, for each $x \in E$ we have $\|x\|_E^A = tr(\rho_A(x))^{\frac{1}{2}} = tr(\phi^{-1}(\rho_B(x)))^{\frac{1}{2}} = tr(\rho_B(x))^{\frac{1}{2}} = \|x\|_E^B$ (2). Furthermore, $\|.\|_E^B$ is a norm on E and so $\|.\|_E^A$ is. Let $a \in A$, $x \in E$, then $\rho_A(ax) = \rho_A(\phi(a)x) = \phi^{-1}(\rho_B(\phi(a)x)) = \phi^{-1}(\phi(a)\rho_B(x)\phi(a)^*) = a\phi^{-1}(\rho_B(x))a^* = a\rho_A(x)a^*$.
Hence E is a pre-Finsler module over A. On the other hand (2) and completeness of $(E, \|.\|_E^B)$ imply that $(E, \|.\|_E^A)$ is complete. Thus E is a Finsler module over A. We will show that E is a full Finsler module over A, i.e. $\overline{\langle\rho_A(E)\rangle}^{\tau_A} = \tau(A)$. Note that by the inverse mapping theorem $(\phi|_{\tau(A)})^{-1} : \tau(B) \to \tau(A)$ is a (τ_B, τ_A)-continuous isomorphism

(and also homeomorphism).

$$\overline{\langle\rho_A(E)\rangle}^{\tau_A} = \overline{\{\sum_{i=1}^{k_n}\lambda_{i,n}\rho_A(x_{i,n}) : \lambda_{i,n}\in\mathbb{C},\ x_{i,n}\in E\}}^{\tau_A}$$
$$= \overline{\{\sum_{i=1}^{k_n}\lambda_{i,n}\phi^{-1}(\rho_B(x_{i,n})) : \lambda_{i,n}\in\mathbb{C},\ x_{i,n}\in E\}}^{\tau_A}$$
$$= \overline{\phi^{-1}\{\sum_{i=1}^{k_n}\lambda_{i,n}\rho_B(x_{i,n}) : \lambda_{i,n}\in\mathbb{C},\ x_{i,n}\in E\}}^{\tau_A}$$
$$= \phi^{-1}\overline{\{\sum_{i=1}^{k_n}\lambda_{i,n}\rho_B(x_{i,n}) : \lambda_{i,n}\in\mathbb{C},\ x_{i,n}\in E\}}^{\tau_B}$$
$$= \phi^{-1}(\ \overline{\langle\rho_B(E)\rangle}^{\tau_B}) = \phi^{-1}(\tau(B)) = \tau(A).$$

■

In the following we shall establish a converse statement to the above theorem.

Theorem 2.6 Let E be a both full Finsler module over A and a full Finsler module over B and let $\phi : A \to B$ be a map such that $ax = \phi(a)x$ and $\phi(\rho_A(x)) = \rho_B(x)$, where $x \in E, a \in A$. Then ϕ is a continuous monomorphism, $\phi|_{\tau(A)} : \tau(A) \to \tau(B)$ is a (τ_A, τ_B)-continuous and it has dense range, i.e. $\overline{\phi|_{\tau(A)}(\tau(A))}^{\tau_B} = \tau(B)$. If for each $x \in E$, $tr(\rho_A(x)) = tr(\rho_B(x))$, then ϕ is isometric on the set $\{a \in A :$ there exists $x \in E$ in which $a^*a = \rho_A(x)\}$.

Proof. For simplicity in writing we put $\phi_1 = \phi|_{\tau(A)}$. Assume that $\{a_n\}_n$ is a sequence in $\tau(A)$ such that $\lim_{n\to\infty}{}^{\tau_A} a_n = 0$ and $\lim_{n\to\infty}{}^{\tau_B} \phi_1(a_n) = b$, $(b \in \tau(B))$. Let x be an arbitrary element in E, then by the comment after Lemma 2.3. $a_n x \to 0$ and $\phi_1(a_n)x \to bx$. By the definition of module action $\phi_1(a_n)x \to 0$. Hence $bx = 0$. Applying Lemma 2.4. $b = 0$. It follows from closed graph theorem that ϕ_1 is (τ_A, τ_B)-continuous. A similar argument shows that ϕ is continuous. Since $(\phi(a + b) - \phi(a) - \phi(b))x = (a + b)x - ax - bx = 0$ for each $x \in E$ and for each $a, b \in A$, so by Lemma 2.4. $\phi(a + b) = \phi(a) + \phi(b)$. Similarly for each $\lambda \in \mathbb{C}$ and for each $a, b \in A$, $\phi(\lambda a) = \lambda\phi(a)$ and $\phi(ab) = \phi(a)\phi(b)$. Now let $a \in \tau(A)$, then we may assume that $a = \lim_{n\to\infty}{}^{\tau_A} u_n$, each u_n is of the form $u_n = \sum_{i=1}^{k_n}\lambda_{i,n}\rho_A(x_{i,n})$ for some $\lambda_{i,n} \in \mathbb{C}$ and $x_{i,n} \in E$. Hence $\phi_1(a^*) = \lim_{n\to\infty}{}^{\tau_B}\phi_1(u_n^*) = \lim_{n\to\infty}{}^{\tau_B}(\phi_1(\sum_{i=1}^{k_n}\overline{\lambda_{i,n}}\rho_A(x_{i,n}))) = \lim_{n\to\infty}{}^{\tau_B}\sum_{i=1}^{k_n}\overline{\lambda_{i,n}}\phi_1(\rho_A(x_{i,n})) = \lim_{n\to\infty}{}^{\tau_B}\sum_{i=1}^{k_n}\overline{\lambda_{i,n}}\rho_B(x_{i,n}) = (\lim_{n\to\infty}{}^{\tau_B}\sum_{i=1}^{k_n}\lambda_{i,n}\rho_B(x_{i,n}))^* = (\lim_{n\to\infty}{}^{\tau_B}\sum_{i=1}^{k_n}\lambda_{i,n}\phi_1(\rho_A(x_{i,n})))^* =$ $(\phi_1(\lim_{n\to\infty}{}^{\tau_A}\sum_{i=1}^{k_n}\lambda_{i,n}\rho_A(x_{i,n})))^* = \phi_1(a)^*$. Therefore ϕ_1 is a morphism. Let $a \in A$, then there exists a sequence $\{a_n\}_n \subseteq \tau(A)$ such that $a = \lim_{n\to\infty} a_n$. By morphism of ϕ_1 and

continuity of ϕ we can write $\phi(a^*) = \phi(\lim_{n\to\infty} a_n^*) = \lim_{n\to\infty} \phi(a_n)^* = (\lim_{n\to\infty} \phi(a_n))^* = (\phi(a))^*$. If $\phi(a) = 0$, then $ax = \phi(a)x = 0$, for all $x \in E$. Hence $a = 0$, by Lemma 2.4. Therefore ϕ is a monomorphism. Given $\epsilon > 0$ and let $b \in \tau(B)$ be arbitrary. Since E is a full Finsler module over B, so $\tau_B(b - \sum_{i=1}^{k_n} \lambda_{i,n}\rho_B(x_{i,n})) < \epsilon$, for some $\lambda_{i,n} \in \mathbb{C}$ and $x_{i,n} \in E$. Hence $\tau_B(b - \phi_1(\sum_{i=1}^{k_n} \lambda_{i,n}\rho_A(x_{i,n}))) < \epsilon$. Therefore ϕ_1 has dense range in $\tau(B)$. Now suppose that for each x in E, $tr(\rho_A(x)) = tr(\rho_B(x))$. Also assume that $a \in A$ and $a^*a = \rho_A(x)$ for some $x \in E$, then $\|a\|^2 = tr(a^*a) = tr(\rho_A(x)) = tr(\rho_B(x)) = tr(\phi(\rho_A(x))) = tr(\phi(a^*a)) = \|\phi(a)\|^2$.
■

We could not drop the condition of fullness. For instance, let $B = l^2$ and $A = E = \{\{a_n\}_n \in B : a_1 = 0\}$. Then E is a full Finsler module over A, when $\rho_A(\{a_n\}_n) = \{|a_n|^2\}_n$ and E is a Finsler module over B when $\rho_B(\{a_n\}_n) = \{|a_n|^2\}_n$. E is not full over B, because let $\{b_nc_n\} \in \tau(B)$ $(= B)$ with b_1c_1 be nonzero. If on the contrary $\overline{\langle\rho_B(E)\rangle}^{\tau_B} = \tau(B)$, then there exist $\lambda_i \in \mathbb{C}$ and $\{a_{i,n}\}_n \in E$ $(i = 1, ..., k)$ in which $\tau_B(\sum_{i=1}^{k} \lambda_i\{|a_{i,n}|^2\}_n - \{b_nc_n\}_n) < \epsilon$ (3). Put $\{d_n\}_n = \sum_{i=1}^{k} \lambda_i\{|a_{i,n}|^2\}_n - \{b_nc_n\}_n$. As we see in Example 2.2. the left side of (3) is equal to $\sum_{n=1}^{\infty} |d_n|$. Hence $|b_1c_1| = |d_1| \leqslant \sum_{n=1}^{\infty} |d_n| < \epsilon$ by (3) and since $\epsilon > 0$ is arbitrary, so $b_1c_1 = 0$, which is a contradiction. Now let $\phi : A \to B$ be the inclusion map, obviously ϕ satisfies in the conditions of Theorem 2.7. i.e, for each $x \in E$ and for each $a \in A$, $ax = \phi(a)x$ and $\phi(\rho_A(x)) = \rho_B(x)$. On the other hand $\overline{\phi(\tau(A))}^{\tau_B} \neq \tau(B)$. Indeed, by a similar argument as above if $\{b_nc_n\}_n \in \tau(B)(= B)$ and $b_1c_1 \neq 0$, then it is not in $\overline{\phi(\tau(A))}^{\tau_B} (= A)$. Thus $\phi|_{\tau(A)}$ does not have dense range in $\tau(B)$.
The following theorem is a version of [3, Lemma 2.2] in the framework of Finsler modules over H^*-algebras.

Theorem 2.7 Let E be a Finsler module over H^*-algebra A, I be a closed two sided ideal of A and x be in E such that $\rho_A(x) \in I$. Then $x = \sum_{\lambda\in\Lambda} e_\lambda x$, where $\{e_\lambda\}_{\lambda\in\Lambda}$ is a maximal family of doubly orthogonal primitive self adjoint idempotents for I.

Proof. Let Λ_0 be a finite subset of Λ. We claim that

$$\rho_A\big(x - \sum_{\lambda\in\Lambda_0} e_\lambda x\big) = \rho_A(x) - \sum_{\lambda\in\Lambda_0} e_\lambda\rho_A(x) - \sum_{\lambda\in\Lambda_0} \rho_A(x)e_\lambda + \sum_{\lambda\in\Lambda_0} e_\lambda[d] \sum_{\gamma\in\Lambda_0} [d]e_\gamma \qquad (4)$$

where $\rho_A(x) = d^*d = [d]^2$ for some $d \in A$ ([9, Lemma 2]). If b is the left side and c is the right side of (4), then obviously b and c are self adjoint and for each $a \in A$, we have

$$\begin{aligned} aca^* &= a\rho_A(x)a^* - a\sum_{\lambda\in\Lambda_0} e_\lambda\rho_A(x)a^* - a\sum_{\lambda\in\Lambda_0}\rho_A(x)e_\lambda a^* + a\sum_{\lambda\in\Lambda_0} e_\lambda[d]\sum_{\gamma\in\Lambda_0}[d]e_\gamma a^* \\ &= \Big(a - \sum_{\lambda\in\Lambda_0} ae_\lambda\Big)\rho_A(x)\Big(a - \sum_{\gamma\in\Lambda_0} ae_\gamma\Big)^* = \rho_A\Big(\big(a - \sum_{\lambda\in\Lambda_0} ae_\lambda\big)x\Big) \\ &= \rho_A\Big(a\big(x - \sum_{\lambda\in\Lambda_0} e_\lambda x\big)\Big) = a\rho_A\Big(x - \sum_{\lambda\in\Lambda_0} e_\lambda x\Big)a^* = aba^*. \end{aligned}$$

Thus for each $a \in A$, $a(c-b)a^* = 0$, specially for $a = c-b$. Hence $(c-b)^3 = 0$ and so $c = b$ by [1, Lemma 2.3]. Consequently $\rho_A(x-\sum_{\lambda\in\Lambda} e_\lambda x) = 0$ and so $tr(\rho_A(x-\sum_{\lambda\in\Lambda} e_\lambda x))^{\frac{1}{2}} = \|x - \sum_{\lambda\in\Lambda} e_\lambda x\|_E = 0$ which implies that, $x = \sum_{\lambda\in\Lambda} e_\lambda x$. ■

Definition 2.8 Let E and F be Finsler modules over proper H^*-algebras A and B respectively and $\phi : A \to B$ be a morphism of H^*-algebras. A linear operator $\Phi : E \to F$ is said to be a ϕ-morphism of Finsler modules if the following conditions are satisfied:
(i) $\Phi(ax) = \phi(a)\Phi(x)$,
(ii) $\rho_B(\Phi(x)) = \phi(\rho_A(x))$,
where $x \in E$ and $a \in A$.
Φ is called a module map if it satisfies in the condition (i). If E, F and G are Finsler modules over proper H^*-algebras A, B and C respectively, $\phi_1 : A \to B$ and $\phi_2 : B \to C$ are morphisms of H^*-algebras, and $\Phi_1 : E \to F$ and $\Phi_2 : F \to G$ are ϕ_1-morphism and ϕ_2-morphism of Finsler modules respectively, then it is straightforward to show that $\Phi_2\Phi_1 : E \to G$ is a $\phi_2\phi_1$-morphism of Finsler modules.

In the following we state some results appeared in [11]Taghavi and Jafarzadeh (2007) to Finsler modules over H^*-algebras.

Theorem 2.9 Let E and F be Finsler modules over H^*-algebras A and B respectively, $\phi : A \to B$ be a morphism in which $\phi|_{\tau(A)} : \tau(A) \to \tau(B)$ be a (τ_A, τ_B)-continuous injective morphism and $\phi(\tau(A))$ be τ_B-closed in $\tau(B)$. Also let $\Phi : E \to F$ be a ϕ-morphism. If $Im(\Phi)$ is a full Finsler module over $Im(\phi)$, then E is a full Finsler module over A.

Proof. Applying inverse mapping theorem, $(\phi|_{\tau(A)})^{-1} : \phi(\tau(A)) \to \tau(A)$ is a (τ_B, τ_A)-continuous morphism. We will show that E is full. Let $a \in \tau(A)$ be arbitrary, then $a = a_1a_2$ for some $a_1, a_2 \in A$. Therefore $\phi(a) = \phi(a_1)\phi(a_2) \in \tau(Im(\phi))$. Since $Im(\Phi)$ is a full Finsler $Im(\phi)$-module, thus we have

$$\begin{aligned} \phi(a) &= \lim_{n\to\infty}^{\tau_B}\sum_{i=1}^{k_n}\lambda_{i,n}\rho_B(\Phi(x_{i,n})) \\ &= \lim_{n\to\infty}^{\tau_B}\sum_{i=1}^{k_n}\lambda_{i,n}\phi(\rho_A(x_{i,n})) \quad (5) \end{aligned}$$

for some $\lambda_{i,n} \in \mathbb{C}$, $x_{i,n} \in E$. Effecting (τ_B, τ_A)-continuous morphism $(\phi|_{\tau(A)})^{-1}$ to both sides of (5), we obtain that $a = \lim_{n\to\infty}^{\tau_A}\sum_{i=1}^{k_n}\lambda_{i,n}\rho_A(x_{i,n})$ by injectivity of the morphism $\phi|_{\tau(A)}$. Thus $a \in \overline{\langle \rho_A(E) \rangle}^{\tau_A}$ and therefore $\tau(A) \subseteq \overline{\langle \rho_A(E) \rangle}^{\tau_A} \subseteq \tau(A)$. So $\tau(A) = \overline{\langle \rho_A(E) \rangle}^{\tau_A}$. Note that ϕ cannot come out in (5). ■

The following lemma is proved in the framework of Finsler modules over C^*-algebras ([11, Lemma 3.1]). It is easy to show this lemma in the Finsler modules over H^*-algebras.

Lemma 2.10 Let E and F be Finsler and full Finsler module over H^*-algebras A and B respectively, $\phi_i \quad (i = 1, 2)$ be maps from A to B and $\Phi : E \to F$ be a surjective map satisfies $\Phi(ax) = \phi_i(a)\Phi(x)$ $(i = 1, 2)$ for all $x \in E$ and $a \in A$. Then $\phi_1 = \phi_2$.

Theorem 2.11 Let E and F be full Finsler modules over H^*-algebras A and B respectively and $\Phi : E \to F$ be a continuous isomorphism satisfies $\Phi(ax) = \phi(a)\Phi(x)$ and $\rho_B(\Phi(x)) = \phi(\rho_A(x))$, for all $x \in E$ and $a \in A$, where $\phi : A \to B$ be a map. Then ϕ is a continuous monomorphism, $\phi|_{\tau(A)}$ is (τ_A, τ_B)-continuous and has dense range in $\tau(B)$. Moreover, ϕ with these conditions is unique.

Proof. Applying a similar argument in the proof of Theorem 2.6. one can see that, ϕ is a continuous monomorphism and $\phi|_{\tau(A)}$ is (τ_A, τ_B)-continuous. We will show that ϕ is one to one. Let $\phi(a) = 0$ for some $a \in A$, so $\phi(a)\Phi(x) = 0$ for each $x \in E$. Hence $\Phi(ax) = 0$ and by injectivity of Φ, $ax = 0$ for each $x \in E$. Then $a = 0$ by fullness of E. So ϕ is a monomorphism. In addition, $\tau(B) = \overline{\langle \rho_B(F) \rangle}^{\tau_B} = \overline{\langle \rho_B(\Phi(E)) \rangle}^{\tau_B} = \overline{\langle \phi(\rho_A(E)) \rangle}^{\tau_B} \subseteq \overline{\langle \phi(\tau(A)) \rangle}^{\tau_B} = \overline{\phi(\langle \tau(A) \rangle)}^{\tau_B} = \overline{\phi(\tau(A))}^{\tau_B} \subseteq \tau(B)$. Therefore $\overline{\phi(\tau(A))}^{\tau_B} = \tau(B)$ and so $\phi|_{\tau(A)}$ has dense range. Uniqueness of ϕ obtains from Lemma 2.10. ■

Remark 2 *Fullness condition can not be dropped in the above theorem. For example let* $B = l^2$, $A = E = \{\{a_n\}_n \in B : a_1 = 0\}$ *and* $F = \{\{a_n\}_n \in B : a_1 = a_2 = 0\}$. *Then E is a full Finsler module over A, when* $\rho_A(\{a_n\}_n) = \{|a_n|^2\}_n$ *and F is a Finsler module over B, when* $\rho_B(\{a_n\}_n) = \{|a_n|^2\}_n$. *As we mentioned before F is not full Finsler module over B. Let* $\Phi : E \to F$ *defined by* $\Phi(\{a_n\}_n) = \{b_n\}_n$, *where* $b_1 = 0$ *and for* $n = 2, ...,$ $b_n = a_{n-1}$ *and* $\phi : A \to B$ *defined by* $\phi(\{a_n\}_n) = \Phi(\{a_n\}_n)$. *Clearly Φ is a continuous isomorphism,* $\Phi(\{a_n\}_n\{b_n\}_n) = \phi(\{a_n\}_n)\Phi(\{b_n\}_n)$ *and* $\rho_B(\Phi(\{a_n\}_n)) = \phi(\rho_A(\{a_n\}_n))$ *for all* $\{a_n\}_n \in A$ *and* $\{b_n\}_n \in E$. *On the other hand* $\phi(\tau(A))$ $(= \phi(A))$ *dose not have dense range in* $\tau(B)$ $(= B)$.

In the following we state [3, Theorem 3.4], in the framework of Finsler modules over the H^*-algebras.

Theorem 2.12 Let E and F be Finsler modules over H^*-algebras A and B respectively, $\phi : A \to B$ be an isometric morphism and $\Phi : E \to F$ be a ϕ-morphism of Finsler modules. Then
(i) $Im(\Phi)$ is a closed subspace of F.
(ii) $Im(\Phi)$ is a Finsler module over H^*-algebra $Im(\phi)$, such that $\rho_{Im\phi}(\Phi(E)) = \phi(\rho_A(E))$.
(iii) If E is a full Finsler module and $\phi\ |_{\tau(A)}: \tau(A) \to \phi(\tau(A))$ is (τ_A, τ_B)-continuous, then $Im(\Phi)$ is a full Finsler module over the H^*-algebra $Im(\phi)$.
(iv) If Φ is surjective, F is full Finsler module over B and $\phi(\tau(A))$ is τ_B-closed, then $\phi|_{\tau(A)}$ is surjective.

Proof. (i) We will show that Φ is isometry and so it has closed range. Let x be an arbitrary element in E. Then $\rho_A(x) = a^*a$ for some $a \in A$, and since ϕ is isometric so $\|\Phi(x)\|_F = tr(\rho_B(\Phi(x)))^{\frac{1}{2}} = tr(\phi(\rho_A(x)))^{\frac{1}{2}} = tr(\phi(a^*a))^{\frac{1}{2}} = tr(a^*a)^{\frac{1}{2}} = tr(\rho_A(x))^{\frac{1}{2}} = \|x\|_E$.
(ii) Straightforward.

(iii) We will show that $Im(\Phi)$ is a full Finsler module over the H^*-algebra $Im(\phi)$ i.e. $\overline{\langle \rho_B(Im\Phi) \rangle}^{\tau_B} = \tau(Im\phi)$. For this, let $b \in \tau(Im\phi)$, then $b = \phi(a_1 a_2)$ for some $a_1, a_2 \in A$. By fullness of E and (τ_A, τ_B)-continuity of $\phi|_{\tau(A)}$ we have $b = \phi(a_1 a_2) = \phi(\lim_{n\to\infty}{}^{\tau_A} \sum_{i=1}^{k_n} \lambda_{i,n} \rho_A(x_{i,n})) = \lim_{n\to\infty}{}^{\tau_B} \sum_{i=1}^{k_n} \lambda_{i,n} \phi(\rho_A(x_{i,n})) = \lim_{n\to\infty}{}^{\tau_B} \sum_{i=1}^{k_n} \lambda_{i,n} \rho_B(\Phi(x_{i,n}))$ for some $\lambda_{i,n} \in \mathbb{C}$ and $x_{i,n} \in E$. It gives the desired result.

(iv) It follows by the argument applied in the proof of Theorem 2.11. ■

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