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On φ-Connes amenability of dual Banach algebras

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Abstract. Let φ be a w^*-continuous homomorphism from a dual Banach algebra to $\mathbb{C}$. The notion of φ-Connes amenability is studied and some characterizations is given. A type of diagonal for dual Banach algebras is defined. It is proved that the existence of such a diagonal is equivalent to φ-Connes amenability. It is also shown that φ-Connes amenability is equivalent to so-called φ-splitting of a certain short exact sequence.

Keywords: Dual Banach algebra, φ-Connes amenability, φ-injectivity.

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra and E be a Banach $\mathcal{A}$-bimodule. A continuous linear operator $D : \mathcal{A} \longrightarrow E$ is a *derivation* if it satisfies $D(ab) = D(a) . b + a . D(b)$ for all $a, b \in \mathcal{A}$. Given $x \in E$, the *inner* derivation $ad_x : \mathcal{A} \longrightarrow E$ is defined by $ad_x(a) = a . x - x . a$. Amenability for Banach algebras as introduced by B. E. Johnson [4], has proved to be an important and fertile notion. A Banach algebra $\mathcal{A}$ is *amenable* if for every Banach $\mathcal{A}$-bimodule E, every derivation from $\mathcal{A}$ into E^*, the dual of E, is inner. We recall that the projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule in the canonical way. Then the map $\pi : \mathcal{A}\hat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ defined by $\pi(a \otimes b) = ab$, is an $\mathcal{A}$-bimodule homomorphism.

Let $\mathcal{A}$ be a Banach algebra. A Banach $\mathcal{A}$-bimodule E is *dual* if there is a closed submodule E_* of E^* such that $E = (E_*)^*$. We say E_* the *predual* of E. A dual Banach $\mathcal{A}$-bimodule E is *normal* if the module actions of $\mathcal{A}$ on E are w^*-continuous. A Banach

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algebra is *dual* if it is dual as a Banach $\mathcal{A}$-bimodule. We write $\mathcal{A} = (\mathcal{A}_*)^*$ if we wish to stress that $\mathcal{A}$ is a dual Banach algebra with predual $\mathcal{A}_*$. Connes amenability, which seems to be a natural variant of amenability for dual Banach algebras, systematically was introduced by V. Runde [7]. Although, it had been studied previously under different names. A dual Banach algebra $\mathcal{A}$ is *Connes amenable* if every w^*-continuous derivation from $\mathcal{A}$ into a normal, dual Banach $\mathcal{A}$-bimodule is inner. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. Then $\sigma wc(E)$ stands for the set of all elements $x \in E$ such that the maps

$$\mathcal{A} \longrightarrow E \quad , \quad a \longmapsto \begin{cases} a \,.\, x \\ x \,.\, a \end{cases} ,$$

are w^*-weak continuous. It is a closed submodule of E.

A generalization of amenability which depends on homomorphisms was introduced by E. Kaniuth, A. T. Lau and J. Pym in [5]. This concept was also studied independently, by M. S. Monfared in [6]. Let $\mathcal{A}$ be a Banach algebra and φ be a homomorphism from $\mathcal{A}$ onto $\mathbb{C}$. We say $\mathcal{A}$ is *φ-amenable* if there exists a bounded linear functional m on $\mathcal{A}^*$ satisfying $m(\varphi) = 1$ and $m(f \,.\, a) = \varphi(a)m(f)$, for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. We write $\Delta(\mathcal{A})$ for the set of all homomorphism from $\mathcal{A}$ onto $\mathbb{C}$.

In this note we study φ-Connes amenability for dual Banach algebras. The organization of the paper is as follow. Firstly, in section 2, we study basic properties of φ-Connes amenability. We characterize it through vanishing of $H^1_{w^*}(\mathcal{A}, E)$ for certain Banach $\mathcal{A}$-bimodule. A number of hereditary properties are also discussed.

In section 3, we define a type of virtual diagonal for a dual Banach algebra $\mathcal{A}$, showing that the existence of such a diagonal is equivalent to φ-Connes amenability of $\mathcal{A}$.

Finally in section 4, we give a characterization of φ-Connes amenability of a dual Banach algebra $\mathcal{A} = (\mathcal{A}_*)^*$ in terms of so-called φ-splitting of the short exact sequence

$$\sum : \; 0 \longrightarrow \mathcal{A}_* \xrightarrow{\pi^*} \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*) \longrightarrow \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)/\pi^*(\mathcal{A}_*) \longrightarrow 0 \,.$$

2. Basic properties

Suppose that $\mathcal{A}$ is a dual Banach algebra and φ is a homomorphism from $\mathcal{A}$ onto $\mathbb{C}$. Then it is an easy observation that φ is w^*-continuous if and only if $\varphi \in \sigma wc(\mathcal{A}^*)$. For a dual Banach algebra $\mathcal{A}$, $\Delta_{w^*}(\mathcal{A})$ will denote the set of all w^*-continuous homomorphism from $\mathcal{A}$ onto $\mathbb{C}$.

Definition 2.1 Suppose that $\mathcal{A}$ is a dual Banach algebra and $\varphi \in \Delta_{w^*}(\mathcal{A})$. We call $\mathcal{A}$ *φ-Connes amenable* if $\mathcal{A}$ admits a *φ-Connes mean* m, i.e., there exists a bounded linear functional m on $\sigma wc(\mathcal{A}^*)$ satisfying $m(\varphi) = 1$ and $m(f \,.\, a) = \varphi(a)m(f)$ for all $a \in \mathcal{A}$ and $f \in \sigma wc(\mathcal{A}^*)$.

We recall some terminology from [7]. Let $\mathcal{A}$ be a dual Banach algebra and E be a normal, dual Banach $\mathcal{A}$-bimodule. We write $Z^1_{w^*}(\mathcal{A}, E)$ for the set of all w^*-continuous derivations from $\mathcal{A}$ to E. Clearly $B^1(\mathcal{A}, E)$, the set of all inner derivations from $\mathcal{A}$ to E, is a subspace of $Z^1_{w^*}(\mathcal{A}, E)$. Whence we have the meaningful definition $H^1_{w^*}(\mathcal{A}, E) = Z^1_{w^*}(\mathcal{A}, E)/B^1(\mathcal{A}, E)$.

Theorem 2.2 Suppose that $\mathcal{A}$ is a dual Banach algebra and $\varphi \in \Delta_{w^*}(\mathcal{A})$. Then the following are equivalent:

(i) $\mathcal{A}$ is φ-Connes amenable;

(ii) If $E = (E_*)^*$ is a normal, dual Banach $\mathcal{A}$-bimodule such that $x \,.\, a = \varphi(a)x$ for all $x \in E$ and $a \in \mathcal{A}$, then $H^1_{w^*}(\mathcal{A}, E) = \{0\}$.

Proof. (i) $\Longrightarrow$ (ii) Let m be φ-Connes mean for $\mathcal{A}$. Take E as in the clause (ii). Let $D : \mathcal{A} \longrightarrow E$ be a w^*-continuous derivation, so that $D(ab) = a \,.\, D(b) + \varphi(b)D(a)$, $a, b \in \mathcal{A}$. From [9, Corollary 4.6], we know that D^* maps E_* into $\sigma wc(\mathcal{A}^*)$. Take $d = D^* \mid_{E_*}$ and $\tilde{D} = d^* : \sigma wc(\mathcal{A}^*)^* \longrightarrow E$. Set $x_0 = \tilde{D}(m) \in E$. Then for all $a \in \mathcal{A}$, $x \in E$ and $f \in E_*$ we have $\langle x, a \,.\, f\rangle = \langle x \,.\, a, f\rangle = \varphi(a)\langle x, f\rangle$, so that $a \,.\, f = \varphi(a)f$ and hence $d(a \,.\, f) = \varphi(a)d(f)$. For all $a \in \mathcal{A}$ and $f \in E_*$, we get for the right action of $\mathcal{A}$ on E_*

$$\begin{aligned}\langle b, d(f \,.\, a)\rangle &= \langle D(b), f \,.\, a\rangle = \langle a \,.\, D(b), f\rangle \\ &= \langle D(ab), f\rangle - \varphi(b)\langle D(a), f\rangle \\ &= \langle b, d(f) \,.\, a\rangle - \varphi(b)\langle f, D(a)\rangle \ .\end{aligned}$$

Therefore $d(f \,.\, a) = d(f) \,.\, a - \langle f, D(a)\rangle\varphi$ for all $a \in \mathcal{A}$ and $f \in E_*$. It follows that

$$\begin{aligned}\langle f, a \,.\, x_0\rangle &= \langle f \,.\, a, \tilde{D}(m)\rangle = \langle d(f \,.\, a), m\rangle \\ &= \langle d(f) \,.\, a, m\rangle - \langle f, D(a)\rangle \\ &= \varphi(a)\langle f, x_0\rangle - \langle f, D(a)\rangle\end{aligned}$$

and hence $D(a) = \varphi(a)x_0 - a \,.\, x_0$. Then we obtain $D(a) = a \,.\, (-x_0) - (-x_0) \,.\, a = ad_{-x_0}(a)$, for all $a \in \mathcal{A}$, as required.

(ii) $\Longrightarrow$ (i) It is easy because in order to prove φ-Connes amenability of $\mathcal{A}$, the condition $H^1_{w^*}(\mathcal{A}, E) = \{0\}$ only exploit for a normal, dual Banach $\mathcal{A}$-bimodule with right action given by $x \,.\, a = \varphi(a)x$ for all $x \in E$ and $a \in \mathcal{A}$. ∎

Let $\mathcal{A}$ be a dual Banach algebra. It is known that its *unitization* $\mathcal{A}^\sharp = \mathcal{A} \oplus \mathbb{C}e$, is a dual Banach algebra as well, where e is the identity of $\mathcal{A}^\sharp$. We define $f_0 : \mathcal{A}^\sharp \longrightarrow \mathbb{C}$ by $f_0(e) = 1$ and $f_0 \mid_{\mathcal{A}} = 0$, so that $(\mathcal{A}^\sharp)^* = \mathcal{A}^* \oplus \mathbb{C}f_0$. Let $\varphi \in \Delta_{w^*}(\mathcal{A})$ and let $\varphi^\sharp$ be its unique extension to $\mathcal{A}^\sharp$, i.e., $\varphi^\sharp(a + \lambda e) = \varphi(a) + \lambda$, $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$. It is obvious that $\varphi^\sharp \in \Delta_{w^*}(\mathcal{A}^\sharp)$. For the right module action in f_0, we have $f_0 \,.\, (a + \lambda e) = \lambda f_0$, since $f_0 \,.\, a = 0$, for all $a \in \mathcal{A}$. We may identify $(\mathbb{C}f_0)^*$ with $\mathbb{C}m_0$, where m_0 is a functional on $(\mathcal{A}^\sharp)^*$ defined by $m_0(f_0) = 1$ and $m_0 \mid_{\mathcal{A}^*} = 0$. Therefore, if we consider $\mathbb{C}f_0$ as a sub $\mathcal{A}^\sharp$-bimodule of $(\mathcal{A}^\sharp)^*$, then we see that $f_0 \in \sigma wc(\mathbb{C}f_0)$ so that $\sigma wc(\mathbb{C}f_0) = \mathbb{C}f_0$. Therefore, we conclude that $\sigma wc((\mathcal{A}^\sharp)^*) = \sigma wc(\mathcal{A}^*) \oplus \mathbb{C}f_0$, so that $\sigma wc((\mathcal{A}^\sharp)^*)^* = \sigma wc(\mathcal{A}^*)^* \oplus \mathbb{C}m_0$.

Now, we are ready to prove the following.

Theorem 2.3 Suppose that $\mathcal{A}$ is a dual Banach algebra and $\varphi \in \Delta_{w^*}(\mathcal{A})$. Then $\mathcal{A}$ is φ-Connes amenable if and only if $\mathcal{A}^\sharp$ is $\varphi^\sharp$-Connes amenable.

Proof. Let $\mathcal{A}$ be φ-Connes amenable and let $m \in \sigma wc(\mathcal{A}^*)^*$ be a φ-Connes mean for $\mathcal{A}$. We define $n \in \sigma wc((\mathcal{A}^\sharp)^*)^*$ by

$$n(f + \lambda f_0) = m(f) \ , \quad (f \in \sigma wc(\mathcal{A}^*), \ \lambda \in \mathbb{C}) \ .$$

Then $n(\varphi^\sharp) = n(\varphi + f_0) = m(\varphi) = 1$, and

$$\begin{aligned} n((f + \lambda f_0) \ . \ (a + \mu e)) &= m(f \ . \ a + \mu f) = \varphi(a)m(f) + \mu m(f) \\ &= (\varphi(a) + \mu)m(f) = \varphi^\sharp(a + \mu e)n(f + \lambda f_0) \end{aligned}$$

for $f \in \sigma wc(\mathcal{A}^*)$, $a \in \mathcal{A}$, and $\lambda, \mu \in \mathbb{C}$. Thus n is a $\varphi^\sharp$-Connes mean for $\mathcal{A}^\sharp$.

Conversely, suppose that there exists $m \in \sigma wc((\mathcal{A}^\sharp)^*)^*$ with $m(\varphi^\sharp) = 1$ and

$$m((f + \lambda f_0) \ . \ (a + \mu e)) = \varphi^\sharp(a + \mu e)m(f + \lambda f_0)$$

for $f \in \sigma wc(\mathcal{A}^*)$, $a \in \mathcal{A}$, and $\lambda, \mu \in \mathbb{C}$. Since $f_0 \ . \ a = 0$ for $a \in \mathcal{A}$, $m(f_0 \ . \ a) = 0$. Choosing $a \in \mathcal{A}$ such that $\varphi(a) = 1$, we conclude that $m(f_0) = 0$. Then $n(\varphi) = 1$ and $n(f \ . \ a) = \varphi^\sharp(a + 0e)m(f + 0e) = \varphi(a)n(f)$ for $f \in \sigma wc(\mathcal{A}^*)$ and $a \in \mathcal{A}$, as required. ■

Theorem 2.4 Suppose that $\mathcal{A}$ and $\mathcal{B}$ are dual Banach algebras, $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ is a continuous and w^*-continuous homomorphism with w^*-dense range, and that $\varphi \in \Delta_{w^*}(\mathcal{B})$. If $\mathcal{A}$ is $\varphi \circ \theta$-Connes amenable, then $\mathcal{B}$ is φ-Connes amenable.

Proof. Notice that $\varphi \circ \theta \in \Delta_{w^*}(\mathcal{A})$. Suppose that $m \in \sigma wc(\mathcal{A}^*)^*$ satisfies $m(\varphi \circ \theta) = 1$ and $m(f \ . \ a) = (\varphi \circ \theta)(a)m(f)$ for all $a \in \mathcal{A}$ and $f \in \sigma wc(\mathcal{A}^*)$. Define $n \in \sigma wc(\mathcal{B}^*)^*$ by $n(g) = m(g \circ \theta)$ for $g \in \sigma wc(\mathcal{B}^*)$. Next, for $a \in \mathcal{A}$ and $g \in \sigma wc(\mathcal{B}^*)$ we have $(g \ . \ \theta(a)) \circ \theta = (g \circ \theta) \ . \ a$, and hence

$$\begin{aligned} n(g \ . \ \theta(a)) &= m((g \ . \ \theta(a)) \circ \theta) = m((g \circ \theta) \ . \ a) \\ &= (\varphi \circ \theta)(a)m(g \circ \theta) = (\varphi \circ \theta)(a)n(g) \ . \end{aligned}$$

Since $\theta(\mathcal{A})$ is w^*-dense in $\mathcal{B}$, the above equation suffices to prove φ-Connes amenability of $\mathcal{B}$. ■

Analogously, we may obtain the following.

Theorem 2.5 Suppose that $\mathcal{A}$ is a Banach algebra, $\mathcal{B}$ is a dual Banach algebra, $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ is a continuous homomorphism with w^*-dense range, and that $\varphi \in \Delta_{w^*}(\mathcal{B})$. If $\mathcal{A}$ is $\varphi \circ \theta$-amenable, then $\mathcal{B}$ is φ-Connes amenable.

Let $\mathcal{A}$ be an Arens regular Banach algebra which is an ideal in $\mathcal{A}^{**}$. It is immediate that $\mathcal{A}^{**}$ is a dual Banach algebra [8]. Let $\varphi \in \Delta(\mathcal{A})$. Then $\tilde{\varphi}$, the extension of φ to $\mathcal{A}^{**}$, belongs to $\Delta_{w^*}(\mathcal{A}^{**})$. To see this, suppose that $\Lambda_\alpha \xrightarrow{w^*} \Lambda$ in $\mathcal{A}^{**}$ and choose $a \in \mathcal{A}$ such that $\varphi(a) \neq 0$. Then $a\Lambda_\alpha \xrightarrow{wk} a\Lambda$ in $\mathcal{A}$, since $\mathcal{A}$ is an ideal of $\mathcal{A}^{**}$. Therefore $\varphi(a) \lim_\alpha \tilde{\varphi}(\Lambda_\alpha) = \lim_\alpha \varphi(a\Lambda_\alpha) = \varphi(a\Lambda) = \varphi(a)\tilde{\varphi}(\Lambda)$, so that $\lim_\alpha \tilde{\varphi}(\Lambda_\alpha) = \tilde{\varphi}(\Lambda)$.

Theorem 2.6 Let $\mathcal{A}$ be an Arens regular Banach algebra which is an ideal in $\mathcal{A}^{**}$, and let $\varphi \in \Delta(\mathcal{A})$. Then the following are equivalent:
(i) $\mathcal{A}$ is φ-amenable.
(ii) $\mathcal{A}^{**}$ is $\tilde{\varphi}$-Connes amenable.

Proof. $(i) \longrightarrow (ii)$ Because $\varphi = \tilde{\varphi} \circ \imath$, where $\imath : \mathcal{A} \hookrightarrow \mathcal{A}^{**}$ is the inclusion map, this is an immediate consequence of Theorem 2.5.

$(ii) \longrightarrow (i)$ By the assumption, there is $m \in \sigma wc(\mathcal{A}^{***})^*$ such that $m(\tilde{\varphi}) = 1$ and $m(F \ . \ u) = \tilde{\varphi}(u)m(F)$, for $u \in \mathcal{A}^{**}$ and $F \in \sigma wc(\mathcal{A}^{***})$. Set $\bar{m} = m|_{\mathcal{A}^*}$, the restriction of m to $\mathcal{A}^*$. Since $\mathcal{A}^{**}$ is a dual Banach algebra, $\mathcal{A}^* \subseteq \sigma wc(\mathcal{A}^{***})$ and therefore $\bar{m}$ is well-defined. Then, it is readily seen that $\bar{m}(\varphi) = m(\tilde{\varphi}) = 1$ and $\bar{m}(f\ .\ a) = \varphi(a)\bar{m}(f)$, $a \in \mathcal{A}$, $f \in \mathcal{A}^*$. ■

Remark 1 *Let $\mathcal{A}$ be a (commutative) dual Banach algebra and let $\varphi \in \Delta_{w^*}(\mathcal{A})$. Suppose that $\mathcal{A}$ admits a non-trivial bounded w^*-point derivation at φ, that is, there exists $0 \neq d \in \mathcal{A}^*$ such that d is w^*-continuous and $d(ab) = \varphi(a)d(b) + d(a)\varphi(b)$, $(a, b \in \mathcal{A})$. Then we say that $\mathcal{A}$ is not φ-Connes amenable. To see this, we consider $\mathbb{C}^* = \mathbb{C}$ as a normal, dual Banach $\mathcal{A}$-bimodule with actions $a\ .\ z = z\ .\ a = \varphi(a)z$, for $a \in \mathcal{A}$ and $z \in \mathbb{C}$. Therefore d is a w^*-continuous derivation and then d is inner, by φ-Connes amenability of $\mathcal{A}$. But any derivation of $\mathcal{A}$ on $\mathbb{C}$ is zero.*

Example 2.7 It is shown that the discrete convolution algebra $\ell^1(\mathbb{Z}^+)$ is isomorphic to the $A^+(\bar{\mathbb{D}})$, the commutative Banach algebra of all functions $f = \sum_{n=0}^{\infty} c_n Z^n$ in the disk algebra $A(\bar{\mathbb{D}})$ which have an absolutely convergent Taylor expansion on $\bar{\mathbb{D}}$, where $\mathbb{D}$ denotes the open unit disk. The map $z \longmapsto \varphi_z$, where φ_z is the point derivation at z, i.e, $\varphi_z(\sum_{n=0}^{\infty} c_n\delta_n) = \sum_{n=0}^{\infty} c_n z^n$, is a bijection between $\bar{\mathbb{D}}$ and $\Delta(\ell^1(\mathbb{Z}^+))$. The reader may see [1] for more information. When $|z| = 1$, we observe that φ_z is not w^*-continuous. On the other hand, if $z \in \mathbb{D}$ then φ_z is w^*-continuous. Therefore $\Delta_{w^*}(\ell^1(\mathbb{Z}^+)) = \mathbb{D}$. For $z \in \mathbb{D}$, the map $d : \ell^1(\mathbb{Z}^+) \longrightarrow \mathbb{C}$ given by

$$d(f) = f'(z) = \sum_{n=0}^{\infty} nc_n z^{n-1}\ , \quad (f = \sum_{n=0}^{\infty} c_n\delta_n \in \ell^1(\mathbb{Z}^+))$$

is a bounded w^*-point derivation at φ_z. We notice that the w^*-continuity of d is a consequence of the fact that $\lim_{n\longrightarrow\infty} nz^{n-1} = 0$. Then by Remark 2.7, we conclude that $\ell^1(\mathbb{Z}^+)$ is not φ_z-Connes amenable for each $z \in \mathbb{D}$.

3. φ-σwc Diagonal

Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra. It is known that $\pi^*(\mathcal{A}_*) \subseteq \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)$ and then taking adjoint, we can extend π to an $\mathcal{A}$-bimodule homomorphism $\pi_{\sigma wc}$ from $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ to $\mathcal{A}$. A *σwc-virtual diagonal* for $\mathcal{A}$ is an element $M \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $a\ .\ M = M\ .\ a$ and $a\pi_{\sigma wc}(M) = a$ for $a \in \mathcal{A}$. It is known that Connes amenability of $\mathcal{A}$ is equivalent to existence of a σwc-virtual diagonal for $\mathcal{A}$. The reader is referred to [9] for the proofs and more details.

From [9], we also know that $\pi^*(\sigma wc(\mathcal{A}^*)) \subseteq \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)$. So if $\varphi \in \Delta_{w^*}(\mathcal{A})$, then $\varphi \otimes \varphi = \pi^*(\varphi) \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)$, where $\varphi \otimes \varphi(a \otimes b) = \varphi(a)\varphi(b)$, for $a, b \in \mathcal{A}$.

Definition 3.1 Let $\mathcal{A}$ be a dual Banach algebra, and let $\varphi \in \Delta_{w^*}(\mathcal{A})$. An element $M \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ is a *φ-σwc virtual diagonal* for $\mathcal{A}$ if
(*i*) $a\ .\ M = \varphi(a)M \quad (a \in \mathcal{A})$;
(*ii*) $\langle \varphi \otimes \varphi, M\rangle = 1$.

Remark 2 *Let $\mathcal{A}$ be a dual Banach algebra. Taking adjoint of the restriction map $\pi^* \mid_{\sigma wc(\mathcal{A}^*)}$, we obtain an $\mathcal{A}$-bimodule homomorphism $\pi^0_{\sigma wc} : \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^* \longrightarrow \sigma wc(\mathcal{A}^*)^*$. Because we choose homomorphisms from $\sigma wc(\mathcal{A}^*)$, which is larger than $\mathcal{A}_*$, working with $\pi^0_{\sigma wc}$ seems more natural than that of $\pi_{\sigma wc}$. As a consequence, we observe that $\langle \varphi \otimes \varphi, M\rangle = \langle \varphi, \pi^0_{\sigma wc}(M)\rangle$, whenever $\varphi \in \Delta_{w^*}(\mathcal{A})$ and $M \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$.*

With these preparations, we can now characterize φ-Connes amenable dual Banach algebras through the existence of φ-σwc virtual diagonals.

Theorem 3.2 Let $\mathcal{A}$ be a dual Banach algebra, and let $\varphi \in \Delta_{w^*}(\mathcal{A})$. Then the following are equivalent:
(i) $\mathcal{A}$ is φ-Connes amenable.
(ii) There is a φ-σwc virtual diagonal for $\mathcal{A}$.

Proof. (i) $\longrightarrow$ (ii) Consider the Banach $\mathcal{A}$-bimodule $\mathcal{A}\hat{\otimes}\mathcal{A}$ with the module actions given by

$$a \,.\, (b \otimes c) = ab \otimes c \quad \text{and} \quad (b \otimes c) \,.\, a = \varphi(a) b \otimes c \quad (a, b, c \in \mathcal{A}) \,.$$

Put $E = \frac{\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)}{\mathbb{C}(\varphi\otimes\varphi)}$. Then $E^* = \mathbb{C}(\varphi \otimes \varphi)^\perp = \{\Lambda \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^* \,:\, \Lambda(\varphi \otimes \varphi) = 0\}$ is a normal, dual Banach $\mathcal{A}$-bimodule for which the right module action is given by $\Lambda \,.\, a = \varphi(a)\Lambda$, $a \in \mathcal{A}$, $\Lambda \in E^*$. Choose $M_0 \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $M_0(\varphi \otimes \varphi) = 1$. Then we see that the image of the inner derivation $ad_{M_0} : \mathcal{A} \longrightarrow \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ is a subset of E^*. By our assumption, there exists $M_1 \in E^*$ such that $ad_{M_0} = ad_{M_1}$. Then it is easy to check that $M := M_0 - M_1$ is a φ-σwc virtual diagonal for $\mathcal{A}$.

(ii) $\longrightarrow$ (i) Suppose that $M \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ is a φ-σwc virtual diagonal for $\mathcal{A}$. It is clear that $\pi^0_{\sigma wc}(M)(\varphi) = 1$. For $f \in \sigma wc(\mathcal{A}^*)$ and $a \in \mathcal{A}$, then we have

$$\pi^0_{\sigma wc}(M)(f \,.\, a) = \langle f, a \,.\, \pi^0_{\sigma wc}(M)\rangle = \langle f, \pi^0_{\sigma wc}(a \,.\, M)\rangle = \varphi(a)\pi^0_{\sigma wc}(M)(f) \,.$$

This shows that $\pi^0_{\sigma wc}(M)$ is a φ-Connes mean for $\mathcal{A}$, as required. ∎

4. φ-splitting

Let $\mathcal{A}$ be a Banach algebra and let X, Y and Z be Banach $\mathcal{A}$-bimodules. We recall that a short exact sequence $\Theta : \; 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is *admissible*, if there exists a bounded linear map $\rho : Y \longrightarrow X$ such that $\rho \circ f$ is the identity map on X. Further, Θ *splits* if we may choose ρ to be an $\mathcal{A}$-bimodule homomorphism.

Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a unital dual Banach algebra. Then the short exact sequence

$$\sum : \; 0 \longrightarrow \mathcal{A}_* \xrightarrow{\pi^*} \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*) \longrightarrow \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)/\pi^*(\mathcal{A}_*) \longrightarrow 0$$

of $\mathcal{A}$-bimodules is admissible (indeed, the map $\rho : \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^* \longrightarrow \mathcal{A}_*$ defined by $\rho(T) = T(e)$ is a bounded left inverse to $\pi^*|_{\mathcal{A}_*}$). In this section we restrict ourselves to the case where $\varphi \in \Delta_{w^*}(\mathcal{A}) \cap \mathcal{A}_*$. In fact, we choose φ's in $\mathcal{A}_*$ because we are interested in the splitting of the short exact sequence $\sum$. Then our result would be comparable to the Daws's theorem; $\mathcal{A}$ is Connes-amenable if and only if $\sum$ splits [2, Proposition 4.4].

Definition 4.1 Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a unital dual Banach algebra, and let $\varphi \in \Delta_{w^*}(\mathcal{A}) \cap \mathcal{A}_*$. We say that $\sum$ *φ-splits* if there exists a bounded linear map $\rho : \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*) \longrightarrow \mathcal{A}_*$ such that $\rho \circ \pi^*(\varphi) = \varphi$ and $\rho(T \,.\, a) = \varphi(a)\rho(T)$, for all $a \in \mathcal{A}$ and $T \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)$.

Theorem 4.2 Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a unital dual Banach algebra, and let $\varphi \in \Delta_{w^*}(\mathcal{A}) \cap \mathcal{A}_*$. Then the following are equivalent:
(i) $\mathcal{A}$ is φ-Connes amenable;
(ii) the short exact sequence $\sum$ φ-splits.

Proof. (i) $\longrightarrow$ (ii) Suppose that M is a φ-σwc virtual diagonal for $\mathcal{A}$. Define the map

$\rho : \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*) \longrightarrow \mathcal{A}^*$ by

$$\langle a, \rho(T)\rangle := \langle T\ .\ a, M\rangle \quad (a \in \mathcal{A},\ T \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)).$$

The same argument as in the proof of [2, Proposition 4.4], shows that ρ maps into $\mathcal{A}_*$. Then for $a \in \mathcal{A}$

$$\langle a, \rho\circ\pi^*(\varphi)\rangle = \langle \pi^*(\varphi)\ .\ a, M\rangle = \langle \pi^*(\varphi), a\ .\ M\rangle = \varphi(a)\langle \pi^*(\varphi), M\rangle = \varphi(a)\ ,$$

hence $\rho\circ\pi^*(\varphi) = \varphi$. Next, for $a, b \in \mathcal{A}$

$$\begin{aligned}\langle b, \rho(T\ .\ a) &= \langle T\ .\ ab, M\rangle = \langle T, ab\ .\ M\rangle = \varphi(ab)\langle T, M\rangle \\ &= \varphi(a)\langle T, b\ .\ M\rangle = \varphi(a)\langle T\ .\ b, M\rangle = \varphi(a)\langle b, \rho(T)\rangle\end{aligned}$$

so that $\rho(T\ .\ a) = \varphi(a)\rho(T)$, as required.

$(ii) \longrightarrow (i)$ Let the map $\rho : \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*) \longrightarrow \mathcal{A}_*$ be such that $\rho\circ\pi^*(\varphi) = \varphi$ and $\rho(T\ .\ a) = \varphi(a)\rho(T)$, for all $a \in \mathcal{A}$ and $T \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)$. Take $M = \rho^*(e)$, where e is the identity of $\mathcal{A}$. Then we have

$$\langle T, a\ .\ M - \varphi(a)M\rangle = \langle \rho(T\ .\ a), e\rangle - \varphi(a)\langle T, M\rangle = \varphi(a)\langle \rho(T), e\rangle - \varphi(a)\langle T, M\rangle = 0\ .$$

We also observe that

$$\langle \varphi\otimes\varphi, M\rangle = \langle \pi^*(\varphi), M\rangle = \langle \rho\circ\pi^*(\varphi), e\rangle = \langle \varphi, e\rangle = 1\ ,$$

so that M is a φ-σwc virtual diagonal for $\mathcal{A}$ and therefore $\mathcal{A}$ is φ-Connes amenable, by Theorem 3.3. ■

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