On (σ, τ)-module extension Banach algebras

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Abstract. Let A be a Banach algebra and X be a Banach A-bimodule. In this paper, we define a new product on $A \oplus X$ and generalize the module extension Banach algebras. We obtain characterizations of Arens regularity, commutativity, semisimplity, and study the ideal structure and derivations of this new Banach algebra.

1.   Introduction

Let A be a Banach algebra and X be a Banach A-bimodule (defined below). The module extension Banach algebra $A \oplus X$ is the l^1-direct sum of A and X with the algebra product that defined as follows;

$$(a, x) \cdot (b, y) = (ab, ay + xb) \quad (a, b \in A, x, y \in X).$$

Y. Zhang in [11], with use of these Banach algebras gave a counterexample to a question that raised and left open by Dales, Gharamani and Gronbaek in [2]. Many authors studied this class of Banach algebras; see for example [6, 7].

We believe that if the structure of this Banach algebra or a more general version of that analyzed appropriately, it can be used as a good source of counterexamples. In the present paper, first we give a generalization of the module extension Banach algebra.

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Then we study some structural properties of this new Banach algebra for aforementioned reason.

In the sequel of this section, we give a brief outline of definitions and known results. For further details one can refer to [1, 4].

Suppose that A is a Banach algebra. A Banach space X is a Banach A-bimodule, if X is an A-bimodule and there exists a constant $K_X > 0$ such that $\|a \cdot x\| \leqslant K_X \|a\| \|x\|$ and $\|x \cdot a\| \leqslant K_X \|a\| \|x\|$ for each $a \in A$ and $x \in X$.

Let X be a Banach A-bimodule. A linear map $D : A \longrightarrow X$ is a *derivation*, if for all $a, b \in A$,

$$D(ab) = D(a) \cdot b + a \cdot D(b).$$

For $x \in X$, $D_x : A \to X$ defined by $D_x(a) = a \cdot x - x \cdot a$ is a derivation called an *inner derivation*. If there exists a net $\{x_\alpha\}$ in X such that for each $a \in A$, $D(a) = \lim_\alpha D_{x_\alpha}(a)$, we say that D is an *approximately inner* derivation.

If X is a Banach A-bimodule, X^* is also a Banach A-bimodule with the following module actions;

$$< a \cdot x^*, x > = < x^*, x \cdot a >, \quad < x^* \cdot a, x > = < x^*, a \cdot x > \quad (a \in A, x \in X, x^* \in X^*).$$

For $a, b \in A$, $\lambda \in A^*$ and $\Phi \in A^{**}$, we define $a \cdot \lambda, \lambda \cdot a, \lambda \cdot \Phi, \Phi \cdot \lambda \in A^*$ by

$$< a \cdot \lambda, b > = < \lambda, ba >, < \lambda \cdot a, b > = < \lambda, ab >,$$

$$< \lambda \cdot \Phi, a > = < \Phi, a \cdot \lambda >, < \Phi \cdot \lambda, a > = < \Phi, \lambda \cdot a > .$$

Define two products $\square$ and $\Diamond$ on A^{**} by

$$< \Phi \square \Psi, \lambda > = < \Phi, \Psi \cdot \lambda >, \quad < \Phi \Diamond \Psi, \lambda > = < \Psi, \lambda \cdot \Phi > \quad (\Phi, \Psi \in A^{**}, \lambda \in A^*).$$

We say that A is *Arens regular*, if for all $\Phi, \Psi \in A^{**}$, $\Phi \square \Psi = \Phi \Diamond \Psi$. This is equivalent to the following double limit criterion. For each $\Phi, \Psi \in A^{**}$ and nets $\{a_\alpha\}, \{b_\beta\}$ in A where converge in w^*-topology to Φ and Ψ respectively,

$$\lim_\alpha \lim_\beta < \lambda, a_\alpha b_\beta > = \lim_\beta \lim_\alpha < \lambda, a_\alpha b_\beta > \quad (\lambda \in A^*),$$

whenever both iterated limits exist.

The Banach algebra A is *semisimple* if $\text{rad}(A) = \{0\}$ which $\text{rad}(A)$ is the intersection of all the maximal modular left ideals of A.

Suppose that B is a Banach algebra and I is a closed ideal of B. An *extension* of A by I is a short exact sequence

$$\sum = \sum(B, I) : 0 \to I \hookrightarrow B \xrightarrow{q} A \to 0,$$

where q is a homomorphism. The extension $\sum$ is *singular*, if I is nilpotent, i.e., $I^2 = \{0\}$. The extension $\sum$ *split strongly*, if there exists a continuous homomorphism $\theta : A \to B$ such that $q \circ \theta = Id_A$ where Id_A is the identity map on A.

This paper is organized as follows. In section 2, we give the main definition of this paper, i.e., we define the (σ, τ)-*module extension Banach algebra* that is a generalization of module extension Banach algebras. Then we investigate some of the important properties of these Banach algebras such as Arens regularity, commutativity, semisimplity, and study

the ideal structure of these algebras. In section 3, we study derivations on (σ, τ)-module extension Banach algebras. As a main result, we specify the range of derivations, and obtain some relations between homomorphisms and (σ, τ)-derivations on these Banach algebras with use of the Singer and Wermer Theorem.

2. Some Structural Properties of $A \oplus_{(\sigma,\tau)} X$

We start this section with the basic definition of the paper as follows.

Let A be a Banach algebra and X be a Banach A-bimodule. Also, let σ and τ be continuous homomorphisms on A.

Suppose that $A \oplus X$ is the direct sum of A and X as vector spaces. We define a bilinear map on $A \oplus X$ as follows:

$$(a, x) \cdot (b, y) = (ab, \sigma(a) \cdot y + x \cdot \tau(b)) \quad (a, b \in A, x, y \in X). \tag{1}$$

Indeed, the above bilinear map is a product on $A \oplus X$ which turns it into an algebra.

Theorem 2.1 With the above product, pointwise addition and the norm $||(a, x)|| = C(||a|| + ||x||)$ where $C = \max\{||\sigma||, ||\tau||, K_X\}$, the vector space $A \oplus X$ is a Banach algebra.

Proof. We only verify the associativity of Product (1). Let $a, b, c \in A$ and $x, y, z \in X$. So

$$(a, x) \cdot [(b, y) \cdot (c, z)] = (a, x) \cdot (bc, \sigma(b) \cdot z + y \cdot \tau(c))$$
$$= (abc, \sigma(a) \cdot (\sigma(b) \cdot z + y \cdot \tau(c)) + x \cdot \tau(bc)). \tag{2}$$
$$[(a, x) \cdot (b, y)] \cdot (c, z) = (ab, \sigma(a) \cdot y + x \cdot \tau(b)) \cdot (c, z)$$
$$= (abc, \sigma(ab) \cdot z + (\sigma(a) \cdot y + x \cdot \tau(b)) \cdot \tau(c)). \tag{3}$$

In view of Relations (2) and (3), and this fact that σ, τ are homomorphisms, we conclude that the product is associative. ∎

We show the above Banach algebra by $A \oplus_{(\sigma,\tau)} X$ and call it the (σ, τ)-*module extension Banach algebra*. It is obvious that $A \oplus_{(Id_A, Id_A)} X$ is the module extension Banach algebra.

Before proceed further, we give the following definitions which will be used throughout. A Banach A-bimodule X is (σ, τ)-symmetric, if

$$\sigma(a) \cdot x = x \cdot \tau(a) \quad (a \in A, x \in X).$$

Also, X is essential if $X = AX = XA$; here $AX = \{a \cdot x : a \in A, x \in X\}$. Recall that X is a symmetric Banach A-bimodule, if $a \cdot x = x \cdot a$ for each $a \in A$ and $x \in X$.

The following example shows that the class of (σ, τ)-symmetric Banach A-bimodules is strictly larger than the class of symmetric Banach A-bimodules.

***Example* 2.2** Let B be a non-zero Banach algebra and let A be a vector space defined by,

$$A = \begin{bmatrix} 0 & B & B \\ 0 & 0 & B \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, A is a Banach algebra equipped with the usual matrix-like operations and l_∞-norm. It is clear that, $A^3 = \{0\}$, but $A^2 \neq 0$. Let $a_0 \in A$. So, $\sigma = L_{a_0}$ and $\tau = R_{a_0}$ are homomorphisms on A. Note that $R_{a_0}, L_{a_0} : A \to A$ are defined by $R_{a_0}(a) = aa_0$ and $L_{a_0}(a) = a_0 a$. If $X = A\widehat{\otimes}A$; the projective tensor product [4], then X is a (σ, τ)-symmetric Banach algebra which is not symmetric.

Recall that a linear map $D : A \to X$ is called a (σ, τ)-*derivation* [8], if

$$D(ab) = \sigma(a) \cdot D(b) + D(a) \cdot \tau(b) \quad (a, b \in A).$$

Theorem 2.3 Let A be a Banach algebra and X be a Banach A-bimodule.

(1) $A \oplus_{(\sigma,\tau)} X$ is commutative if and only if A is commutative and X is (σ, τ)-symmetric.
(2) If we identify X with $0 \oplus_{(\sigma,\tau)} X$, then X is a nilpotent ideal of $A \oplus_{(\sigma,\tau)} X$ and $A \oplus_{(\sigma,\tau)} X$ is semisimple if and only if A is semisimple and $X = 0$.
(3) If $d : A \to X$ is a (σ, τ)-derivation, then $D : (a, x) \to (0, d(a))$ is a derivation on $A \oplus_{(\sigma,\tau)} X$. Moreover, $D(A \oplus_{(\sigma,\tau)} X) \subseteq \mathrm{rad}(A \oplus_{(\sigma,\tau)} X)$.
(4) Let $D : A \to X$ be a linear map. Then $\theta : a \to (a, D(a))$ from A into $A \oplus_{(\sigma,\tau)} X$ is a homomorphism if and only if D is a (σ, τ)-derivation.

Proof. (1): Let $(a, x), (b, y) \in A \oplus_{(\sigma,\tau)} X$. So

$$(a, x) \cdot (b, y) = (ab, \sigma(a) \cdot y + x \cdot \tau(b)), \tag{4}$$

$$(b, y) \cdot (a, x) = (ba, \sigma(b) \cdot x + y \cdot \tau(a)). \tag{5}$$

Let A be commutative and X be (σ, τ)-symmetric. Then from the above relations, $A\oplus_{(\sigma,\tau)} X$ is commutative.

Conversely, if (4)=(5), we have $ab = ba$ and $\sigma(a) \cdot y + x \cdot \tau(b) = \sigma(b) \cdot x + y \cdot \tau(a)$ for all $a, b \in A$ and $x, y \in X$. In particular, for $x = 0$, we have $\sigma(a) \cdot y = y \cdot \tau(a)$. Therefore, A is commutative and X is (σ, τ)-symmetric.

(2): If $(a, x) \in A \oplus_{(\sigma,\tau)} X$ and $(0, y) \in 0 \oplus_{(\sigma,\tau)} X$, then

$$(a, x) \cdot (0, y) = (0, \sigma(a) \cdot y), \quad (0, y) \cdot (a, x) = (0, y.\tau(a)).$$

So, $0 \oplus_{(\sigma,\tau)} X$ is an ideal in $A \oplus_{(\sigma,\tau)} X$. With a simple calculation we can see that $(0, x) \cdot (0, y) = 0$ for all $x, y \in X$ and hence $0 \oplus_{(\sigma,\tau)} X$ is a nilpotent ideal in $A \oplus_{(\sigma,\tau)} X$.

If we identify every element $(0, x)$ of $0 \oplus_{(\sigma,\tau)} X$ with x, then $X \cong 0 \oplus_{(\sigma,\tau)} X$ as Banach spaces.

By [1, Theorem 1.5.6], we know that every nilpotent ideal in a Banach algebra A is a subset of $\mathrm{rad}(A)$. So, $0 \oplus_{(\sigma,\tau)} X \subseteq \mathrm{rad}(A \oplus_{(\sigma,\tau)} X)$. On the other hand, every ideal I in Banach algebra A such that $I \subseteq \mathrm{rad}(A)$ satisfies the relation $\mathrm{rad}(A/I) = \mathrm{rad}(A)/I$([1, Theorem 1.5.4]). So

$$\mathrm{rad}(A) = \mathrm{rad}\left(\frac{A \oplus_{(\sigma,\tau)} X}{0 \oplus_{(\sigma,\tau)} X}\right) = \frac{\mathrm{rad}(A \oplus_{(\sigma,\tau)} X)}{0 \oplus_{(\sigma,\tau)} X}.$$

Hence, $\mathrm{rad}(A \oplus_{(\sigma,\tau)} X) = \mathrm{rad}(A) \oplus X$. Therefore, $A \oplus_{(\sigma,\sigma)} X$ is semisimple if and only if A is semisimple and $X=0$.

(3): Let d be a (σ, τ)-derivation. Then for $(a, x), (b, y) \in A \oplus_{(\sigma, \tau)} X$ we have

$$
\begin{aligned}
D((a, x) \cdot (b, y)) = D(ab, \sigma(a) \cdot y + x \cdot \tau(b)) &= (0, d(ab)) \\
&= (0, \sigma(a) \cdot d(b) + d(a) \cdot \tau(b)) \\
&= (0, \sigma(a) \cdot d(b)) + (0, d(a) \cdot \tau(b)) \\
&= (a, x) \cdot D(b, y) + D(a, x) \cdot (b, y).
\end{aligned}
$$

Therefore, D is a derivation. For the second part, since for all $(a, x) \in A \oplus_{(\sigma, \tau)} X$, $D(a, x) = (0, da) \in \mathrm{rad}(A) \oplus X$, we conclude that $D(A \oplus_{(\sigma, \tau)} X) \subseteq \mathrm{rad}(A) \oplus X$.

(4): Let θ be a homomorphism. So, for $(a, x), (b, y) \in A \oplus_{(\sigma, \tau)} X$ we have,

$$
\begin{aligned}
(ab, D(ab)) = \theta(ab) = \theta(a)\theta(b) &= (a, Da) \cdot (b, Db) \\
&= (ab, \sigma(a) \cdot Db + Da \cdot \tau(b)).
\end{aligned}
$$

Hence, $D(ab) = \sigma(a) \cdot Db + Da \cdot \tau(b)$ and this implies that D is a (σ, τ)-derivation. The converse is also holds in view of the above relations. ∎

The following theorem shows the relations between (closed) ideals of A and $A \oplus_{(\sigma, \tau)} X$. Our proof is a mimic of [9, Proposition 2.7] in the setting of the (σ, τ)-module extension Banach algebras.

Theorem 2.4 Let A be a Banach algebra and X be a Banach A-bimodule.

(1) Let I be a left ideal of A and J be a left A-submodule of X. Then $M = I \times J$ is a left ideal of $A \oplus_{(\sigma, \tau)} X$ if $I \subseteq \ker(\tau)$.

(2) Let M be a left ideal of $A \oplus_{(\sigma, \tau)} X$ and

$$
I = \{a \in A : (a, x) \in M \text{ for some } x \in X\}
$$
$$
J = \{x \in X : (a, x) \in M \text{ for some } a \in A\}.
$$

Then I is a left ideal of A. Moreover, if M is closed and A has a left approximate identity (a_α) such that $(\sigma(a_\alpha))$ is a left approximate identity for X, then $M = I \times J$.

Proof. (1): Suppose that $I \subseteq \ker(\tau)$. Let $(a, x) \in A \oplus_{(\sigma, \tau)} X$ and $(c, z) \in I \times J$. So, we have

$$
(a, x) \cdot (c, z) = (ac, \sigma(a) \cdot z + x \cdot \tau(c)) = (ac, \sigma(a) \cdot z) \in I \times J.
$$

(2): Let $a_0 \in I$ and $a \in A$. So, there exists $x \in X$ such that $(a_0, x) \in M$. We have, $(a, 0).(a_0, x) = (aa_0, \sigma(a) \cdot x)$. Hence, I is a left ideal of A.

For the second part, let M be a closed ideal and (a_α) be a left approximate identity for A such that $\sigma(a_\alpha) \cdot x \to x$ for all $x \in X$. It is clear that $M \subseteq I \times J$ and now we show the reverse inclusion. Let $a_0 \in I$ and $x_0 \in J$. Therefore, we have

$$
(a_\alpha, 0) \cdot (a_0, x_0) = (a_\alpha a_0, \sigma(a_\alpha) \cdot x_0) \to (a_0, x_0).
$$

Since M is a closed ideal we conclude this part.

∎

The following corollary shows that $A \oplus_{(\sigma,\tau)} X$ is a strongly splitting Banach algebra extension of A by X.

Corollary 2.5 The short exact sequence

$$\sum(A \oplus_{(\sigma,\tau)} X, X) : 0 \to X \hookrightarrow A \oplus_{(\sigma,\tau)} X \xrightarrow{q} A \to 0$$

where $q(a,x) = a$, is a singular extension of A by X that splits strongly.

Proof. From part (2) of Theorem 2.3, we conclude that the extension is singular. Also, the map $\theta : A \to A \oplus_{(\sigma,\tau)} X$ defined by $\theta(a) = (a,0)$ is a continuous homomorphism such that $q \circ \theta = Id_A$, hence the extension splits strongly. ∎

One of the important properties of Banach algebras is Arens regularity. Here we investigate the Arens regularity of $A \oplus_{(\sigma,\tau)} X$, but first we give a definition which will be used.

Definition 2.6 Let A be a Banach algebra and X be a Banach A-bimodule. We say that X has (σ,τ)-*double limit commutative property*, if for any nets (a_α) and (x_β) respectively in A and X,

$$\lim_\beta \lim_\alpha \sigma(a_\alpha).x_\beta = \lim_\alpha \lim_\beta \sigma(a_\alpha).x_\beta$$

$$\lim_\beta \lim_\alpha x_\beta.\tau(a_\alpha) = \lim_\alpha \lim_\beta x_\beta.\tau(a_\alpha),$$

whenever both double limits exist.

For convenience we use briefly (σ,τ)-DLC for every Banach A-bimodule X which has the (σ,τ)-double limit commutative property.

As an example, every Arens regular Banach algebra A is a (σ,τ)-DLC Banach A-bimodule.

Recall that if X is a Banach A-bimodule, then X^{**} is a Banach A^{**}-bimodule; see [2].

Theorem 2.7 The Banach algebra $A \oplus_{(\sigma,\tau)} X$ is Arens regular if and only if A is Arens regular and X has the (σ,τ)-DLC property.

Proof. We know that the underlying set of $(A \oplus_{(\sigma,\tau)} X)^{**}$ and $(A \oplus_{(\sigma,\tau)} X)^*$ are $A^{**} \oplus X^{**}$ and $A^* \oplus X^*$ respectively.

Now, let $(U, \Phi), (V, \Psi)$ be two arbitrary element of $A^{**} \oplus_{(\sigma,\tau)} X^{**}$. Goldestine's Theorem ([1], Theorem A.3.29 (i)) gives nets $(u_\alpha), (v_\beta)$ in A and $(x_\alpha), (y_\beta)$ in X such that

$$(U, \Phi) = w^* - \lim_\alpha (u_\alpha, x_\alpha), \quad (V, \Psi) = w^* - \lim_\beta (v_\beta, y_\beta).$$

So, we have

$$(U, \Phi)\square(V, \Psi) = w^* - \lim_\alpha w^* - \lim_\beta (u_\alpha, x_\alpha) \cdot (v_\beta, y_\beta),$$

$$(U, \Phi)\lozenge(V, \Psi) = w^* - \lim_\beta w^* - \lim_\alpha (u_\alpha, x_\alpha) \cdot (v_\beta, y_\beta).$$

First, let A be Arens regular and X has the (σ, τ)-DLC property. Hence, for (a^*, x^*) in $A^* \oplus X^*$ we have,

$$
\begin{aligned}
< (a^*, x^*), (U, \Phi) \square (V, \Psi) > &= \lim_\alpha \lim_\beta < (u_\alpha, x_\alpha) \cdot (v_\beta, y_\beta), (a^*, x^*) > \\
&= \lim_\alpha \lim_\beta < (u_\alpha v_\beta, \sigma(u_\alpha) \cdot y_\beta + x_\alpha \cdot \tau(v_\beta)), (a^*, x^*) > \\
&= \lim_\alpha \lim_\beta (< u_\alpha v_\beta, a^* > + < \sigma(u_\alpha) \cdot y_\beta + x_\alpha \cdot \tau(v_\beta), x^* >) \\
&= < a^*, U \square V > + < x^*, \sigma^{**}(U) \cdot \Psi + \Phi \cdot \tau^{**}(V) > \\
&= < (a^*, x^*), (U \square V, \sigma^{**}(U) \cdot \Psi + \Phi \cdot \tau^{**}(V)) > .
\end{aligned}
$$

Therefore,

$$
(U, \Phi) \square (V, \Psi) = (U \square V, \sigma^{**}(U) \cdot \Psi + \Phi \cdot \tau^{**}(V)), \tag{6}
$$

where σ^{**}, τ^{**} are w^*-extensions of σ, τ. Similarly, for the second Arens product we have,

$$
\begin{aligned}
< (a^*, x^*), (U, \Phi) \lozenge (V, \Psi) > &= \lim_\beta \lim_\alpha < (u_\alpha, x_\alpha) \cdot (v_\beta, y_\beta), (a^*, x^*) > \\
&= \lim_\beta \lim_\alpha < (u_\alpha v_\beta, \sigma(u_\alpha) \cdot y_\beta + x_\alpha \cdot \tau(v_\beta)), (a^*, x^*) > \\
&= \lim_\beta \lim_\alpha (< u_\alpha v_\beta, a^* > + < \sigma(u_\alpha) \cdot y_\beta + x_\alpha \cdot \tau(v_\beta), x^* >) \\
&= < a^*, U \lozenge V > + < x^*, \sigma^{**}(U) \cdot \Psi + \Phi \cdot \tau^{**}(V) > \\
&= < (a^*, x^*), (U \lozenge V, \sigma^{**}(U) \cdot \Psi + \Phi \cdot \tau^{**}(V)) > .
\end{aligned}
$$

Hence

$$
(U, \Phi) \lozenge (V, \Psi) = (U \lozenge V, \sigma^{**}(U) \cdot \Psi + \Phi \cdot \tau^{**}(V)). \tag{7}
$$

Thus, from relations (6) and (7) we conclude that $A \oplus_{(\sigma, \tau)} X$ is Arens regular. The converse of the theorem is just a simple and similar calculation. ■

3. Derivations on $A \oplus_{(\sigma, \tau)} X$

In this section, we study some properties of derivations on $A \oplus_{(\sigma, \tau)} X$.

Theorem 3.1 Let A be an Arens regular Banach algebra, σ and τ be continuous homomorphisms on A, and X be a Banach A-bimodule which has the (σ, τ)-DLC property. If every continuous derivation on $(A \oplus_{(\sigma, \tau)} X)^{**}$ is inner, then each continuous derivation on $A \oplus_{(\sigma, \tau)} X$ is inner or approximately inner.

Proof. Let $S = A \oplus_{(\sigma, \tau)} X$ and $D : S \to S$ be a continuous derivation. So, $i_S \circ D : S \to \widehat{S} \subseteq S^{**}$ is also a continuous derivation where $i_S : S \to \widehat{S}$ is the canonical embedding.

Since A is Arens regular and X has the (σ, τ)-DLC property, by Theorem 2.7, S is Arens regular. In view of [1, Theorem 2.7.17], the derivation $i_S \circ D$ extended to a continuous derivations $\overline{i_S \circ D}$ on S^{**} such that $\overline{i_S \circ D}_{|S} = i_S \circ D$.

So, there exists $s^{**} \in S^{**}$ such that

$$i_S \circ D(s) = s \cdot s^{**} - s^{**} \cdot s \qquad (s \in S).$$

Now, two case might happen; $s^{**} \in \widehat{S}$ or $s^{**} \in S^{**} \setminus \widehat{S}$. In the first case, there exists $s_0 \in S$ such that $s^{**} = \widehat{s_0}$. Since the canonical embedding is one-to-one, we have

$$D(s) = s \cdot s_0 - s_0 \cdot s \qquad (s \in S).$$

In the second case, in view of Goldstine's Theorem there exists a net (s_α) in S such that for all α, $\|\widehat{s_\alpha}\| \leqslant s^{**}$ and $s^{**} = w^* - \lim_\alpha \widehat{s_\alpha}$.

So, we have

$$D(s) = w - \lim_\alpha (s \cdot s_\alpha - s_\alpha \cdot s) \qquad (s \in S).$$

Hence, the above relation and Mazur's Theorem ([1], Theorem A.3.29 (ii)) yield

$$\begin{aligned}
D(s) &\in w - \mathrm{closure}\{s \cdot s_\alpha - s_\alpha \cdot s : \alpha\} \\
&\subseteq w - \mathrm{closure}(\mathrm{conv}\{s \cdot s_\alpha - s_\alpha \cdot s : \alpha\}) \\
&= \|.\| - \mathrm{closure}(\mathrm{conv}\{s \cdot s_\alpha - s_\alpha \cdot s : \alpha\}) \quad (s \in S).
\end{aligned}$$

Therefore, we conclude that there exists a bounded net (y_β) in S such that

$$D(s) = \lim_\beta (s \cdot y_\beta - y_\beta \cdot s) \qquad (s \in S).$$

So, D is approximately inner. ∎

Singer and Wermer proved that if D is a continuous derivation on a Banach algebra A such that $D(A) \subseteq Z(A)$, then $D(A) \subseteq \mathrm{rad}(A)$ ([1], Theorem 2.7.20); here $Z(A)$ is the center of A, i.e., $Z(A) = \{a \in A : ab = ba \quad (b \in A)\}$.

Theorem 3.2 Let A be a commutative Arens regular Banach algebra, σ and τ be continuous homomorphisms on A, and X be a (σ, τ)-symmetric Banach A-bimodule such that has the (σ, τ)-DLC property. Then for every continuous derivation $D : A \oplus_{(\sigma,\tau)} X \to (A \oplus_{(\sigma,\tau)} X)^{**}$ we have $D(A \oplus_{(\sigma,\tau)} X) \subseteq \mathrm{rad}(A \oplus_{(\sigma,\tau)} X)^{**}$.

Proof. Let $S = A \oplus_{(\sigma,\tau)} X$. By Theorem 2.7, S is Arens regular and by Theorem 2.3, S is commutative. Hence, $(S^{**}, \square)$ is commutative.

Therefore, by [1, Theorem 2.7.17 (iv)], there exists a continuous derivation $\widetilde{D} : S^{**} \to S^{**}$ such that $\widetilde{D}_{|S} = D$. Now, by using of the Singer and Wermer Theorem, $\widetilde{D}(S^{**}) \subseteq \mathrm{rad}(S^{**})$. Thus, $D(S) \subseteq \mathrm{rad}(S)^{**}$.

∎

To proceed further for the convenience of the reader, we state a result of [10]. Recall the Lie product that defined by $[a, b] = ab - ba$ for all $a, b \in A$.

Theorem 3.3 [10, Theorem 2.5] Let A be a non-commutative Banach algebra and $D : A \to A$ be a continuous derivation such that, $[[Da, a], Da] \in \mathrm{rad}(A)$. Then $D(A) \subseteq \mathrm{rad}(A)$.

Theorem 3.4 Let A be a commutative Banach algebra and X be a non-(σ, τ)-symmetric Banach A-bimodule. Then for each continuous derivation D on $A \oplus_{(\sigma,\tau)} X$ we have $D(A \oplus_{(\sigma,\tau)} X) \subseteq \mathrm{rad}(A \oplus_{(\sigma,\tau)} X)$.

Proof. Theorem 2.3 shows that the Banach algebra $A \oplus_{(\sigma,\tau)} X$ is non-commutative.

Let $(a, x) \in A \oplus_{(\sigma,\tau)} X$. First we compute the Lie product

$$(a,x)^{'} = [[D(a,x),(a,x)], D(a,x)].$$

If we show that $(a,x)^{'} \in \mathrm{rad}(A) \oplus X = \mathrm{rad}(A \oplus_{(\sigma,\tau)} X)$, then Theorem 3.3 implies the conclusion.

But for each $(a,x) \in A \oplus_{(\sigma,\tau)} X$ there exists $(a^{'}, x^{'}) \in A \oplus_{(\sigma,\tau)} X$ such that $D(a,x) = (a^{'}, x^{'})$. Now, we have

$$[[D(a,x),(a,x)], D(a,x)] = [[(a^{'},x^{'}),(a,x)],(a^{'},x^{'})]$$
$$= (a^{'}aa^{'} - aa^{'}a^{'} - a^{'}a^{'}a + a^{'}aa^{'}, x_0),$$

for some $x_0 \in X$.

Since A is commutative, $a^{'}aa^{'} - aa^{'}a^{'} - a^{'}a^{'}a + a^{'}aa^{'} = 0$. Therefore,

$$(a,x)^{'} \in 0 \oplus_{(\sigma,\tau)} X \subseteq \mathrm{rad}(A) \oplus_{(\sigma,\tau)} X = \mathrm{rad}(A \oplus_{(\sigma,\tau)} X).$$

∎

We know that if each homomorphism from A into a Banach algebra is continuous, then each derivation from A into a Banach A bimodule is continuous ([1], Theorem 2.7.5). We conclude the paper with the following result concerning to homomorphisms and (σ, τ)-derivations.

Theorem 3.5 Let A be a Banach algebra and σ, τ be continuous homomorphism on A.

(1) Suppose that each homomorphism from A into a Banach algebra is continuous. Then each (σ, τ)-derivation from A into a Banach A-bimodule is continuous.

(2) Let A be commutative and each homomorphism from A into a commutative Banach algebra is continuous. Then each (σ, τ)-derivation from A into a (σ, τ)-symmetric Banach A-bimodule is continuous.

Proof. (1): Let E be a Banach A-bimodule and $D : A \to E$ be a (σ, τ) derivation. So, by part (4) of Theorem 2.3, $\theta : a \to (a, Da)$ is a homomorphism from A into the Banach algebra $A \oplus_{(\sigma,\tau)} E$. By hypothesis, θ is continuous and hence there exists a positive constant C such that $||(a, Da)|| \leqslant C||a||$. Therefore, $||Da|| \leqslant C||a||$ and this completes the proof.

(2): Let E be a (σ, τ)-symmetric Banach A-bimodule and $D : A \to E$ be a (σ, τ)-derivation. As in the above proof, let $\theta : A \to A \oplus_{(\sigma,\tau)} E$ be the homomorphism specified by $\theta(a) = (a, Da)$. By Theorem 2.3, we know that $A \oplus_{(\sigma,\tau)} E$ is a commutative Banach algebra. So, θ is continuous by hypothesis and therefore D is continuous. ∎

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