Journal of
Linear and Topological Algebra
Vol. 02*, No.* 03*,* 2013*,* 161- 167

On the nonnegative inverse eigenvalue problem of traditional matrices

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Abstract. In this paper, at first for a given set of real or complex numbers σ with nonnegative summation, we introduce some special conditions that with them there is no nonnegative tridiagonal matrix in which σ is its spectrum. In continue we present some conditions for existence such nonnegative tridiagonal matrices.

Keywords: Inverse eigenvalue problem, Tridiagonal matrix, Nonnegative matrix.

2010 AMS Subject Classification: 15A29, 15A18.

1. Introduction

The nonnegative inverse eigenvalue problem (NIEP) asks for necessary and sufficient conditions on a list $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of complex numbers in order that it be the spectrum of a nonnegative matrix. In terms of n the NIEP solve only for $n \leqslant 5$ [1,2,3,4,5].

The problem of constructing a symmetrical tridiagonal matrix from certain spectral information is important in many applications, such as vibration theory, structural design, control theory, and it has attracted the attention of many authors [7,8,9]. In this paper we discuss about inverse eigenvalue problem for nonnegative tridiagonal matrices.

The spectral radius of nonnegative matrix A denoted by $\rho(A)$. There is a right and a left eigenvector associated with the Perron eigenvalue with nonnegative entries. In addition s_k the k-th power sum of the eigenvalues λ_i and in the list σ, λ_1 is the Perron element.

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

Some necessary conditions on the list of complex number $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ to be the spectrum of a nonnegative matrix are listed below.
(1) The Perron eigenvalue $\max\{|\lambda_i|; \lambda_i \in \sigma\}$ belongs to σ (Perron-Frobenius theorem).
(2) $s_k = \sum_{i=1}^{n} \lambda_i^k \geq 0$.
(3) $s_k^m \leq n^{m-1} s_{km}$ for $k, m = 1, 2, \ldots$ (JLL inequality)[2,6].

We recall below Theorem 2.1 of [5] that is similar to Lemma 5 of [3] and by using this Theorem, we construct a $n \times n$ nonnegative tridiagonal matrix for a given set which satisfies in the special conditions in a recursive method for $n \leqslant 5$.

Theorem 1.1 Let B be a $m \times m$ nonnegative matrix, $M_1 = \{\mu_1, \mu_2, \ldots, \mu_m\}$ be its eigenvalues and μ_1 be Perron eigenvalue of B. Also assume that A is a $n \times n$ nonnegative matrix in following form $A = \begin{pmatrix} A_1 & a \\ b^T & \mu_1 \end{pmatrix}$, where A_1 is a $(n-1) \times (n-1)$ matrix, a and b are arbitrary vectors in $\mathbb{R}^{n-1}$ and $M_2 = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ is the set of eigenvalues of A. Then there exist a $(m+n-1) \times (m+n-1)$ nonnegative matrix such that $M = \{\mu_2, \ldots, \mu_m, \lambda_1, \lambda_2, \ldots, \lambda_m\}$ is its eigenvalues.

In section 2 of this paper we show that for a given set of real or complex numbers with nonnegative summation that satisfies in following conditions:

$$\alpha_1 = \sum_{1 \leqslant i < j \leqslant n} \lambda_i \lambda_j < 0, \tag{1.1}$$

$$\alpha_2 = \sum_{1 \leqslant i < j < k \leqslant n} \lambda_i \lambda_j \lambda_k > 0, \tag{1.2}$$

there is no nonnegative tridiagonal matrix that σ is spectrum.

In section 3 for $n \leqslant 5$ and for set of real numbers σ we introduce some necessary conditions for existence nonnegative tridiagonal matrix that realizes σ. We also present some cases that there is no solution of problem.

2. Absence solution

Theorem 2.1 Let $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a set of complex numbers that satisfies in (1.1) and (1.2) and following conditions

$$\begin{array}{l} (1)\ \lambda_1 > 0 \\ (2)\ \sum_{i=1}^{n} \lambda_i \geqslant 0 \\ (3)\ \lambda_1 > |\lambda_i|,\ \text{i} = 2,\ \ldots,\ \text{n}. \end{array} \tag{2.1}$$

Then there is no any nonnegative tridiagonal matrix with spectrum σ.

Proof. If λ_i for $i = 1, 2, \ldots, n$ are the eigenvalues of $n \times n$ matrix, then the its characteristic polynomials is as follows

$$p(\lambda) = \lambda^n - (\sum_{i=1}^{n} \lambda_i)\lambda^{n-1} + (\sum_{1 \leqslant i < j \leqslant n} \lambda_i \lambda_j)\lambda^{n-2} - (\sum_{1 \leqslant i < j < k \leqslant n} \lambda_i \lambda_j \lambda_k)\lambda^{n-3} + \ldots. \tag{2.2}$$

We continue proof by reductio ad absurdum. Assume that there exist the $n \times n$ nonnegative tridiagonal matrix $A = (a_{ij})_{n \times n}$ as

$$\begin{pmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

such that λ_i for $i = 1, 2, \ldots, n$ are its spectrum. Therefore the characteristics polynomial of A is

$$\begin{aligned} P_A(\lambda) &= \lambda^n - (\textstyle\sum_{i=1}^{n} a_{ii})\lambda^{n-1} + (\sum_{1\leqslant i<j\leqslant n} a_{ii}a_{jj} - \sum_{i=1}^{n-1} a_{i,i+1}a_{i+1,i})\lambda^{n-2} \\ &-(\textstyle\sum_{1\leqslant i<j<k\leqslant n} a_{ii}a_{jj}a_{kk} - \sum_{\substack{i=1\\ i\neq j}}^{n} \sum_{j=1}^{n-1} a_{ii}a_{j,j+1}a_{j+1,j})\lambda^{n-3} + \ldots . \end{aligned} \tag{2.3}$$

By relations (1.1) and (1.2) we have

$$\sum_{i=1}^{n-1} a_{i,i-1}a_{i+1,i} > \sum_{1\leqslant i<j\leqslant n} a_{ii}a_{jj}, \tag{2.4}$$

$$\sum_{1\leqslant i<j<k\leqslant n} a_{ii}a_{jj}a_{kk} > \sum_{\substack{i=1\\ i\neq j}}^{n} \sum_{j=1}^{n-1} a_{ii}a_{j,j+1}a_{j+1,j}. \tag{2.5}$$

With lose of generality we assume that

$$\left\{ \begin{array}{ll} a_{33} = \min\{a_{11}, a_{33}\} & \text{for } n = 3 \\ a_{44} = \min\{a_{22}, a_{44}\}, a_{33} = \min\{a_{11}, a_{33}\} & \text{for } n = 4 \\ a_{55} = \min\{a_{33}, a_{55}\}, a_{44} = \min\{a_{22}, a_{44}\}, a_{33} = \min\{a_{11}, a_{33}\} & \text{for } n = 5 \\ \vdots & \\ a_{n,n} = \min\{a_{n-2,n-2}, a_{n,n}\}, a_{n-1,n-1} = & \\ \min\{a_{n-3,n-3}, a_{n-1,n-1}\}, \ldots, a_{33} = \min\{a_{11}, a_{33}\}. & \end{array} \right\} \tag{2.6}$$

If we replace the relation (2.6) to the right hand side of (2.5) we can reach to the relation that includes the right hand side of relation (2.4). i.e.

$$\sum_{1\leqslant i<j<k\leqslant n} a_{ii}a_{jj}a_{kk} > \sum_{\substack{i=1\\ i\neq j}}^{n} \sum_{j=1}^{n-1} a_{ii}a_{j,j+1}a_{j+1,j} =$$
$$a_{11}a_{23}a_{32} + a_{11}a_{34}a_{43} + \ldots + a_{11}a_{n-1,n}a_{n,n-1}+$$
$$a_{22}a_{34}a_{43} + a_{22}a_{45}a_{54} + \ldots + a_{22}a_{n-1,n}a_{n,n-1}+$$
$$a_{33}a_{12}a_{21} + a_{33}a_{45}a_{54} + \ldots + a_{33}a_{n-1,n}a_{n,n-1} + \ldots + a_{n-1,n-1}a_{12}a_{21}+$$
$$a_{n-1,n-1}a_{23}a_{32} + \ldots + a_{n-1,n-1}a_{n-3,n-2}a_{n-2,n-3} + a_{n,n}a_{12}a_{21} + a_{n,n}a_{23}a_{32}+$$
$$\ldots + a_{n,n}a_{n-3,n-2}a_{n-2,n-3} + a_{n,n}a_{n-2,n-1}a_{n-1,n-2} >$$
$$a_{33}(\sum_{i=1}^{n-1} a_{i,i-1}a_{i+1,i}) + a_{44}(\sum_{i=1}^{n-1} a_{i,i-1}a_{i+1,i})+$$
$$a_{55}(\sum_{i=1}^{n-1} a_{i,i-1}a_{i+1,i}) + \ldots + a_{n,n}(\sum_{i=1}^{n-1} a_{i,i-1}a_{i+1,i}) >$$
$$a_{33}(\sum_{1\leqslant i<j\leqslant n} a_{ii}a_{jj}) + a_{44}(\sum_{1\leqslant i<j\leqslant n} a_{ii}a_{jj}) + a_{55}(\sum_{1\leqslant i<j\leqslant n} a_{ii}a_{jj}) + \ldots + a_{n,n}$$
$$(\sum_{1\leqslant i<j\leqslant n} a_{ii}a_{jj})$$

After simplifying of the above relations we reach to the following relation

$$a_{33}^2(a_{11} + a_{22} + a_{44} + \ldots + a_{n,n}) + a_{44}^2(a_{11}+$$
$$a_{22} + a_{33} + a_{55} + \ldots + a_{n,n}) + \ldots + a_{n,n}^2(a_{11} + a_{22} + a_{33}+$$
$$\ldots + a_{n-2,n-2} + a_{n-1,n-1}) + a_{33}a_{44}(a_{11} + a_{22} + a_{55}+$$
$$a_{66} + \ldots + a_{n,n}) + a_{33}a_{55}(a_{11} + a_{22} + a_{44} + a_{66} + a_{77} + \ldots + a_{n,n}) + a_{33}a_{n,n}(a_{11}+$$
$$a_{22} + a_{44} + \ldots + a_{n-1,n-1}) + a_{44}a_{55}(a_{11} + a_{22} + a_{66} + a_{77} + \ldots + a_{n,n}) + a_{44}a_{66}(a_{11}+$$
$$a_{22} + a_{55} + a_{77} + \ldots + a_{n,n}) + \ldots + a_{44}a_{n,n}(a_{11} + a_{22}+$$
$$a_{55} + a_{66} + \ldots + a_{n-1,n-1}) + a_{55}a_{66}(a_{11}+$$
$$a_{22} + a_{77} + a_{88} + \ldots + a_{n,n}) + a_{55}a_{77}(a_{11} + a_{22} + a_{66} + a_{88} + a_{99} + \ldots + a_{n,n})+$$
$$\ldots + a_{55}a_{n,n}(a_{11} + a_{22} + a_{66} + a_{77} + \ldots + a_{n-1,n-1}) + \ldots + a_{n-1,n-1}a_{n,n}(a_{11} + a_{22}) < 0.$$

And this means the summation of sum of nonnegative numbers is strictly negative and this is impossible.□

Corollary 2.2 Let $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a set of real numbers that holds in (2.1) and for $i = 2, 3, \ldots, n$ we have $\lambda_i < 0$, then there is no nonnegative tridiagonal matrix that σ is its spectrum.

Proof. If $\lambda_i < 0$ for $i = 2, 3, \ldots, n$, it is obvious the relations (1.1) and (1.2) are hold, and therefore by Theorem 2.1 proof is complete. □

3. Existence and construction

In this section we study existence (with construction) or absence of nonnegative tridiagonal matrix of order maximum 5, for a given set of real numbers σ with $|\sigma| = n \leqslant 5$ that σ is its spectrum.

- **The case $n = 2$**

Theorem 3.1 Let $\sigma = \{\lambda_1, \lambda_2\}$ be a set of real numbers such that satisfies relation (2.1) then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.

Proof. σ has only one of following cases:
(a) If $\lambda_2 \geqslant 0$, then $A = \mathrm{diag}(\lambda_1, \lambda_2)$ is a solution of problem.

(b) If $\lambda_2 < 0$, then the matrix

$$A = \begin{pmatrix} 0 & -\lambda_1\lambda_2 \\ 1 & \lambda_1 + \lambda_2 \end{pmatrix}, \tag{3.1}$$

solves the problem.□

- **The case** $n = 3$

Theorem 3.2 Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ be a set of real numbers that satisfies relation (2.1). Then we have the following cases:
(a) If $\lambda_2, \lambda_3 \geqslant 0$, then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.
(b) If $\lambda_2, \lambda_3 < 0$, then there is no nonnegative tridiagonal matrix with spectrum σ.
(c) If $\lambda_2 < 0$ and $\lambda_3 \geqslant 0$, then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.

Proof.
(a) $B = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ is a solution of our problem.
(b) Corollary 2.2
(c) The nonnegative tridiagonal matrix

$$B = \begin{pmatrix} A & 0 \\ 0^T & \lambda_3 \end{pmatrix}, \tag{3.2}$$

is a solution of our problem where A is matrix (3.1) and o is zero vector with dimension 2×1. □

- **The case** $n = 4$

Theorem 3.3 Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be a set of real numbers that satisfies relation (2.1). Then we have the following cases:
(a) If $\lambda_2, \lambda_3, \lambda_4 \geqslant 0$, then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.
(b) If $\lambda_2, \lambda_3, \lambda_4 < 0$, then there is no nonnegative tridiagonal matrix with spectrum σ.
(c) If $\lambda_2 < 0$ and $\lambda_3, \lambda_4 \geqslant 0$, then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.
(d) If $\lambda_2, \lambda_3 \leqslant 0$, $\lambda_4 > 0$ and at least for one of the eigenvalues λ_2 and λ_3, for example λ_3, we have $\lambda_3 + \lambda_4 \geqslant 0$ then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.
(e) If $\lambda_2, \lambda_3 \leqslant 0$, $\lambda_4 > 0$ and we have $\lambda_2 + \lambda_4 \leqslant 0$, $\lambda_3 + \lambda_4 \leqslant 0$ and relation (1.2) then there is no nonnegative tridiagonal matrix with spectrum σ.

Proof.
(a) $C = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is a solution of our problem.
(b) Corollary 2.2
(c) The nonnegative tridiagonal matrix

$$C = \begin{pmatrix} B & o \\ o^T & \lambda_4 \end{pmatrix}, \tag{3.3}$$

is a solution of our problem where B is matrix (3.2) and o is zero vector with dimension of 3×1.
(d) The nonnegative tridiagonal matrix

$$C = \begin{pmatrix} 0 & -\lambda_1\lambda_2 & 0 & 0 \\ 1 & \lambda_1 + \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_3\lambda_4 \\ 0 & 0 & 1 & \lambda_3 + \lambda_4 \end{pmatrix}, \tag{3.4}$$

is a solution of our problem.
(e) If $\lambda_2 + \lambda_4 \leqslant 0$, $\lambda_3 + \lambda_4 \leqslant 0$ and $\lambda_2, \lambda_3 \leqslant 0$, $\lambda_4 > 0$ then $\lambda_2, \lambda_3 < 0$ and $\alpha_1 = \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4 = \lambda_1(\lambda_3+\lambda_4)+\lambda_2(\lambda_3+\lambda_4)+\lambda_1\lambda_2+\lambda_3\lambda_4 < 0$ and if we have relation (1.2) then by Theorem 2.1 proof is complete. □

- **The case** $n = 5$

Theorem 3.4 Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ be a set of real numbers that satisfies relation (2.1). Then we have the following cases:
(a) If $\lambda_2, \lambda_3, \lambda_4, \lambda_5 \geqslant 0$, then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.
(b) If $\lambda_2, \lambda_3, \lambda_4, \lambda_5 < 0$, then there is no nonnegative tridiagonal matrix with spectrum σ.
(c) If $\lambda_2 < 0$ and $\lambda_3, \lambda_4, \lambda_5 \geqslant 0$, then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.
(d) If $\lambda_2, \lambda_3 < 0$ and $\lambda_4, \lambda_5 \geqslant 0$ and at least for one of the eigenvalues λ_2 and λ_3, for example λ_3, we have $\lambda_3 + \lambda_4 \geqslant 0$ or $\lambda_3 + \lambda_5 \geqslant 0$ then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.
(e) If $\lambda_2, \lambda_3 < 0$, $\lambda_4, \lambda_5 \geqslant 0$ and we have $\lambda_2 + \lambda_4 \leqslant 0$, $\lambda_3 + \lambda_4 \leqslant 0$ and $\lambda_2 + \lambda_5 \leqslant 0$, $\lambda_3 + \lambda_5 \leqslant 0$ and relation (1.2) then there is no nonnegative tridiagonal matrix with spectrum σ.
(f) If $\lambda_2, \lambda_3, \lambda_4 < 0$, $\lambda_5 \geqslant 0$ and we have $\lambda_2 + \lambda_5 < 0$, $\lambda_3 + \lambda_5 < 0$, $\lambda_4 + \lambda_5 < 0$ then there is no nonnegative tridiagonal matrix with spectrum σ.

Proof.
(a) $D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ is a solution of our problem.
(b) Corollary 2.2
(c) The nonnegative tridiagonal matrix

$$D = \begin{pmatrix} C & o \\ o^T & \lambda_5 \end{pmatrix}, \tag{3.5}$$

is a solution of our problem where C is matrix (3.3) and o is zero vector with dimension of 4×1.

(d) The nonnegative tridiagonal matrix

$$D = \begin{pmatrix} 0 & -\lambda_1\lambda_2 & 0 & 0 & 0 \\ 1 & \lambda_1+\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_3\lambda_4 & 0 \\ 0 & 0 & 1 & \lambda_3+\lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}, \tag{3.6}$$

is a solution of our problem.
(e) If $\lambda_2+\lambda_4 \leqslant 0$, $\lambda_3+\lambda_4 \leqslant 0$ and $\lambda_2+\lambda_5 \leqslant 0$, $\lambda_3+\lambda_5 \leqslant 0$ then $\alpha_1 = \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_1\lambda_5+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_2\lambda_5+\lambda_3\lambda_4+\lambda_3\lambda_5+\lambda_4\lambda_5 = \lambda_1(\lambda_3+\lambda_4)+\lambda_2(\lambda_3+\lambda_4)+\lambda_1(\lambda_2+\lambda_5)+\lambda_5(\lambda_2+\lambda_4)+\lambda_3(\lambda_4+\lambda_5) < 0$ and if we have relation (1.2) then by theorem 2.1 proof is complete.
(f) The relations (1.1) and (1.2) are hold because
$\alpha_1 = \sum_{1\leqslant i<j\leqslant 5}\lambda_i\lambda_j = \lambda_1(\lambda_2+\lambda_5)+\lambda_3(\lambda_1+\lambda_4)+\lambda_4(\lambda_1+\lambda_2)+\lambda_5(\lambda_2+\lambda_4)+\lambda_3(\lambda_2+\lambda_5) < 0$
and
$\alpha_2 = \sum_{1\leqslant i<j<k\leqslant 5}\lambda_i\lambda_j\lambda_k = \lambda_1\lambda_2(\lambda_3+\lambda_5)+\lambda_1\lambda_4(\lambda_2+\lambda_5)+\lambda_1\lambda_3(\lambda_4+\lambda_5)+\lambda_2\lambda_3(\lambda_4+\lambda_5)+\lambda_4\lambda_5(\lambda_2+\lambda_3) > 0$
Therefore by Theorem 2.1 proof is complete. □

4. The conjecture

In this section we introduce a conjecture that is proved above with some conditions:

If relation (1.2) is not hold, again there is no any nonnegative tridiagonal matrix for (e) of The case $n = 4$ and The case $n = 5$.

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