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Solution of the first order fuzzy differential equations with generalized differentiability

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Abstract. In this paper, we study first order linear fuzzy differential equations with fuzzy coefficient and initial value. We use the generalized differentiability concept and apply the exponent matrix to present the general form of their solutions. Finally, one example is given to illustrate our results.

Keywords: First order fuzzy differential equations; Generalized differentiability; Fuzzy linear differential equations; Exponent matrix

1. Introduction

First order linear fuzzy differential equations are one of the simplest fuzzy differential equations which may appear in many applications [13]. Strongly generalized differentiability was introduced in [4] and studied in [3], and under this generalized differentiability concept the solutions of the first order linear fuzzy differential equations in some especial cases were presented (see [2]). In this paper, we consider first order linear fuzzy differential equations under generalized differentiability concept and apply the exponent matrix to present the solutions of this problem, in fact our results extend the results in [13].

The structure of the paper is organized as follows:
In Section 2, some basic definitions which will be used later in the paper are provided.

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In Section 3, we consider the initial value problem

$$\begin{cases} y'(t) = aty(t) + b(t), & t \geqslant 0 \\ y(0) = y_0, \end{cases}$$

so that the coefficients a and initial value are fuzzy numbers, also inhomogeneous term i.e., $b(t)$ is fuzzy valued function and propose a method for obtaining the solution of the above problem by applying exponent matrix and discuss the proposed method in detail. Two examples are presented to illustrate the applicability of our results in Section 4, and the conclusion is drawn in Section 5.

2. Preliminaries

In this paper let $\Re_F$ be a set of all fuzzy numbers on $\Re$.

Definition 2.1 [1], An arbitrary fuzzy number in the parametric form is represented by an ordered pair of functions $(u_1(\alpha), u_2(\alpha))$, $0 \leqslant \alpha \leqslant 1$, which satisfy the following requirements:

(i) $u_1(\alpha)$ is a bounded left-continuous non-decreasing function over $(0, 1]$ and right continuous at 0,
(ii) $u_2(\alpha)$ is a bounded left-continuous non-increasing function over $(0, 1]$ and right continuous at 0,
(iii) $u_1(\alpha) \leqslant u_2(\alpha)$, $0 \leqslant \alpha \leqslant 1$.

The notation

$$[u]^\alpha = [u_1(\alpha), u_2(\alpha)],$$

denotes explicitly the α-level set of u. We refer to $u_1(\alpha)$ and $u_2(\alpha)$ as the lower and upper branches of the parametric form , respectively. For $u \in \Re_F$, we define the length of u as

$$diam(u) = u_2(\alpha) - u_1(\alpha).$$

For $u, v \in \Re_F$ and $\lambda \in \Re$ the sum $u + v$ and the product λu are defined by $[u + v]^\alpha = [u]^\alpha + [u]^\alpha, [\lambda u]^\alpha = \lambda[u]^\alpha, \forall \alpha \in [0, 1]$ where $[u]^\alpha + [u]^\alpha$ means the usual addition of two intervals (subsets) of $\Re$ and $\lambda[u]^\alpha$ means the usual product between a scalar and a subset of $\Re$. (see e.g.[6])
We say u is a positive fuzzy number if $u_1(\alpha) > 0$, $0 \leqslant \alpha \leqslant 1$.
The metric structure is given by the Hausdorff distance

$$D : \Re_F \times \Re_F \longrightarrow \Re_+ \cup \{0\},$$

$$D(u, v) = \sup\ _{\alpha \in [0,1]} \max\{|u_1(\alpha) - v_1(\alpha)|, |u_2(\alpha) - v_2(\alpha)|\}.$$

$(\Re_F, D)$ is a complete metric space and the following properties are well known:(see e.g.[6])

(1) $D(u+w, v+w) = D(u,v) \qquad \forall u, v, w \in \Re_F$,
(2) $D(ku, kv) = |k|D(u,v) \qquad \forall k \in \Re, u, v \in \Re_F$,
(3) $D(u+v, w+e) \leqslant D(u,w) + D(v,e) \qquad \forall u, v, w, e \in \Re_F$.

Definition 2.2 Consider $x, y \in E$. If there exists $z \in E$ such that $x = y + z$, then z is called the H-difference of x and y and it is denoted by $x \ominus y$.

In this paper, the sign "$\ominus$" always stands for H-difference and note that $x \ominus y \neq x + (-y)$. Also throughout of paper is assumed the Hukuhara difference and Hukuhara generalized differentiability are existed. Let us recall the definition of strongly generalized differentiability introduced in [3].

Definition 2.3 Let $\Re_F$ be a set of all fuzzy numbers, we say that $F(x)$ is a fuzzy valued function if $F : I \to \Re_F$.

Definition 2.4 [3], Let $F : I \to \Re_F$. For $t_0 \in I$, we say F is differentiable at t_0 , if there exists an element $F'(t_0) \in \Re_F$ such that either

(i) For all $h > 0$ sufficiently close to 0, the H-differences $F(t_0+h) \ominus F(t_0), F(t_0) \ominus F(t_0-h)$ exist and the limits (in the metric D)

$$lim_{h\to 0^+} \frac{F(t_0+h) \ominus F(t_0)}{h} = lim_{h\to 0^+} \frac{F(t_0)(t_0-h)}{h} = F'(t_0),$$

(ii) For all $h > 0$ sufficiently close to 0, the H-differences $F(t_0) \ominus F(t_0+h), F(t_0-h) \ominus F(t_0)$ exist and the limits (in the metric D)

$$lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0+h)}{-h} = lim_{h\to 0^+} \frac{F(t_0-h) \ominus F(t_0)}{-h} = F'(t_0).$$

Remark 1 *In [3], the authors consider four cases for derivatives. Here we only consider the two first cases of Definition (2.3) in [3]. In the other cases, the derivative is trivial because it is reduced to a crisp element (for details see Theorem 7 in [3]).*

Definition 2.5 Let $F : I \to \Re_F$. We say F is (1)-differentiable on I if F be differentiable in the sense (1) of Definition 2.4 and similarly F is (2)-differentiable on I if F be differentiable in the sense (2) of Definition 2.4.

Theorem 2.6 [5], Let $F : I \to \Re_F$ and get $[F(t,\alpha)] = [f(t,\alpha), g(t,\alpha)]$ for each $\alpha \in [0,1]$.

(i) If F is (1)-differentiable then $f(t,\alpha)$ and $g(t,\alpha)$ are differentiable functions and $[F'(t,\alpha)] = [f'(t,\alpha), g'(t,\alpha)]$.
(ii) If F is (2)-differentiable then $f(t,\alpha)$ and $g(t,\alpha)$ are differentiable functions and $[F'(t,\alpha)] = [g'(t,\alpha), f'(t,\alpha)]$.

Definition 2.7 Consider the exponent matrix

$$e^{A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}}.$$

The characteristic equation of tghe above matrix is

$$x^2 - trace(A)x + det(A) = 0.$$

Now suppose that α, β are the roots of the characteristic equation. We introduce

$$s_0 = \frac{\alpha e^{\beta} - \beta e^{\alpha}}{\alpha - \beta},$$

$$s_1 = \frac{e^{\alpha} - e^{\beta}}{\alpha - \beta},$$

then

$$e^{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = s_0(I) + s_1(A), \tag{1}$$

and if A be a diagonal matrix then

$$e^{A=\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}} = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}. \tag{2}$$

3. Solving fuzzy linear differential equations

Consider the first order linear fuzzy differential equation

$$\begin{cases} y'(t) = aty(t) + b(t), \; t \geqslant 0 \\ y(0) = y_0, \end{cases} \tag{3}$$

so that the coefficients a, and initial value y_0 are fuzzy numbers and $b(t)$ is fuzzy valued function.

Definition 3.1 Let $y : I \to \Re_F$ be a fuzzy function such that it be (1)-differentiable or (2)-differentiable. If y and (1)-differentiability satisfy in the problem (3), we say that y is a (1)-solution of that. Similarly if y and (2)-differentiability satisfy in the problem (3), we say that y is a (2)-solution of that.

Since the coefficients are fuzzy numbers, we study those in the two cases. Notice that, for sake of simplicity, we consider that $y(t)$ is a positive fuzzy number for all $t \geqslant 0$.

3.1 *Case one: $a \prec 0$*

It means that the support of the a is belong to $(-\infty, 0)$. Problem (3) by considering (1)-differentiability, is transformed into the following ordinary differential equations (ODEs) system

$$\begin{cases} y_1'(t,\alpha) = a_1(\alpha)ty_2(t,\alpha) + b_1(t,\alpha), \\ y_2'(t,\alpha) = a_2(\alpha)ty_1(t,\alpha) + b_2(t,\alpha), \\ y_1(0) = y_{01}, \\ y_2(0) = y_{02}. \end{cases}$$

or it can be written as

$$\begin{pmatrix} y_1'(t,\alpha) \\ y_2'(t,\alpha) \end{pmatrix} = \begin{pmatrix} 0 & a_1(\alpha)t \\ a_2(\alpha)t & 0 \end{pmatrix} \begin{pmatrix} y_1(t,\alpha) \\ y_2(t,\alpha) \end{pmatrix} + \begin{pmatrix} b_1(t,\alpha) \\ b_2(t,\alpha) \end{pmatrix}, \tag{4}$$

thus

$$Y'(t,\alpha) = A(t,\alpha)Y(t,\alpha) + B(t,\alpha).$$

By the variation of constants formula for ODEs, we have

$$Y(t,\alpha) = e^{\int_0^t A(u,\alpha)du} \left[Y_0 + \int_0^t b(s,\alpha) e^{-\int_0^s A(u,\alpha)du} ds \right],$$

therefore

$$Y(t,\alpha) = e^{\begin{pmatrix} 0 & \int_0^t a_1(\alpha)udu \\ \int_0^t a_2(\alpha)udu & 0 \end{pmatrix}}$$
$$\times \left[Y_0 + \int_0^t e^{\begin{pmatrix} 0 & -\int_0^t a_1(\alpha)udu \\ -\int_0^t a_2(\alpha)udu & 0 \end{pmatrix}} \begin{pmatrix} b_1(s,\alpha) \\ b_2(s,\alpha) \end{pmatrix} ds \right].$$

Using Definition (2.7) the characteristic equation of above matrix is

$$x^2 - \left(a_1(\alpha)\frac{t^2}{2} \right) \left(a_2(\alpha)\frac{t^2}{2} \right) = 0,$$

that the roots are

$$x = \pm\sqrt{a_1(\alpha)a_2(\alpha)}\frac{t^2}{2}.$$

If

$$\alpha = +\sqrt{a_1(\alpha)a_2(\alpha)}\ \frac{t^2}{2},$$

and

$$\beta = -\sqrt{a_1(\alpha)a_2(\alpha)}\ \frac{t^2}{2},$$

then we can calculate

$$\begin{aligned} s_0 &= \frac{\sqrt{a_1(\alpha)\ a_2(\alpha)}\frac{t^2}{2} e^{-\sqrt{a_1(\alpha)\ a_2(\alpha)}\frac{t^2}{2}} + \sqrt{a_1(\alpha)\ a_2(\alpha)}\frac{t^2}{2} e^{\sqrt{a_1(\alpha)\ a_2(\alpha)}\frac{t^2}{2}}}{2\sqrt{a_1(\alpha)\ a_2(\alpha)}\frac{t^2}{2}} \\ &= \cosh\left(\sqrt{a_1(\alpha)\ a_2(\alpha)}\frac{t^2}{2} \right) \\ &:= \cosh(M_t), \end{aligned}$$

and

$$\begin{aligned}
s_1 &= \frac{e^{\sqrt{a_1(\alpha)\ a_2(\alpha)}\frac{t^2}{2}} - e^{-\sqrt{a_1(\alpha)\ a_2(\alpha)}\frac{t^2}{2}}}{2\sqrt{a_1(\alpha)\ a_2(\alpha)}\frac{t^2}{2}} \\
&= \frac{1}{\sqrt{a_1(\alpha)\ a_2(\alpha)}\frac{t^2}{2}} \times \sinh\left(\sqrt{a_1(\alpha)\ a_2(\alpha)}\frac{t^2}{2}\right) \\
&:= \frac{1}{M_t}\sinh(M_t).
\end{aligned}$$

Since

$$e^{\begin{pmatrix} 0 & a_1(\alpha)\frac{t^2}{2} \\ a_2(\alpha)\frac{t^2}{2} & 0 \end{pmatrix}} = s_0(t)I + s_1(t)A,$$

so, we have

$$\begin{aligned}
e^{\begin{pmatrix} 0 & a_1(\alpha)\frac{t^2}{2} \\ a_2(\alpha)\frac{t^2}{2} & 0 \end{pmatrix}} &= \begin{pmatrix} \cosh(M_t) & 0 \\ 0 & \cosh(M_t) \end{pmatrix} \\
&+ \begin{pmatrix} 0 & \frac{a_1(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}\sinh(M_t) \\ \frac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}\sinh(M_t) & 0 \end{pmatrix}.
\end{aligned}$$

then the solution can be written as

$$\begin{aligned}
\begin{pmatrix} y_1(t,\alpha) \\ y_2(t,\alpha) \end{pmatrix} &= \begin{pmatrix} \cosh(M_t) & \frac{a_1(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}\sinh(M_t) \\ \frac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}\sinh(M_t) & \cosh(M_t) \end{pmatrix} \\
&\times \left(\begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} + \begin{pmatrix} \int_0^t \left(b_1(s,\alpha)\cosh(M_s)\ - b_2(s,\alpha)\frac{a_1(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}\sinh(M_s)\right) ds \\ \int_0^t \left(-b_1(s,\alpha)\frac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}\sinh(M_s) + b_2(s,\alpha)\cosh(M_s)\right) ds \end{pmatrix} \right).
\end{aligned}$$

therefore

$$\begin{aligned}
y_1(t,\alpha) &= \cosh(M_t)\left[y_{01} + \int_0^t \left(b_1(s,\alpha)\cosh(M_s)\ - b_2(s,\alpha)\frac{a_1(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}\sinh(M_s)\right) ds\right] \\
&+ \frac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}\sinh(M_t)\left[y_{02} + \int_0^t \left(-b_1(s,\alpha)\frac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}\sinh(M_s) + b_2(s,\alpha)\cosh(M_s)\right) ds\right],
\end{aligned}$$

and

$$y_2(t,\alpha) = \tfrac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}} \sinh(M_t) \left[y_{01} + \textstyle\int_0^t \left(b_1(s,\alpha)\cosh(M_s) - b_2(s,\alpha) \tfrac{a_1(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}} \sinh(M_s) \right) ds \right]$$
$$+ \cosh(M_t) \left[y_{02} + \textstyle\int_0^t \left(-b_1(s,\alpha) \tfrac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}} \sinh(M_s) + b_2(s,\alpha)\cosh(M_s) \right) ds \right].$$

Thus for $a \prec 0$, the (1)-solution of the problem (3) is

$$\begin{aligned} y(t) &= \cosh(M_t) \left[y_0 + \textstyle\int_0^t \left(b(s)\cosh(M_s) \ominus \tfrac{b(s)a}{\sqrt{a_1(\alpha)\ a_2(\alpha)}} \sinh(M_s) \right) ds \right] \\ &\quad + \tfrac{a}{\sqrt{a_1(\alpha)\ a_2(\alpha)}} \sinh(M_t) \left[y_0 + \textstyle\int_0^t \left(b(s)\cosh(M_s) \ominus \tfrac{b(s)a}{\sqrt{a_1(\alpha)\ a_2(\alpha)}} \sinh(M_s) \right) ds \right], \end{aligned} \tag{5}$$

Now, we show that the H-differences in the integral terms of the Eq. (5) exist. So, it is sufficient to show that

$$diam[b(s)\cosh(M_t)] > diam[\frac{b(s)a}{\sqrt{a_1(\alpha)\ a_2(\alpha)}} \sinh(M_t)]$$

Always we have

$$\cosh(x) > \sinh(x) \qquad \forall x$$

and since $a \prec 0$ then, $\sqrt{a_1(\alpha)a_2(\alpha)}$ has positive value thus

$$\cosh(M_t) > \frac{a}{\sqrt{a_1(\alpha)\ a_2(\alpha)}} \sinh(M_t)$$

then, the H-difference exist.

Theorem 3.2 For $a \prec 0$ the (1)-solution of the problem (3) i.e., Eq. (5) is (1)- differentiable.

Proof. It is dificult to show that

$$\lim_{h \to 0^+} D \left(\frac{y(t+h,\alpha) \ominus y(t,\alpha)}{h}, aty(t) + b(t) \right) = 0.$$

If $\cosh(M_t) := g_1(t),\ \sinh(M_t) := g_2(t)$, thus

$$
\begin{aligned}
&\tfrac{y_1(t+h,\alpha)-y_1(t,\alpha)}{h} - (a_1(\alpha)\ t\ y_2(t,\alpha) + b_1(t,\alpha))\\
&= \tfrac{g_1(t+h)}{h}\left\{y_{01} + \int_0^{t+h}\left(b_1(s,\alpha)g_1(s) - b_2(s,\alpha)\tfrac{a_1(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(s)\right)\ ds\right\}\\
&+\tfrac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}\tfrac{g_2(t+h)}{h}\left\{y_{02} + \int_0^{t+h}\left(b_2(s,\alpha)g_1(s) - b_1(s,\alpha)\tfrac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(s)\right)\ ds\right\}\\
&+\tfrac{-g_1(t)}{h}\left\{y_{01} + \int_0^{t}\left(b_1(s,\alpha)g_1(s) - b_2(s,\alpha)\tfrac{a_1(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(s)\right)\ ds\right\}\\
&-\tfrac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}\tfrac{g_2(t)}{h}\left\{y_{02} + \int_0^{t}\left(b_2(s,\alpha)g_1(s) - b_1(s,\alpha)\tfrac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(s)\right)\ ds\right\}\\
&-a_1(\alpha)\ t\tfrac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(t)\left\{y_{01} + \int_0^{t}\left(b_1(s,\alpha)g_1(s) - b_2(s,\alpha)\tfrac{a_1(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(s)\right)\ ds\right\}\\
&+g_1(t)\left\{y_{02} + \int_0^{t}\left(b_2(s,\alpha)g_1(s) - b_1(s,\alpha)\tfrac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(s)\right)\ ds\right\} - b_1(t,\alpha)\\
&= \left(\tfrac{g_1(t+h)-g_1(t)}{h} - \tfrac{ta_1(\alpha)a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(t)\right)\\
&\times\left\{y_{01} + \int_0^{t}\left(b_1(s,\alpha)g_1(s) - b_2(s,\alpha)\tfrac{a_1(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(s)\right)\ ds\right\}\\
&+\tfrac{g_1(t+h)}{h}\left\{\int_t^{t+h}\left(b_1(s,\alpha)g_1(s) - b_2(s,\alpha)\tfrac{a_1(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(s)\right)\ ds\right\}\\
&+\left(\tfrac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}\left(\tfrac{g_2(t+h)-g_2(t)}{h}\right) - ta_1(\alpha)g_1(t)\right)\\
&\times\left\{y_{02} + \int_0^{t}\left(b_2(s,\alpha)g_1(s) - b_1(s,\alpha)\tfrac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(s)\right)\ ds\right\}\\
&+\tfrac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}\tfrac{g_2(t+h)}{h}\left\{\int_t^{t+h}\left(b_2(s,\alpha)g_1(s) - b_1(s,\alpha)\tfrac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(s)\right)\ ds\right\} - b_1(t,\alpha)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\tfrac{g_1(t+h)-g_1(t)}{h} - \tfrac{ta_1(\alpha)a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(t)\right)\ \beta_1(t,\alpha)\\
&+\left(\tfrac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}\left(\tfrac{g_2(t+h)-g_2(t)}{h}\right) - ta_1(\alpha)g_1(t)\right)\ \beta_2(t,\alpha)\\
&+g_1(t+h)\ \tfrac{\int_t^{t+h} b_1(s,\alpha)g_1(s)ds}{h} - g_1(t+h)\tfrac{\int_t^{t+h}\frac{a_1(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}b_2(s,\alpha)g_2(s)ds}{h}\\
&+\tfrac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(t+h)\ \tfrac{\int_t^{t+h} b_2(s,\alpha)g_1(s)ds}{h}\\
&-\tfrac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(t+h)\tfrac{\int_t^{t+h}\frac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}b_1(s,\alpha)g_2(s)ds}{h} - b_1(t,\alpha)
\end{aligned}
$$

so that

$$y_{01}+\int_0^t\left(b_1(s,\alpha)g_1(s)-b_2(s,\alpha)\frac{a_1(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(s)\right)\,ds:=\beta_1(t,\alpha),$$

and

$$y_{02}+\int_0^t\left(-b_2(s,\alpha)g_1(s)+b_1(s,\alpha)\frac{a_2(\alpha)}{\sqrt{a_1(\alpha)\ a_2(\alpha)}}g_2(s)\right)\,ds:=\beta_2(t,\alpha).$$

Now, it is easy to check that

$$\lim_{h\to 0_+}\sup_{\alpha\in[0,1]}\left|\frac{y_1(t+h,\alpha)-y_1(t,\alpha)}{h}-(a_1(\alpha)ty_2(t,\alpha)+b_1(t,\alpha))\right|=0,$$

In as much as $g_1'(t)=M'g_2(t)$, $g_2'(t)=M'g_1(t)$ and $g_1^2(t)-g_2^2(t)=1$, $t\geqslant 0$, the following limits exist for every $\alpha\in[0,1]$

$$\lim_{h\to 0_+}\frac{\int_t^{t+h}b_k(s,\alpha)g_1(s)ds}{h}=b_k(t,\alpha)g_1(t),\qquad k=1,2,$$
$$\lim_{h\to 0_+}\int_t^{t+h}g_k(s)b_k(s,\alpha)a_k(\alpha)ds=g_k(t)b_k(t,\alpha)a_k(t)\qquad k=1,2,$$

and they are uniformly in $\alpha\in[0,1]$. Since $b(t)$ is a continuous fuzzy valued function, and hence $\int_0^t b(s,\alpha)g_k(s)ds$, is (1)-differentiable with derivative $b(t,\alpha)g_k(t)$, $k=1,2$.(see [13]) $\beta_1(t,\alpha)$ and $\beta_2(t,\alpha)$ are bounded (in the variable α for each t fixed). Indeed, the support of y_0 is bounded, and the endpoints of the support of b are continuous functions on the compact interval $[0,t]$and, thus, bounded. Also, g_1 and g_2 are bounded on the compact interval $[0,t]$.(see [13])

Analogously it can be proved that

$$\lim_{h\to 0_+}\sup_{\alpha\in[0,1]}\left|\frac{y_1(t+h,\alpha)-y_1(t,\alpha)}{h}-(a_1(\alpha)ty_2(t,\alpha)+b_1(t,\alpha))\right|=0,$$

therefore

$$\lim_{h\to 0_+}\frac{y(t+h)\ \ominus\ y(t)}{h}=aty(t)+b(t).$$

And similarly it can be showed that

$$\lim_{h\to 0_+}\frac{y(t)\ \ominus\ y(t-h)}{h}=aty(t)+b(t).$$

Hence, $y(t)$ is (1)-differentiable and

$$\lim_{h\to 0_+}D\left(\frac{y(t)\ominus y(t-h)}{h},aty(t)+b(t)\right)=0,$$

thus the proof is completed. ■

The ODEs system by considering (2) - differentiability is

$$\begin{cases} y_1'(t,\alpha) = a_2(\alpha)ty_1(t,\alpha) + b_2(t,\alpha), \\ y_2'(t,\alpha) = a_1(\alpha)ty_2(t,\alpha) + b_1(t,\alpha), \\ y_1(0) = y_{01}, \\ y_2(0) = y_{02}, \end{cases}$$

or

$$\begin{pmatrix} y_1'(t,\alpha) \\ y_2'(t,\alpha) \end{pmatrix} = \begin{pmatrix} a_2(\alpha)t & 0 \\ 0 & a_1(\alpha)t \end{pmatrix} \begin{pmatrix} y_1(t,\alpha) \\ y_2(t,\alpha) \end{pmatrix} + \begin{pmatrix} b_2(t,\alpha) \\ b_1(t,\alpha) \end{pmatrix}. \tag{6}$$

Since, we get a diagonal matrix then by Eq. (2) and by Theorem 6 in [2], the (2)-solution is

$$\begin{aligned} Y(t,\alpha) &= \begin{pmatrix} e^{a_2(\alpha)\ \frac{t^2}{2}} & 0 \\ 0 & e^{a_1(\alpha)\ \frac{t^2}{2}} \end{pmatrix} \left[Y_0 \ominus \int_0^t \begin{pmatrix} e^{-a_2(\alpha)\frac{t^2}{2}} & 0 \\ 0 & e^{-a_1(\alpha)\frac{t^2}{2}} \end{pmatrix} \begin{pmatrix} -b_2(s,\alpha) \\ -b_1(s,\alpha) \end{pmatrix} ds \right] \\ &= \begin{pmatrix} e^{a_2(\alpha)\frac{t^2}{2}} & 0 \\ 0 & e^{a_1(\alpha)\frac{t^2}{2}} \end{pmatrix} \begin{pmatrix} y_{01} - \int_0^t -e^{-a_2(\alpha)\ \frac{s^2}{2}} b_2(s,\alpha)ds \\ y_{02} - \int_0^t -e^{-a_1(\alpha)\ \frac{s^2}{2}} b_1(s,\alpha)ds \end{pmatrix} \\ &= \begin{pmatrix} e^{a_2(\alpha)\ \frac{t^2}{2}} \{y_{01} + \int_0^t e^{-a_2(\alpha)\ \frac{s^2}{2}}\ b_2(s,\alpha)ds\} \\ e^{a_1(\alpha)\ \frac{t^2}{2}} \{y_{02} + \int_0^t e^{-a_1(\alpha)\ \frac{s^2}{2}}\ b_1(s,\alpha)ds\} \end{pmatrix}. \end{aligned}$$

Now we prove that

$$\left| \frac{y_1(t,\alpha) - y_1(t+h,\alpha)}{-h} - (a_2(\alpha)t\ y_1(t,\alpha) + b_2(t,\alpha)) \right|,$$

and similarly

$$\left| \frac{y_2(t,\alpha) - y_2(t+h,\alpha)}{-h} - (a_1(\alpha)t\ y_2(t,\alpha) + b_1(t,\alpha)) \right|,$$

tend to 0 as $h \to 0^+$ and then the H- differences $y(t,\alpha) \ominus y(t+h,\alpha)$ and $y(t-h,\alpha) \ominus y(t,\alpha)$ exist and thus

$$\lim_{h\to 0^+} D\left(\frac{y(t,\alpha) \ominus y(t+h,\alpha)}{-h}, aty(t) + b(t) \right) = 0.$$

Therefore

$$\frac{y_1(t,\alpha)-y_1(t+h,\alpha)}{-h} - (a_2(\alpha)ty_1(t,\alpha)+b_2(t,\alpha)) = \frac{e^{a_2(\alpha)\ \frac{t^2}{2}}}{-h}\left\{y_{01}+\int_0^t e^{-a_2(\alpha)\ \frac{s^2}{2}} b_2(s,\alpha)ds\right\}$$

$$+\frac{-e^{a_2(\alpha)\,\frac{(t+h)^2}{2}}}{-h}\left\{y_{01}+\int_0^{t+h} e^{-a_2(\alpha)\ \frac{(s+h)^2}{2}}\ b_2(s,\alpha)ds\right\}$$

$$-a_2(\alpha)t\left[e^{a_2(\alpha)\frac{t^2}{2}}\left\{y_{01}+\int_0^t e^{-a_2(\alpha)\frac{s^2}{2}} b_2(s,\alpha)ds\right\}\right]-b_2(t,\alpha)$$

$$=\left(\frac{e^{a_2(\alpha)\ \frac{t^2}{2}}-e^{a_2(\alpha)\frac{(t+h)^2}{2}}}{-h}-a_2(\alpha)te^{a_2(\alpha)\ \frac{t^2}{2}}\right)\left\{y_{01}+\int_0^t e^{-a_2(\alpha)\frac{s^2}{2}}\ b_2(s,\alpha)ds\right\}$$

$$+\frac{e^{a_2(\alpha)\ \frac{(t+h)^2}{2}}}{h}\ \int_t^{t+h} e^{-a_2(\alpha)\ \frac{(s+h)^2}{2}}\ b_2(s,\alpha)ds - b_2(t,\alpha).$$

The limit of the above equation is equal to 0, because

$$\begin{aligned}\lim_{h\to 0^+}\left(\frac{e^{a_2(\alpha)\frac{t^2}{2}}-e^{a_2(\alpha)\frac{(t+h)^2}{2}}}{-h}\right) &= \left(e^{a_2(\alpha)\frac{t^2}{2}}\right)' \\ &= \left(a_2(\alpha)\frac{t^2}{2}\right)' e^{a_2(\alpha)\frac{t^2}{2}} \\ &= a_2(\alpha)te^{a_2(\alpha)\frac{t^2}{2}},\end{aligned}$$

then the first term is equal to 0, and by applying part 3 in page 9 of [13], we will have

$$\begin{aligned}&\lim_{h\to 0^+}\left(\frac{\int_t^{t+h} e^{-a_2(\alpha)\frac{(s+h)^2}{2}} b_2(s,\alpha)ds}{h}\right) \\ &= e^{-a_2(\alpha)\frac{(t+h)^2}{2}} b_2(t,\alpha)\end{aligned}$$

thus,

$$e^{a_2(\alpha)\frac{(t+h)^2}{2}} e^{-a_2(\alpha)\frac{(t+h)^2}{2}} b_2(t,\alpha)-b_2(t,\alpha)=0$$

3.2 *Case 2:* $a \succ 0$

It means that the support of the a is belong to $[0,\infty)$. The corresponding ODEs system of problem (3) by (1)-differentiability, is

$$\begin{cases} y_1'(t,\alpha)=a_1(\alpha)ty_1(t,\alpha)+b_1(t,\alpha), \\ y_2'(t,\alpha)=a_2(\alpha)ty_2(t,\alpha)+b_2(t,\alpha), \\ y_1(0)=y_{01}, \\ y_2(0)=y_{02}, \end{cases}$$

as like the previous section and by Theorem 6 in [2], the (1)-solution can be obtained as follows:

$$Y(t,\alpha)=\begin{pmatrix} e^{a_1(\alpha)\ \frac{t^2}{2}} & 0 \\ 0 & e^{a_2(\alpha)\ \frac{t^2}{2}} \end{pmatrix}\left[Y_0 \ominus \int_0^t \begin{pmatrix} e^{-a_1(\alpha)\frac{t^2}{2}} & 0 \\ 0 & e^{-a_2(\alpha)\frac{t^2}{2}} \end{pmatrix}\begin{pmatrix} b_1(s,\alpha) \\ b_2(s,\alpha) \end{pmatrix} ds\right]$$

$$=\begin{pmatrix} e^{a_1(\alpha)\frac{t^2}{2}} & 0 \\ 0 & e^{a_2(\alpha)\frac{t^2}{2}} \end{pmatrix}\begin{pmatrix} y_{01}+\int_0^t -e^{-a_1(\alpha)\ \frac{s^2}{2}} b_1(s,\alpha)ds \\ y_{02}+\int_0^t -e^{-a_2(\alpha)\ \frac{s^2}{2}} b_2(s,\alpha)ds \end{pmatrix}$$

$$=\begin{pmatrix} e^{a_1(\alpha)\ \frac{t^2}{2}}\{y_{01}+\int_0^t e^{-a_1(\alpha)\ \frac{s^2}{2}}\ b_1(s,\alpha)ds\} \\ e^{a_2(\alpha)\ \frac{t^2}{2}}\{y_{02}+\int_0^t e^{-a_2(\alpha)\ \frac{s^2}{2}}\ b_2(s,\alpha)ds\} \end{pmatrix}.$$

Now if we consider problem (3) with (2) -differentiability, thecorresponding ODEs system is

$$\begin{cases} y_1'(t,\alpha)=a_2(\alpha)ty_2(t,\alpha)+b_2(t,\alpha), \\ y_2'(t,\alpha)=a_1(\alpha)ty_1(t,\alpha)+b_1(t,\alpha), \\ y_1(0)=y_{01}, \\ y_2(0)=y_{02}, \end{cases}$$

Computing the (2)-solution by using Theorem 6 in [2] is simple.

4. Examples

***Example* 4.1** A first order fuzzy differential equation with negative coefficient is considered as follows:

$$\begin{cases} y'(t)=(\alpha-3,-1-\alpha)ty \\ y(0)=(-1.5+0.5\alpha,-2.5-0.5\alpha), \qquad t\geqslant 0 \end{cases} \tag{7}$$

The solution is

$$y(t)=\left((-1.5+0.5\alpha)e^{(-1-\alpha)\frac{t^2}{2}},\ (-2.5-0.5\alpha)e^{(-3+\alpha)\frac{t^2}{2}}\right).$$

It is clear that $y(t)$ is the (2)-solution of Eq. (7).

Concluding remarks

The authors, as a new look, solved a full - fuzzy differential equations system actually, and by generalized differentiability concept and apply the exponent matrix could obtain (1)-differentiability and (2)-differentiability solutions.

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