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s-Topological vector spaces

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Abstract. In this paper, we have defined and studied a generalized form of topological vector spaces called s-topological vector spaces. s-topological vector spaces are defined by using semi-open sets and semi-continuity in the sense of Levine. Along with other results, it is proved that every s-topological vector space is generalized homogeneous space. Every open subspace of an s-topological vector space is an s-topological vector space. A homomorphism between s-topological vector spaces is semi-continuous if it is s-continuous at the identity.

Keywords: s-Topological vector space; semi-open set; semi-closed set; semi-continuous mapping; s-continuous mapping; left (right) translation; generalized homeomorphism; generalized homogeneous space.

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1. Introduction

If a set is endowed with algebraic and topological structures, then it is always fascinating to probe relationship between these two structures. The most formal way for such a study is to require algebraic operations to be continuous. This is the case we are investigating here for algebraic and topological structures on a set X, where algebraic operations (addition and scalar multiplication mappings) fail to be continuous. We join these two structures through weaker form of continuity.

A topological vector space [10, 17] is a basic structure in topology in which a vector space X over a topological field F ($\mathbb{R}$ or $\mathbb{C}$) is endowed with a topology τ such that:

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(i) the vector addition mapping $m : X \times X \to X$ defined by $m((x, y)) = x + y$ and
(ii) scalar multiplication mapping $M : F \times X \to X$ defined by $M((\lambda, x)) = \lambda \cdot x$ for all $\lambda \in F$ and $x, y \in X$

are continuous with respect to τ. Equivalently, $(X_{(F)}, \tau)$ is a topological vector space if:

(i) for each $x, y \in X$, and for each open neighbourhood W of $x + y$ in X, there exist open neighbourhoods U of x and V of y in X such that $U + V \subset W$, and
(ii) for each $\lambda \in F$, $x \in X$ and for each open neighbourhood W in X containing $\lambda \cdot x$, there exist open neighbourhoods U of λ in F and V of x in X such that $U \cdot V \subset W$.

The axioms for a space to become a topological vector space or linear topological space have been given and studied by Kolmogroff [13] in 1934, and von Neumann [15], in 1935. The relation between the axioms of topological vector space has been discussed by Wehausen [18], in 1938 and Hyers [11], in 1939. Also Kelly [12] has done classical work on topological vector spaces. In the last decade, we can see the work of Chen [4] on fixed points of convex maps in topological vector spaces. Bosi [2] and Clark [5] has researched on conics in topological vector spaces. More work, in recent years has been done by Drewnowski [9], Alsulami and Khan [1].

The beautiful interaction between linearity and topology is explored in the present paper where the compatibility is studied under semi-continuity. The basic idea in our mind is to study such structures in which the topology is endowed upon a vector space which fails to satisfy the continuity condition for vector addition and scalar multiplication or either. We intend to study such structures for the weaker form of continuity such as semi-continuity in the sense of Levine [14]. The concept of semi-continuity was introduced by Norman Levine [14] in 1963 as a consequence of the study of semi-open sets. In this paper, several new facts concerning topologies of s-topological vector spaces are established.

2. Preliminaries

Throughout in this paper X and Y are always topological spaces with no separation axioms considered until otherwise mentioned.

In 1963, N. Levine [14] defined semi-open sets in topological spaces. Since than many mathematicians explored different concepts and generalized them by using semi-open sets. A subset A of a topological space X is said to be semi-open if there exists an open set O in X such that $O \subset A \subset Cl(O)$, or equivalently if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. $\mathsf{SO}(X)$ denotes the collection of all semi-open sets in X. The complement of a semi-open set is said to be *semi-closed*; the *semi-closure* of $A \subset X$, denoted by $\mathrm{sCl}(A)$, is the intersection of all semi-closed subsets of X containing A [6, 7]. Let us mention that $x \in \mathrm{sCl}(A)$ if and only if for any semi-open set U containing x, $U \cap A \neq \emptyset$. Every open (closed) set is semi-open (semi-closed). It is known that the union of any collection of semi-open sets is semi-open set, while the intersection of two semi-open sets may not be semi-open. The intersection of an open set and a semi-open set is semi-open set. Basic properties of semi-open sets are given in [14] and of semi closed sets in [6–8].

Recall that a set $U \subset X$ is a semi-open neighbourhood of a point $x \in X$ if there exists $A \in SO(X)$ such that $x \in A \subset U$. A set $A \subset X$ is semi open in X if and only if A is semi open neighbourhood of each of its points. If a semi open neighbourhood U of a point x is a semi open set, we say that U is a semi open neighbourhood of x.

If $X_{(F)}$ is a vector space then e denotes its identity element, and for a fixed $x \in X$, ${}_xT : X \to X; y \longmapsto x+y$ and $T_x : X \to X, y \mapsto y+x$, denote the left and the right translation by x, respectively. The operator $+$ we call the addition mapping $m : X \times X \to X$ defined by $m((x,y)) = x+y$, and the scalar multiplication mapping $M_\lambda : F \times X \to X$ defined by $M((\lambda, x)) = \lambda \cdot x$.

Definition 2.1 Suppose (X, τ_X) and (Y, τ_Y) are topological spaces. A function $f : X \to Y$ is called:

(1) *semi-continuous* [14] if for each open set O in Y, the inverse image $f^{-1}(O) \in SO(X)$. Equivalently, a mapping $f : X \to Y$ is semi-continuous if and only if for each $x \in X$ and each open neighbourhood V of $f(x)$ there is a semi-open neighbourhood U of x with $f(U) \subset V$. Clearly, continuity implies semi-continuity; the converse need not be true.
(2) *semi-Open* if for every open set A of X, the set $f(A)$ is semi-open in Y;
(3) *s-continuous* [3] if the pre-image of every semi-open set is open;
(4) *pre-semi-open* [8] if for every semi-open set A of X, the set $f(A)$ is semi-open in Y;

Lemma 2.2 [16] Let A and X_0 be subsets of a topological space X such that $A \subseteq X_0$ and $X_0 \in SO(X)$. Then, $A \in SO(X)$ if and only if $A \in SO(X_0)$.

3. s-TOPOLOGICAL VECTOR SPACES

Definition 3.1 An s-topological vector space $(X_{(F)}, \tau)$ is a vector space X over the field F ($\mathbb{R}$ or $\mathbb{C}$) with a topology τ defined on $X_{(F)}$ and standard topology on F such that:

1) for each x, $y \in X$, and for each open neighbourhood W of $x+y$ in X, there exist semi-open neighbourhoods U and V of x and y respectively in X , such that

$$U + V \subseteq W$$

2)for each $\lambda \in F$, $x \in X$ and for each open neighbourhood W of $\lambda \cdot x$ in X, there exist semi-open neighbourhoods U of λ in F and V of x in X such that

$$U \cdot V \subseteq W$$

It follows from the definition that every topological vector space is s- topological vector space. The example below shows that the converse is not true in general.

***Example* 3.2** Let $X = \mathbb{R}$ be a vector space of real numbers over the field $F = \mathbb{R}$ and let τ be a topology on X induced by open intervals (a,b) and the sets $[1,c)$ where $a,b,c \in \mathbb{R}$.

In this case, $(\mathbb{R}_{(\mathbb{R})}, \tau)$ is an s-topological vector space over the field $\mathbb{R}$ with the topology τ defined on $\mathbb{R}$. We note that for each $x, y \in \mathbb{R}$ and each open neighbourhood W of $x+y$ in τ, there exist semi-open neighbourhoods U and V of x and y respectively in τ , such that $U + V \subseteq W$. Also for each $\lambda \in \mathbb{R}$, $x \in \mathbb{R}$ and for each open neighbourhood W of $\lambda \cdot x$ in τ, there exist semi-open neighbourhoods U of λ in $\mathbb{R}$ and V of x in τ such that $U \cdot V \subseteq W$. However, $(\mathbb{R}_{(\mathbb{R})}, \tau)$ is not a topological vector space because, for instance if, we choose $x = -3$, $y = 4$ and an open neighbourhood $W = [1,2)$ of $x+y$ in τ, we can not find open neighbourhoods U and V containing x and y respectively in τ which satisfy the condition $U + V \subseteq W$.

Theorem 3.3 Let $(X_{(F)}, \tau)$ be an s-topological vector space. Suppose T_x: $X \to X$ is a right translation and M_λ: $X \to X$ is multiplication mapping, then T_x and M_λ both are semi-continuous.

Proof. Let y be an arbitrary element in X and let W be an open neighbourhood of $T_x(y) = y + x$. By definition of s-topological vector spaces, there exist semi-open neighbourhoods U and V containing y and x respectively, such that $U + V \subseteq W$. In particular, we have $U + x \subseteq W$ which means $T_x(U) \subseteq W$. The inclusion shows that T_x is semi-continuous at y. Hence T_x is semi-continuous on X.

Now we prove the statement for multiplication mapping. Let $\lambda \in F$ and $x \in X$. Let W be an open neighbourhood of $M_\lambda(x) = \lambda \cdot x$. By definition of s-topological vector spaces, there exist semi-open neighbourhoods U and V containing λ and x respectively, such that $U \cdot V \subseteq W$. In particular, we have $\lambda \cdot V \subseteq W$, which means $M_\lambda(V) \subseteq W$. The inclusion shows that M_λ is semi-continuous at x. Hence M_λ is semi-continuous on X. ■

Theorem 3.4 Let $(X_{(F)}, \tau)$ be an s-topologicaal vector space. If A $\in \tau$ then

(1) for every y $\in$ X, A+y $\in SO(X)$,
(2) for every non zero $\lambda \in F$, $\lambda \cdot A \in SO(X)$.

Proof. 1) Let $z \in A + y$. We have to show that z is a semi-interior point of $A + y$. Now $z = x + y$, where x is some point in A. By Theorem, 3.3 $T_{-y} : X \to X$ is semi-continuous for $z \in X$. Thus, for the open set A containing x; $x = T_{(-y)}(z)$, there exists semi-open neighbourhood M_z of z such that $T_{-y}(M_z) = M_z + (-y) \subseteq A$. This implies $M_z \subseteq A + y$ which shows that z is a semi-interior point of $A + y$. Hence $A + y \in SO(X)$.

2) Let $z \in \lambda \cdot A$. We have to show that z is a semi-interior point of $\lambda \cdot A$. Now $z = \lambda \cdot x$, for some x in A. We have multiplication mapping $M_{\lambda^{-1}} : X \to X$ is semi continuous. Thus, for the set $A \in \tau$ containing $M_{\lambda^{-1}}(z) = \lambda^{-1} \cdot z = x$, there exists semi-open neighbourhood U_z of z in X such that $M_{\lambda^{-1}}(U_z) = \lambda^{-1} \cdot U_z \subseteq A$. This implies $U_z \subseteq \lambda \cdot A$. This shows that z is a semi-interior point of $\lambda \cdot A$. Hence $\lambda \cdot A \in SO(X)$. ■

Theorem 3.5 Let (X$_{(F)}$, τ) be an s-topological vector space. If A $\in \tau$ and B is any subset of X, then $A + B \in SO(X)$.

Proof. We have by Theorem 3.4, $T_{x_i}(A) = A + x_i \in SO(X)$ for each $x_i \in B$. Since union of any number of semi open sets is semi open, therefore $A + B = \cup_{x_i \in B}(A + x_i)$ is semi open in X. ■

Corollary 3.6 Suppose $(X_{(F)}, \tau_X)$ is an s-topological vector space and $A \in \tau$. Then the set $U = \cup_{n=1}^{\infty} nA$ is a semi-open set in X.

Definition 3.7 A bijective mapping f from a topological space to itself is called generalized homeomorphism if it is semi-continuous and semi-open.

Definition 3.8 An s-topological vector space $(X_{(F)}, \tau)$ is said to be generalized- homogenous space if for all x,$y \in X$, there is a generalized-homeomorphism f of the space X onto itself such that $f(x) = y$.

Theorem 3.9 Let $(X_{(F)}, \tau)$ be an s-topological vector space. For given y $\in$ X and λ in F with $\lambda \neq 0$, the right (left) translation map $T_y : x \mapsto x + y$ and multiplication map $M_\lambda : x \mapsto \lambda \cdot x$, where $x \in X$, are generalized-homeomorphisms onto itself.

Proof. It is obvious that right translations are bijective mappings. By Theorem 3.3, the translations T_y and M_λ are semi-continuous mappings. We prove that the translation T_y

is semi-open mapping. Let U be any open negihbourhood of x. Then $T_y(U) = U + y$. By Theorem 3.4, $U + y$ is semi-open in X. This proves that T_y is semi-open mapping.

Similarly, we can prove that $M_\lambda : x \mapsto \lambda \cdot x$ is a generalized homeomorphism. ■

Theorem 3.10 Every s-topological vector space $(X_{(F)}, \tau)$ is a generalized-homogenous space.

Proof. Take any x, y $\in X$ and put z = (-x) + y. Then $T_z : X \to X$ is a generalized homeomorphism and $T_z(x) = x + z = y$. ■

Theorem 3.11 Suppose $(X_{(F)}, \tau)$ is an s-topological vector space and S is a subspace of X. If S contains a non-empty open set, then S is semi-open in (X, τ).

Proof. Suppose $U \neq \phi$ is open subset in X such that U $\subseteq$ S. For any $y \in S$ the set $T_y(U) = U + y$ is semi-open in X and is a subset of S. Therefore, the subspace $S = \bigcup_{y \in S}(U + y)$ is semi-open in X as the union of semi-open sets. ■

Theorem 3.12 Every open subspace S of an s-topological vector space $(X_{(F)}, \tau)$ is also an s-topological vector space (called s-topological subspace of X).

Proof. Let $x, y \in S$ and W be an open neighbourhood of $x + y$ in S. This gives W is an open neighbourhood of $x + y$ in X. Hence, there exist semi-open neighbourhoods $U \subseteq X$ of x and $V \subseteq X$ of y such that $U + V \subseteq W$. Now by Lemma 2.2, the sets $A = U \cap S$ and $B = V \cap S$ are semi-open neighbourhoods of x and y respectively in S because S is open in X. Also $A + B \subseteq U + V \subseteq W$.

Again, let $\lambda \in F$ and $x \in S$. Let W be an open neighbourhood of $\lambda \cdot x$ in S. Since S is open in X, therefore W is open neighbourhood of $\lambda \cdot x$ in X. Hence, there exist semi open neighbourhoods $U \subseteq$ F of λ and $V \subseteq X$ of y such that $U \cdot V \subseteq W$. Now by Lemma 2.2, the set $A = U \cap F$ is semi open neighbourhood of λ in F and the set $B = V \cap S$ is semi-open neighbourhood of y in S. Also $A \cdot B \subseteq U \cdot V \subseteq W$, which means that S is an s-topological vector space. ■

Theorem 3.13 In an s-topological vector space, for any open neighbourhood U of e, there is a semi-open neighbourhood V of e such that $V + V \subseteq U$.

Proof. Proof is simple, therefore omitted. ■

Theorem 3.14 Let A and B be subsets of an s-topological vector space $(X_{(F)}, \tau)$. Then $sCl(A) + sCl(B) \subseteq Cl(A + B)$.

Proof. Suppose that x $\in$ sCl(A), y$\in$ sCl(B). Let W be an open neighbourhood of $x + y$. Then there are semi-open neighbourhoods U and V of x and y respectively, such that $U + V \subseteq W$. Since x $\in$ sCl(A), y $\in$ sCl(B), there are a $\in$ A$\cap$ U and b $\in$ B $\cap$ V. Then $a + b \in (A + B) \cap (U + V) \subseteq (A + B) \cap W$. This means $x + y \in Cl(A + B)$, that is $sCl(A) + sCl(B) \subseteq Cl(A + B)$. ■

Theorem 3.15 Let f : X $\to$ Y be a homomorphism of s-topological vetor spaces. If f is s-continuous at the identity e of $(X_{(F)}, \tau)$, then f is semi-continuous on X.

Proof. Let $x \in X$. Suppose W is open neighbourhood of $y = f(x)$ in Y. Since $T_y : Y \to Y$ is semi-continuous, therefore there is a semi-open neighbourhood V of e such that $T_y(V) = V + y \subseteq W$. Now from s-continuity of f at e of X, there exists open neighbourhood U of e in X such that $f(U) \subseteq V$. Since $T_x : X \to X$ is semi-open, therefore the set $U + x$ is semi-open neighbourhood of x. So $f(U + x) = f(U) + f(x) = f(U) + y \subset V + y \subseteq W$. Therefore f is semi-continuous at x of X, and hence on X. ■

Theorem 3.16 Let $(X_{(F)}, \tau)$ be an s-topological vector space. Then every open subspace of X is semi-closed in X.

Proof. Let S be an open subspace of X. As right translation $T_x : X \to X$ is semi-open, therefore $S + x$ is semi-open in X. Then $Y = \bigcup_{x \in X-S}(S + x)$ is also semi-open. Now $S = X - Y$, is semi-closed. ■

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