Journal of
Linear and Topological Algebra
Vol. 02, *No.* 03, 2013, 135- 140

Some results of semilocally simply connected property

A. Etemad Dehkordy[a,*] and M. Malek Mohamad[a]

[a] *Department of Mathematical sciences, Isfahan University of Technology, Isfahan, Iran.*

Abstract. If we consider some special conditions, we can assume fundamental group of a topological space as a new topological space. In this paper, we will present a number of theorems in topological fundamental group related to semilocally simply connected property for a topological space.

Keywords: Semilocally simply connected, topological fundamental group, discrete space.

2010 AMS Subject Classification: Primary 55U40; Secondary 54H11, 57M10.

1. Introduction

There are some results about topological fundamental group in paper [1] by Daniel K. Biss. In mentioned paper, he attempted to define a topology on fundamental group $\pi_1(X)$ (denoted by $\pi_1^{Top}(X)$) for special topological space X, then he proved X is semiloclly simply connected if and only if $\pi_1^{Top}(X)$ is discrete.

In this paper we will state and prove other theorems in this subject. Let topological spaces denote by X, Y, $E, \ldots$. A lift for a continuous map $f : Y \to X$ relative to a map $p : E \to X$, is a continuous map $\tilde{f} : Y \to E$ such that $p\, o\tilde{f} = f$. Moreover, if $f : (X, x) \to (Y, y)$ be a continuous map between pointed topological spaces, then by $f^{Top} : \pi_1^{Top}(X, x) \to \pi_1^{Top}(Y, y)$, we mean the induced map from f, between topological fundamental groups. In this paper , the map $p : E \to X$ is a *(weak) fibration*, means that p has *homotopy lifting property* relative to members of set $\{I^n | n \in N\}$, where $I = [0, 1]$. Also, we say a fibration $p : E \to X$ is a *covering fibration* if $p_\sharp : \pi_i(E) \to \pi_i(X)$ be

*Corresponding author.
E-mail address: ae110mat@cc.iut.ac.ir (A. Etemad Dehkordy).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

an isomorphism for $i \geqslant 2$ and one to one for $i = 1$. For example every covering map is a covering fibration. Furthermore, if for every two paths $\tilde{f_1}$ and $\tilde{f_2}$, two conditions $p\ o\tilde{f_1} = p\ o\tilde{f_2}$ and $\tilde{f_1}(0) = \tilde{f_2}(0)$ conclude that $\tilde{f_1} = \tilde{f_2}$, we say that the map $p : E \to X$, has *unique path lifting property.*

2. Some Results

At first, by use of Lemma (7.6.15) and Lemma (7.6.13) of [7] we state a simple proof for the following Theorem.

Theorem 2.1 Let X be a topological space and $f : S^{n-1} \to X$ be a continuous map. If $X' = X \cup_f E^n$ and Y be a CW-complex with dimension little than n, then for every continuous map $g : Y \to X'$, there is a map $g' : Y \to X$ such that g is homotopic to g'.

Proof. By assumption for Y, we have $dim(Y - \emptyset) = dim\ Y \leqslant n - 1$. So by Lemma (7.6.15) of [7], $(X \cup_f E^n, X)$ is $(n-1)$-connected for all $n \geqslant 1$. Now, $X \subset X \cup_f E^n$ is close if and only if $X \subset X \cup E^n/\sim$ is close. It is true if and only if $p^{-1}(X) = X$ is close in $X \cup E^n$, where $p : X \cup E^n \to X \cup E^n/\sim$ is natural projection. But we have $X \subset X \cup E^n$ is close if and only if $X \cap E^n \subset E^n$ is close. Since $S^{n-1} = X \cap E^n \subset E^n$ is close, so space X is close in $X \cup_f E^n$. On the other hand the space $X \cup_f E^n$ has weak topology respect to $\{X \cup_f E^n, X\}$, therefore $\{X \cup_f E^n, X\}$ has a relative CW-structure . Now by Lemma (7.6.13) of [7], there is a continuous map $g' : Y \to X$, such that g is homotopic to g'. ∎

For fibration $p : E \to X$, if each fiber F of p has no nonconstant path, then path connected component of F has at most one member. Therefore $\pi_1(F) = 0$ and consequently

$$0 = \pi_2(F) = \pi_3(F) = ...$$

Using this statement and Theorem (2.2.5) of [7], the Definition 4.1 of [1], can be brief in the following manner.

Definition 2.2 A fibration $p : E \to X$ is called a *rigid covering fibration* if each fiber of it, has no nonconstant path.

Following Theorem has useful role in the rest of his paper.

Theorem 2.3 Let $p_1 : (E_1, e_1) \to (X, x)$ and $p_2 : (E_2, e_2) \to (X, x)$ be two locally path connected, rigid covering fibration for connected space X.The following statements are equivalent:

i) There is an isomorphism $f : (E_1, e_1) \to (E_2, e_2)$ such that it is an isomorphism in category of rigid fibration,

ii) $p_{1\sharp}(\Pi_1(E_1, e_1)) = p_{2\sharp}(\Pi_1(E_2, e_2))$.

Proof. For conclusion *ii)* from *i)*, since f is bijective, continuous map with continuous inverse and $p_2o\ f = p_1,\ \ p_1o\ f^{-1} = p_2$, therefor we have,

$$p_{1\sharp}(\Pi_1(E_1, e_1)) \subset p_{2\sharp}(\Pi_1(E_2, e_2)) \quad , \quad p_{2\sharp}(\Pi_1(E_2, e_2)) \subset p_{1\sharp}(\Pi_1(E_1, e_1)).$$

So the result is obvious. Conversely, using the fact that every rigid covering fiberation has unique path lifting property and *ii)*, by Theorem (2.2.5) of [7], there is (continuous) lifts $\tilde{p_1} : (E_1, e_1) \to (E_2, e_2)$ and $\tilde{p_2} : (E_2, e_2) \to (E_1, e_1)$ such that $p_2o\tilde{p_1} = p_1$ and $p_1o\tilde{p_2} = p_2$ respectively. So we have $p_2o(\tilde{p_1}o\tilde{p_2}) = p_2$. On the other hand $p_2oId_{E_2} = p_2$

and $(\tilde{p_1} o \tilde{p_2})(e_2) = e_2 = Id_{E_2}(e_2)$, therefor $\tilde{p_1} o \tilde{p_2} = Id_{E_2}$ and similarly $\tilde{p_2} o \tilde{p_1} = Id_{E_1}$. It yields $\tilde{p_1}$ is continuous, bijective with continuous inverse and we have $i)$. ∎

Following Theorem states an important property for the map p that define as follows.

$$p : Hom((S^1, 1), (X, x)) \longrightarrow \pi_1^{top}(X, x)$$
$$f \longmapsto [f]$$

Theorem 2.4 For a locally path connected, pointed topological space X, the map p is open.

Proof. Let $U \subset Hom((S^1, 1), (X, x))$ be open. By definition of quotient topology, we must be show $p^{-1}(p(U)) \subset Hom((S^1, 1), (X, x))$ is open . But we have,

$$p^{-1}(p(U)) = p^{-1}(\{[f] | f \in U\}) = \{\bar{f} \in Hom((S^1, 1), (X, x)) | \exists f \in U \ s.t \ \bar{f} \simeq_p f\}$$

Since above set equal to union of the path components of space $Hom((S^1, 1), (X, x))$ that each of them contains at least one member of U, so we put $Hom((S^1, 1), (X, x)) = \bigcup_{\lambda \in \Lambda} Hom_\lambda((S^1, 1), (X, x))$ such that $Hom_\lambda((S^1, 1), (X, x))$ is a path component in $Hom((S^1, 1), (X, x))$ for every $\lambda \in \Lambda$. Therefore we can assume that $p^{-1}(p(U)) = \cup_{\lambda \in \Omega \subset \Lambda} Hom_\lambda((S^1, 1), (X, x))$. Let f_λ be represnta-tive of homotopic class of component $Hom_\lambda((S^1, 1), (X, x))$ for every $\lambda \in \Lambda$. So we have a homotopy $F_\lambda : f \simeq_p f_\lambda$ in λ-th component such that F transforms f to f_λ continuously for every $f \in Hom_\lambda((S^1, 1), (X, x))$. We define the map $r_\lambda : Hom_\lambda((S^1, 1), (X, x)) \longrightarrow Hom((S^1, 1), (X, x))$ by $r_\lambda(f) = F_\lambda(f)|_{S^1 \times \{1\}} = f_\lambda$ for every $\lambda \in \Lambda \backslash \Omega$. Since restriction of all homotopy between f and f_λ in $S^1 \times \{1\}$ is f_λ, so r_λ is well defined and also a constant map respect to f_λ. In addition r_λ is continuous and in fact it is a contraction. We assume two cases for $\lambda \in \Omega$:

1) If $U \cap Hom_\lambda((S^1, 1), (X, x))$ has only one member, we choose it as f_λ and r_λ defines as before.
2) If $U \cap Hom_\lambda((S^1, 1), (X, x))$ has more than one member.

So there are two continuous map:

$a)$ Choosing $f_\lambda \in U \cap Hom_\lambda((S^1, 1), (X, x))$, we assume the non empty set $Z_\lambda := (U \cap Hom_\lambda((S^1, 1), (X, x))) \backslash \{f_\lambda\}$ and we consider the continuous identity map Id_{Z_λ}.
$b)$ We define r_λ as before.

Now by gluing Lemma the continuous map

$$r : Hom((S^1, 1), (X, x)) \to Hom((S^1, 1), (X, x))$$

can be define as,

$$r(f) = \begin{cases} Id_{Z_\lambda} & \text{If } \lambda \in \Omega, \ f \in U \text{ and } f \text{ be homotopic with another} \\ & \text{member and not be the represenive of } \lambda \text{ component} \\ r_\lambda(f) & \text{otherwise} \end{cases}$$

for every $f \in Hom_\lambda((S^1, 1), (X, x))$. But we have $Im \ r = U \cup (\cup_{\lambda \in \Lambda \backslash \Omega} f_\lambda)$ and since $U \subset Hom((S^1, 1), (X, x)))$ is open, therefor $U \subset Im \ r$ is open. We have r is continuous, so $r^{-1}(U)$ is open in $Hom((S^1,1),(X,x)))$. In fact we have,

$$r^{-1}(U)=\bigcup_{\lambda\in\Omega_1\subset\Omega} Hom_\lambda((S^1,1),(X,x))) \bigcup_{\lambda\in\Omega_2\subset\Omega} Hom_\lambda((S^1,1),(X,x))\backslash Z_\lambda) \bigcup_{\lambda\in\Omega_2} Z_\lambda$$

where $\Omega=\Omega_1\cup\Omega_2$ and $\emptyset=\Omega_1\cap\Omega_2$. So

$$r^{-1}(U)=\bigcup_{\lambda\in\Omega} Hom_\lambda((S^1,1),(X,x))$$

is open in $Hom((S^1,1),(X,x))$. Note that for all $\lambda\in\Omega_1$, $Hom_\lambda((S^1,1),(X,x))$ is an inverse image of the element of U that consider as representative of λ-th component and not homotopic to any element of U. Also, for every $\lambda\in\Omega_2$, $Hom_\lambda((S^1,1),(X,x))\backslash Z_\lambda$ is inverse image of a member of U that is at least homotopic to a member of itself and it choose as representative of λ-th. Moreover Z_λ is the inverse image of the elements of $U\cap Hom_\lambda((S^1,1),(X,x))$ that they are homotopic with a member of themselves and they not choose as representative of λ component. This conclude that p is an open map. ■

3. Main Theorems

The following Theorem states a necessary and sufficient condition for products of topological spaces to be semilocally simply connected.

Theorem 3.1 Let $X=\prod_{i\in I}X_i$ and X_is are metrizable, path connected and locally path connected spaces. In addition let the index set I be countable, then the space X is semilocally simply connected if and only if it satisfies in the following conditions.

(1) All but finitely many X_i are simply connected.
(2) All X_i are semilocally simply connected.

Proof. Let X be semilocally simply connected. By assumption and product topology X is a metrizable, path connected and locally path connected space. Therefore by [4], $\pi_1^{Top}(X)$ is discrete space. Product topology again, yields $\pi_1^{Top}(X_i)$ is discrete for all $i\in I$. Therefore by [4] , X_i is semilocally simply connected and condition (2) is satisfy. Since the product of infinity discrete space is discrete iff all but finitely many of them have one point, so all but finitely many X_is are simply connected and we have condition (1). Conversely, let X_i be semilocally simply connected for all $i\in I$. So by [4], $\pi_1^{Top}(X_i)$ have discrete topology for all $i\in I$. On the other hand, consider index set $J\subset I$ with $|J|<\infty$, such that $\pi_1^{Top}(X_i)=0$ for all $i\in I\setminus J$. It yields $\prod_{i\in I}\pi_1^{Top}(X_i)=\prod_{i\in J}\pi_1^{Top}(X_i)$, therefore by product topology $\prod_{i\in I}\pi_1^{Top}(X_i)$ is discrete. Since we have $\pi_1^{Top}(\prod_{i\in I}X_i)\cong\prod_{i\in I}\pi_1^{Top}(X_i)$ and the righthand of this congruent is discrete, so the left hand of this congruent is also discrete and by use of another conditions (2) and [4], we have the result. ■

Using proposition (1.36) of [5], the following Theorem present a classification for connected covering of a triply connected (i.e.connected, locally path connected, semilocally simply connected) space. At first some primary notes must be state. Let ΩX be the space of all loops in X based at x with compact-open topology. By some notes after (7.2.4) in [7], there is an isomorphism $\psi:\pi_1(X)\cong\pi_0(\Omega X)$ such that image of $[f]\in\pi(X)$ by ψ is a set of all members of ΩX that are path homotopic to f i.e. $\psi[f] = \{g \in \Omega X | g \simeq_p f\}$ as an element of $\pi_0(\Omega X)$. Therefore the map $\delta = \gamma^{Top} o \psi : \pi_1(X) \to \pi_0^{Top}(F)$, is a continuous map for all fiber F of fiberation $p : E \to X$ that has unique path lifting property.

Theorem 3.2 Suppose that (X, x) is a path connected, locally path connected, semilocally simply connected pointed space. Therefore connected covers of X are classified by conjugacy classes of open subgroups of $\pi_1^{Top}(X, x)$.

Proof. By general topology, the statement of Theorem is equal to prove two following statements.

(a) Let $q : (\tilde{X}, \tilde{x}) \to (X, x)$ is a covering map such that $\tilde{X}$ is connected and furthermore $q_\sharp(\pi_1(\tilde{X}, \tilde{x})) = \pi$. We will prove that the homogeneous space $\pi_1(X, x)/\pi$ has discrete topology. By above note, we know that for this fibration and all fiber F, the mentioned map $\delta : \pi_1^{Top}(X) \to \pi_0^{Top}(F)$ is continuous. Moreover by this condition, the space $\tilde{X}$ is also path connected, so $\pi_0(\tilde{X}) = 0$. Therefore the end of exact homotopy sequence, with induced topology is

$$... \to \pi_1^{Top}(F) \to \pi_1^{Top}(\tilde{X}) \to \pi_1^{Top}(X) \xrightarrow{\delta} \pi_0^{Top}(F) \to 0,$$

so δ is onto. On the other hand, since $\pi_0(F)$ is set of connected component of F, therefore by assumptions we have isomorphism $\psi : \pi_0(F) \cong F$ with inverse $p_1 : F \to \pi_0^{Top}(F)$, that maps members of a path component to corresponding path component as a member of $\pi_0^{Top}(F)$. Now the map q is continuous covering projection , so the topology on F is discrete and since the topology of $\pi_0^{Top}(F)$ is quotient topology that induced by p_1, therefore it is also a discrete topological space. Obviously p_1 is an open map, so $\psi = p_1^{-1}$ is continuous. Define $\bar{\delta} : \dfrac{\pi_1^{Top}(X)}{Ker\ \delta} \to \pi_0^{Top}(F)$ by $\bar{\delta}([g]+Ker\ \delta) = \delta([g])$. Using assumptions, $\bar{\delta}$ is continuous and since δ is onto, therefore $\bar{\delta}$ is bijective . Now we consider the image of all single points of $\dfrac{\pi_1^{Top}(X)}{Ker\ \delta}$ by $\bar{\delta}_0$. We know that it is a member of space $\pi_0^{Top}(F)$. So by use of discreteness of this space, it must be has one point and it is an open set. Therefore the inverse of this set with single point is open in $\dfrac{\pi_1^{Top}(X)}{Ker\ \delta_0}$ and we have the result. For the rest of proof it is enough we prove $Ker\ \delta = \pi = q_\sharp(\pi_1(\tilde{X}, \tilde{x}))$ or equivalently we must prove $\{[f] \in \pi_1^{Top}(X, x) | \tilde{f}(1) = \tilde{x}\} = q_\sharp(\pi_1(\tilde{X}, \tilde{x}))$. Let $[e_{e_0}]$ be trivial element of $\pi_0^{Top}(F)$. Since $\delta(\pi) = [e_{e_0}]$, so obviously $\pi \subset Ker\delta$. On the other hand if $[f] \in\ Ker\delta$, then $\tilde{f}$,the lift of $[f]$ based at $\tilde{x}$, is a loop at $\tilde{x}$ i.e. $[\tilde{f}] \in \pi_1(\tilde{X}, \tilde{x})$. Therefore $[f] =\ [qo\tilde{f}] \in\ q_\sharp(\pi_1(\tilde{X}, \tilde{x}) = \pi$, so $Ker\delta \subset \pi$. Furthermore, if we substituted $\tilde{x}$ by $\tilde{y}$ and considering covering map $g : (\tilde{X}, \tilde{y}) \to (X, x)$, then two subgroups $q_\sharp(\pi_1(\tilde{X}, \tilde{x}))$ and $g_\sharp(\pi_1(\tilde{X}, \tilde{y}))$ are conjugate (Theorem (1.38) of [6]).

(b) Let X be a path connected, locally path connected, semilocally simply connected, so for all subgroup $\pi \leqslant \pi_1(X, x)$, there is a covering projection map $q : \tilde{X} \to X$, such that for a suitable point $\tilde{x} \in \tilde{X}$, we have $q_\sharp(\pi_1(\tilde{X}, \tilde{x})) = \pi$ (Proposition (1.36) of [6]). There is one note about this part, if $\tilde{X}$ is connected, then by (a), the space $\dfrac{\pi_1^{Top}(X)}{Ker\ \delta}$ has discrete topology. Moreover if π' is a conjugate subgroup for π in $\pi_1(X, x)$, then there is a covering projection map $g : \tilde{Y} \to X$ such that for suitable point $\tilde{y} \in \tilde{Y}$ we have $g_\sharp(\pi_1(\tilde{Y}, \tilde{y})) = \pi'$. Now if we assume $\tilde{Y}$ is connected, by use of part 3 of Theorem (2.5.2) of [7], we conclude that there is a covering projection map $f : \tilde{Y} \to \tilde{X}$. ∎

References

[1] D.K. Biss, The topological fundamental group and generalization covering space, Topology and its application 124(2002), 355-371.
[2] W.A. Bogley and A.J. Sieradski, Wieghted combinatorial group theory and wiled metric complexces A.C. Kim(Ed) Groups-Korea'98 de Gruyter, Pusan(2005), 53-80.
[3] J.S. Calcut and J.D. McCarthy, Discreteness and homogeneity of the topological fundamental group, Topology Proc. 34(2009) 339-349.
[4] P. Fabel, Metric spaces with discrete topological fundamental group, Topology Appl., 154(2007), 635-638.
[5] R. Fritsch and R.A. Piccinini, Cellular Structures in Topology, Cambridge University press(1990).
[6] A. Hatcher, Algebraic Topology, Cambridge University press(2002).
[7] E.H. Spanier, Algebraic Topology, McGraw-Hill, New york(1966).