Journal of
Linear and Topological Algebra
Vol. 03*, No.* 03*,* 2014, 131- 147

Tripled coincidence point under ϕ-contractions in ordered G_b-metric spaces

R. Jalal Shahkoohi[a*], S. A. Kazemipour[a] and A. Rajabi Eyvali[a]

[a] *Department of Mathematics and Statistics, Aliabad Katoul Branch, Islamic Azad University, Aliabad Katoul, Iran.*

Received 27 July 2014; Revised 25 November 2014; Accepted 16 December 2014.

Abstract. In this paper, tripled coincidence points of mappings satisfying ψ-contractive conditions in the framework of partially ordered G_b-metric spaces are obtained. Our results extend the results of Aydi *et al.* [H. Aydi, E. Karapınar and W. Shatanawi, *Tripled fixed point results in generalized metric space*, *J. Applied Math.*, Volume 2012, Article ID 314279, 10 pages]. Moreover, some examples of the main result are given.

Keywords: Tripled fixed point, Generalized weakly contraction, Generalized metric spaces, Partially ordered set.

2010 AMS Subject Classification: Primary 47H10; Secondary 54H25.

1. Introduction and Preliminaries

Berinde and Borcut [7] introduced the concept of tripled fixed point and obtained some tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. For a survey of tripled fixed point theorems and related topics we refer the reader to [7, 8, 9, 17, 24, 25].

Definition 1.1 [7] An element $(x, y, z) \in X^3$ is called a tripled fixed point of $F : X^3 \to X$ if $F(x, y, z) = x$, $F(y, x, y) = y$, and $F(z, y, x) = z$.

Definition 1.2 [8] An element $(x, y, z) \in X^3$ is called a tripled coincidence point of the mappings $F : X^3 \to X$ and $g : X \to X$ if $F(x, y, z) = g(x)$, $F(y, x, y) = gy$ and $F(z, y, x) = gz$.

*Corresponding author.
E-mail address: rog.jalal@gmail.com (R. Jalal Shahkoohi).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.iauctb.ac.ir

Definition 1.3 [8] An element $(x, y, z) \in X^3$ is called a tripled common fixed point of $F : X^3 \to X$ and $g : X \to X$ if $x = g(x) = F(x, y, z)$, $y = g(y) = F(y, x, y)$ and $z = g(z) = F(z, y, x)$.

Definition 1.4 [17] Let X be a non-empty set. We say that the mappings $F : X^3 \to X$ and $g : X \to X$ are commutative if $g\,(F(x, y, z)) = F(gx, gy, gz)$, for all $x, y, z \in X$.

Definition 1.5 ([7, 8]) Let $(\mathcal{X}, \preceq)$ be a partially ordered set, $F : \mathcal{X}^3 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$.

We say that F has the mixed g-monotone property if $F(x, y, z)$ is g-nondecreasing in x, g-nonincreasing in y and g-nondecreasing in z, that is if, for any $x, y, z \in \mathcal{X}$,

$$x_1, x_2 \in \mathcal{X}, gx_1 \preceq gx_2 \Rightarrow F(x_1, y, z) \preceq F(x_2, y, z),$$
$$y_1, y_2 \in \mathcal{X}, gy_1 \preceq gy_2 \Rightarrow F(x, y_1, z) \succeq F(x, y_2, z)$$

and

$$z_1, z_2 \in \mathcal{X}, gz_1 \preceq gz_2 \Rightarrow F(x, y, z_1) \preceq F(x, y, z_2).$$

The concept of a generalized metric space, or a G-metric space, was introduced by Mustafa and Sims [20].
Czerwik in [12] introduced the concept of a b-metric space. Since then, several papers dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces.(for instance, see([10]) Cone metric spaces were introduced in ([14]). A similar notion was also considered by Rzepecki in ([29]). After carefully defining convergence and completeness in cone metric spaces, the authors proved some fixed point theorems of contractive mappings. (for instance, see([13], [18], [27], [23], [31], [33]).

Definition 1.6 ([12]) Let X be a (nonempty) set and $s \geqslant 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

(b_1) $d(x, y) = 0$ iff $x = y$,
(b_2) $d(x, y) = d(y, x)$,
(b_3) $d(x, z) \leqslant s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric if (and only if) $s = 1$. We present an easy example to show that in general a b-metric need not be a metric.

***Example* 1.7** Let (X, ρ) be a metric space, and $d(x, y) = (\rho(x, y))^p$, where $p \geqslant 1$ is a real number. Then d is a b-metric with $s = 2^{p-1}$.

Definition 1.8 ([20]) Let X be a nonempty set and let $G : X^3 \to \mathbb{R}^+$ be a function satisfying the following properties:
(G1) $G(x, y, z) = 0$ if $x = y = z$;
(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leqslant G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables);
(G5) $G(x, y, z) \leqslant G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Theorem 1.9 ([4]) Let $(X, \preceq)$ be a partially ordered set and (X, G) be a G-metric space such that (X, G) is G-complete. Let $F : X^3 \to X$ be a continuous mapping having the mixed monotone property on X. Assume there exists $\phi \in \Phi$ such that for all $x, y, z, u, v, w, r, s, t \in X$, with $x \preceq u \preceq r$, $y \succeq v \succeq s$ and $z \preceq w \preceq t$, we have,

$$G(F(x, y, z), F(u, v, w), F(r, s, t)) \leqslant \phi(\max\{G(x, u, r), G(y, v, s), G(z, w, t)\}).$$

Suppose there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then F has a tripled fixe point in X, i.e., there exist $x, y, z \in X$ such that $F(x, y, z) = x$, $F(y, x, y) = y$ and $F(z, y, x) = gz$.

Also, they proved that the above theorem is still valid for F not necessarily continuous, assuming the following hypothesis (see, Theorem 2.4 of [5]).

I. If $\{x_n\}$ is a nondecreasing sequence with $x_n \to x$, then $x_n \preceq x$, for all $n \in \mathbb{N}$.

II. If $\{y_n\}$ is a nonincreasing sequence with $y_n \to y$, then $y_n \succeq y$, for all $n \in \mathbb{N}$.

A partially ordered G-metric space (X, G) with the above properties is called regular.

In this paper, we obtain some tripled coincidence point theorems for nonlinear ϕ-contractive mappings in partially ordered G_b-metric spaces. This results generalize and modify several comparable results in the literature. First, we recall the concept of generalized b-metric spaces, or G_b-metric spaces.

Definition 1.10 [3] Let X be a nonempty set and $s \geqslant 1$ be a given real number. Suppose that a mapping $G : X^3 \to \mathbb{R}^+$ satisfies:

(G_b1) $G(x, y, z) = 0$ if $x = y = z$,
(G_b2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G_b3) $G(x, x, y) \leqslant G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(G_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),
(G_b5) $G(x, y, z) \leqslant s[G(x, a, a) + G(a, y, z)]$ for all $x, y, z, a \in X$ (rectangle inequality).

Then, G is called a generalized b-metric and the pair (X, G) is called a generalized b-metric space or a G_b-metric space.

Obviously, each G-metric space is a G_b-metric space with $s = 1$. But, the following example shows that a G_b-metric on X need not to be a G-metric on X.

***Example* 1.11** [3] Let (X, G) be a G-metric space and $G_*(x, y, z) = G(x, y, z)^p$, where $p > 1$ is a real number.

Note that G_* is a G_b-metric with $s = 2^{p-1}$.

Also, in the above example, (X, G_*) is not necessarily a G-metric space. For example, let $X = \mathbb{R}$ and G-metric G be defined by

$$G(x, y, z) = \frac{1}{3}(|x - y| + |y - z| + |x - z|),$$

for all $x, y, z \in \mathbb{R}$ (see [20]). Then $G_*(x, y, z) = G(x, y, z)^2 = \frac{1}{9}(|x - y| + |y - z| + |x - z|)^2$ is a G_b-metric on $\mathbb{R}$ with $s = 2^{2-1} = 2$, but it is not a $G-$metric. To see this, let $x = 3$, $y = 5$, $z = 7$ and $a = \frac{7}{2}$. Hence, we get, $G_*(3, 5, 7) = \frac{64}{9}$, $G_*(3, \frac{7}{2}, \frac{7}{2}) = \frac{1}{9}$ and $G_*(\frac{7}{2}, 5, 7) = \frac{49}{9}$, therefore, $G_*(3, 5, 7) = \frac{64}{9} \nleq \frac{50}{9} = G_*(3, \frac{7}{2}, \frac{7}{2}) + G_*(\frac{7}{2}, 5, 7)$.

***Example* 1.12** [3] Let $X = \mathbb{R}$ and $d(x,y) = |x-y|^2$. We know that (X,d) is a $b-$metric space with $s=2$. Let $G(x,y,z) = d(x,y)+d(y,z)+d(z,x)$, then (X,G) is not a G_b-metric space.

However, $G(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$ is a G_b-metric on $\mathbb{R}$ with $s=2$. Similarly, if $d(x,y) = |x-y|^p$ is selected with $p \geqslant 1$, then

$$G(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$$

is a G_b-metric on $\mathbb{R}$ with $s = 2^{p-1}$.

Now we present some definitions and propositions in a G_b-metric space.

Definition 1.13 [3] A G_b-metric G is said to be symmetric if $G(x,y,y) = G(y,x,x)$, for all $x, y \in X$.

Definition 1.14 [3] Let (X,G) be a G_b-metric space. Then for $x_0 \in X$ and $r > 0$, the G_b-ball with center x_0 and radius r is

$$B_G(x_0, r) = \{y \in X \mid G(x_0, y, y) < r\}.$$

Proposition 1.15 [3] Let X be a G_b-metric space. Then for each $x, y, z, a \in X$ it follows that:
(1) if $G(x,y,z) = 0$ then $x = y = z$,
(2) $G(x,y,z) \leqslant s(G(x,x,y) + G(x,x,z))$,
(3) $G(x,y,y) \leqslant 2sG(y,x,x)$,
(4) $G(x,y,z) \leqslant s(G(x,a,z) + G(a,y,z))$.

Definition 1.16 [3] Let X be a G_b-metric space. We define $d_G(x,y) = G(x,y,y) + G(x,x,y)$, for all $x, y \in X$. It is easy to see that d_G defines a $b-$metric d on X, which we call it the $b-$metric associated with G.

Proposition 1.17 [3] Let X be a G_b-metric space. Then for any $x_0 \in X$ and $r > 0$, if $y \in B_G(x_0, r)$, then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B_G(x_0, r)$.

From the above proposition the family of all G_b-balls

$$F = \{B_G(x, r) \mid x \in X,\ r > 0\}$$

is a base of a topology $\tau(G)$ on X, which we call it the G_b-metric topology.

Proposition 1.18 [3] Let X be a G_b-metric space. Then for any $x_0 \in X$ and $r > 0$, we have,

$$B_G(x_0, \frac{r}{2s+1}) \subseteq B_{d_G}(x_0, r) \subseteq B_G(x_0, r).$$

Thus every G_b-metric space is topologically equivalent to a $b-$metric space.

Definition 1.19 [3] Let X be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be:

(1) G_b-Cauchy, if for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n, l \geqslant n_0, G(x_n, x_m, x_l) < \varepsilon$;
(2) G_b-convergent to a point $x \in X$, if for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m, n \geqslant n_0, G(x_n, x_m, x) < \varepsilon$.

Proposition 1.20 [3] Let X be a G_b-metric space. Then the following are equivalent:

(1) the sequence $\{x_n\}$ is G_b-Cauchy.
(2) for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geqslant n_0$.

Proposition 1.21 [3] Let X be a G_b-metric space. The following are equivalent:
(1) $\{x_n\}$ is G_b-convergent to x.
(2) $G(x_n, x_n, x) \to 0$, as $n \to +\infty$.
(3) $G(x_n, x, x) \to 0$, as $n \to +\infty$.

Definition 1.22 [3] A G_b-metric space X is called complete if every G_b-Cauchy sequence is G_b-convergent in X.

Definition 1.23 [21] Let (X, G) and (X', G') be two G_b-metric spaces. Then a function $f : X \to X'$ is G_b-continuous at a point $x \in X$ if and only if it is G_b-sequentially continuous at x, that is, whenever $\{x_n\}$ is G_b-convergent to x, $\{f(x_n)\}$ is G'_b-convergent to $f(x)$.

Mustafa and Sims proved that each $G-$metric function $G(x, y, z)$ is jointly continuous in all three of its variables (see proposition 8 in [20]). But in general, a G_b-metric function $G(x, y, z)$ for $s > 1$ is not jointly continuous in all its variables. Now, we present an example of a discontinuous G_b-metric.

***Example* 1.24** Let $X = \mathbb{N} \cup \{\infty\}$ and let $D : X^2 \to \mathbb{R}$ be defined by,

$$D(m,n) = \begin{cases} 0, & \text{if } m = n, \\ \left|\frac{1}{m} - \frac{1}{n}\right|, & \text{if } m, n \text{ are even or } mn = \infty, \\ 5, & \text{if } m \text{ and } n \text{ are odd and } m \neq n, \\ 2, & \text{otherwise.} \end{cases}$$

Then it is easy to see that for all $m, n, p \in X$, we have

$$D(m, p) \leqslant 3(D(m, n) + D(n, p)).$$

Thus, (X, D) is a $b-$metric space with $s = 3$ (see example 3 in [15]).

Let $G(x, y, z) = \max\{D(x, y), D(y, z), D(z, x)\}$. It is easy to see that G is a G_b-metric with $s = 3$. Now, we show that $G(x, y, z)$ is not a continuous function. Take $x_n = 2n, y_n = z_n = 1$, then we have, $x_n \to \infty, y_n \to 1$ and $z_n \to 1$. Also,

$$\begin{aligned} G(2n, \infty, \infty) &= \max\{D(2n, \infty), D(\infty, \infty), D(\infty, 2n)\} \\ &= \max\{D(2n, \infty), D(\infty, \infty)\} = \frac{1}{2n} \to 0, \end{aligned}$$

and

$$G(y_n, 1, 1) = G(z_n, 1, 1) = 0 \to 0.$$

On the other hand,

$$G(x_n, y_n, z_n) = \max\{D(x_n, 1), D(1, 1), D(1, x_n)\} = D(x_n, 1) = 2,$$

and

$$G(\infty, 1, 1) = \max\{D(\infty, 1), D(1, 1), D(1, \infty)\} = 1.$$

Hence, $\lim_{n\to\infty} G(x_n, y_n, z_n) \neq G(x, y, z)$.

So, from the above discussion we need the following simple lemma about the G_b-convergent sequences in the proof of our main result.

Lemma 1.25 [3] Let (X, G) be a G_b-metric space with $s > 1$, and Suppose that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are G_b-convergent to x, y and z, respectively. Then we have,

$$\frac{1}{s^3} G(x, y, z) \leqslant \liminf_{n\longrightarrow\infty} G(x_n, y_n, z_n) \leqslant \limsup_{n\longrightarrow\infty} G(x_n, y_n, z_n) \leqslant s^3 G(x, y, z).$$

2. Main results

A mapping $\phi : [0, \infty) \to [0, \infty)$ is called a *comparison function* if it is increasing and $\phi^n(t) \to 0$, as $n \to \infty$ for any $t \in [0, \infty)$.

let Φ be the set of all comparison functions ϕ, that is,

$$\Phi = \{\phi : [0, \infty) \to [0, \infty) : \phi \ is\ a\ comparison\ function\}.$$

The following lemma is an essential result.

Lemma 2.1 ([19]) If $\phi : [0, \infty) \to [0, \infty)$ is a comparison function, then:

(1) each iterate ϕ^k of ϕ, $k \geqslant 1$, is also a comparison function;
(2) ϕ is continuous at 0;
(3) $\phi(t) < t$, for any $t > 0$.

In [4], Aydi *et al.* established some tripled coincidence point results for mappings $F : X^3 \to X$ and $g : X \to X$ involving nonlinear contractions in the setting of ordered G-metric spaces.

Definition 2.2 ([3])Let X be a nonempty set.Then $(X, G, \preceq)$ is called partially ordered G_b-metric space if G is a G_b-metric on partially ordered set $(X, \preceq)$.

Theorem 2.3 Let $(X, \preceq, G)$ be a partially ordered G_b-metric space with $s \geqslant 1$and $F : X^3 \to X$and $g : X \to X$ be such that $F(X^3) \subseteq g(X)$. Assume

$$sG(F(x, y, z), F(u, v, w), F(r, s, t)) \leqslant \phi(\max\{G(gx, gu, gr), G(gy, gv, gs), G(gz, gw, gt)\}) \tag{1}$$

for every $x, y, z, u, v, w, r, s, t \in X$ with $gx \preceq gu \preceq gr$, $gy \succeq gv \succeq gs$ and $gz \preceq gw \preceq gt$, or, $gr \preceq gu \preceq gx$, $gs \succeq gv \succeq gy$ and $gt \preceq gw \preceq gz$, where $\phi \in \Phi$.

Assume

(1) F has the mixed g-monotone property.

(2) g is G_b-continuous and commutes with F.

Also, suppose,

(a) Either F is G_b-continuous and (X, G) is G_b-complete, or,

(b) (X, G) is regular and $(g(X), G)$ is G_b-complete.

If there exists $x_0, y_0, z_0 \in X$ such that $gx_0 \preceq F(x_0, y_0, z_0)$, $gy_0 \succeq F(y_0, x_0, y_0)$ and $gz_0 \preceq F(z_0, y_0, x_0)$, then F and g have a tripled coincidence point in X.

Proof. Let $x_0, y_0, z_0 \in X$ be such that $gx_0 \preceq F(x_0, y_0, z_0)$, $gy_0 \succeq F(y_0, x_0, y_0)$ and $gz_0 \preceq F(z_0, y_0, x_0)$. Define $x_1, y_1, z_1 \in X$ such that $gx_1 = F(x_0, y_0, z_0)$, $gy_1 = F(y_0, x_0, y_0)$ and $gz_1 = F(z_0, y_0, x_0)$. Then, $gx_0 \preceq gx_1$, $gy_0 \succeq gy_1$ and $gz_0 \preceq gz_1$. Similarly, define $gx_2 = F(x_1, y_1, z_1)$, $gy_2 = F(y_1, x_1, y_1)$ and $gz_2 = F(z_1, y_1, x_1)$. Since F has the mixed g-monotone property, we have $gx_0 \preceq gx_1 \preceq gx_2$, $gy_0 \succeq gy_1 \succeq gy_2$ and $gz_0 \preceq gz_1 \preceq gz_2$.

In this way, we construct the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ as

$$a_n = gx_n = F(x_{n-1}, y_{n-1}, z_{n-1}),$$

$$b_n = gy_n = F(y_{n-1}, x_{n-1}, y_{n-1})$$

and

$$c_n = gz_n = F(z_{n-1}, y_{n-1}, x_{n-1}),$$

for all $n \geqslant 1$.

We will finish the proof in two steps.

Step I. We will show that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are G_b-Cauchy.

Let

$$\delta_n = \max\{G(a_{n-1}, a_n, a_n), G(b_{n-1}, b_n, b_n), G(c_{n-1}, c_n, c_n)\}.$$

As $gx_{n-1} \preceq gx_n$, $gy_{n-1} \succeq gy_n$ and $gz_{n-1} \preceq gz_n$, from (1),

$$\begin{aligned} G(a_n, a_{n+1}, a_{n+1}) &\leqslant sG(a_n, a_{n+1}, a_{n+1}) \\ &= sG(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n), F(x_n, y_n, z_n)) \\ &\leqslant \phi(\max\{G(gx_{n-1}, gx_n, gx_n), G(gy_{n-1}, gy_n, gy_n), G(gz_{n-1}, gz_n, gz_n)\}) \\ &= \phi(\max\{G(a_{n-1}, a_n, a_n), G(b_{n-1}, b_n, b_n), G(c_{n-1}, c_n, c_n)\}), \end{aligned} \tag{2}$$

$$\begin{aligned} G(b_n, b_{n+1}, b_{n+1}) &\leqslant sG(b_n, b_{n+1}, b_{n+1}) \\ &= sG(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_n, x_n, y_n), F(y_n, x_n, y_n)) \\ &\leqslant \phi(\max\{G(gy_{n-1}, gy_n, gy_n), G(gx_{n-1}, gx_n, gx_n)\}) \\ &\leqslant \phi(\max\{G(gy_{n-1}, gy_n, gy_n), G(gx_{n-1}, gx_n, gx_n), G(gz_{n-1}, gz_n, gz_n)\}) \\ &= \phi(\max\{G(a_{n-1}, a_n, a_n), G(b_{n-1}, b_n, b_n), G(c_{n-1}, c_n, c_n)\}) \end{aligned} \tag{3}$$

and

$$\begin{aligned} G(c_n, c_{n+1}, c_{n+1}) &\leqslant sG(c_n, c_{n+1}, c_{n+1}) \\ &= sG(F(z_{n-1}, y_{n-1}, x_{n-1}), F(z_n, y_n, x_n), F(z_n, y_n, x_n)) \\ &\leqslant \phi(\max\{G(gz_{n-1}, gz_n, gz_n), G(gy_{n-1}, gy_n, gy_n), G(gx_{n-1}, gx_n, gx_n)\}) \\ &= \phi(\max\{G(a_{n-1}, a_n, a_n), G(b_{n-1}, b_n, b_n), G(c_{n-1}, c_n, c_n)\}). \end{aligned} \tag{4}$$

From the above inequalities, it follows that

$$\begin{aligned} &\max\{G(a_n, a_{n+1}, a_{n+1}), G(b_n, b_{n+1}, b_{n+1}), G(c_n, c_{n+1}, c_{n+1})\} \\ &\leqslant \phi(\max\{G(a_{n-1}, a_n, a_n), G(b_{n-1}, b_n, b_n), G(c_{n-1}, c_n, c_n)\}). \end{aligned} \tag{5}$$

Repeating (5) $n-$times and using the fact that ϕ is non-decreasing, we get that

$$\begin{aligned}
&\max\{G(a_n, a_{n+1}, a_{n+1}), G(b_n, b_{n+1}, b_{n+1}), G(c_n, c_{n+1}, c_{n+1})\}\\
&\leqslant \phi(\max\{G(a_{n-1}, a_n, a_n), G(b_{n-1)}, b_n, b_n), G(c_{n-1}, c_n, c_n)\})\\
&\leqslant \phi^2(\max\{G(a_{n-2}, a_{n-1}, a_{n-1}), G(b_{n-2}, b_{n-1}, b_{n-1}), G(c_{n-2}, c_{n-1}, c_{n-1})\})\\
&\cdots\\
&\leqslant \phi^n(\max\{G(a_0, a_1, a_1), G(b_0, b_1, b_1), G(c_0, c_1, c_1)\}).
\end{aligned} \tag{6}$$

from (5) we have $\delta_{n+1} \leqslant \phi(\delta_n)$. Since $\phi(\delta_n) < \delta_n$ we have $\delta_{n+1} < \delta_n$, that is, $\{\delta_n\}$ is a non-increasing sequence of nonnegative real numbers. Thus, there is an $r \geqslant 0$ such that

$$\lim_{n\to\infty} \delta_n = r.$$

Since $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$, we obtain from (6) that $\lim\limits_{n\to\infty} \delta_n = 0$.

Next, we claim that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are G_b-Cauchy.

We will show that for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that if $m \geqslant n \geqslant k$,

$$\max\{G(a_m, a_n, a_n), G(b_m, b_n, b_n), G(c_m, c_n, c_n)\} < \varepsilon. \tag{7}$$

Let $\varepsilon > 0$ be arbitrary. As $\lim\limits_{n\to\infty} \delta_n = 0$ and $G(x, y, y) \leqslant 2sG(y, x, x)$ and $0 < \varepsilon - \phi(\varepsilon) \leqslant \varepsilon$ we conclude that

$$\max\{G(a_{n+1}, a_n, a_n), G(b_{n+1}, b_n, b_n), G(c_{n+1}, c_n, c_n)) < \frac{\varepsilon - \phi(\varepsilon)}{s} \leqslant \varepsilon. \tag{8}$$

Therefore, 7 holds when $m = n + 1$.

Now suppose that 7 holds for $m = k$. For $m = k + 1$, we have

$$\begin{aligned}
&G(a_{k+1}, a_n, a_n) \leqslant s[G(a_{k+1}, a_{n+1}, a_{n+1}) + G(a_{n+1}, a_n, a_n)]\\
&< sG(F(x_k, y_k, z_k), F(x_n, y_n, z_n), F(x_n, y_n, z_n)) + s.\tfrac{\varepsilon-\phi(\varepsilon)}{s}\\
&\leqslant \phi(\max\{G(a_k, a_n, a_n), G(b_k, b_n, b_n), G(c_k, c_n, c_n))\}) + s.\tfrac{\varepsilon-\phi(\varepsilon)}{s}\\
&\leqslant \phi(\varepsilon) + \varepsilon - \phi(\varepsilon) = \varepsilon.
\end{aligned} \tag{9}$$

Consequently, $\{a_n\}$ and similarly, $\{b_n\}$ and $\{c_n\}$ are G_b-Cauchy.

Step II. We will show that F and g have a tripled coincidence point.

First, let (a) holds, that is, F is G_b-continuous and (X, G) is G_b-complete.

Since X is G_b-complete and $\{a_n\}$ is G_b-Cauchy, there exists $a \in X$ such that,

$$\lim_{n\to\infty} G(a_n, a_n, a) = \lim_{n\to\infty} G(gx_n, gx_n, a) = 0. \tag{10}$$

Similarly, there exist $b, c \in X$ such that

$$\lim_{n\to\infty} G(b_n, b_n, b) = \lim_{n\to\infty} G(gy_n, gy_n, b) = 0 \tag{11}$$

and

$$\lim_{n\to\infty} G(c_n, c_n, c) = \lim_{n\to\infty} G(gz_n, gz_n, c) = 0. \tag{12}$$

Now, we prove that (a, b, c) is a tripled coincidence point of F and g.

G_b-Continuity of g and Lemma 1.25 yields that,

$$0 = \frac{1}{s^3} G(ga, ga, ga) \leqslant \lim \inf_{n\to\infty} G(g(gx_n), g(gx_n), ga)$$
$$\leqslant \lim \sup_{n\to\infty} G(g(gx_n), g(gx_n), ga) \leqslant s^3 G(ga, ga, ga) = 0.$$

Hence,

$$\lim_{n\to\infty} G(g(gx_n), g(gx_n), ga) = 0 \tag{13}$$

and similarly,

$$\lim_{n\to\infty} G(g(gy_n), g(gy_n), gb) = 0 \tag{14}$$

and

$$\lim_{n\to\infty} G(g(gz_n), g(gz_n), gc) = 0. \tag{15}$$

Since $gx_{n+1} = F(x_n, y_n, z_n)$, $gy_{n+1} = F(y_n, x_n, y_n)$ and $gz_{n+1} = F(z_n, y_n, x_n)$, the commutativity of F and g yields that,

$$g(gx_{n+1}) = g(F(x_n, y_n, z_n)) = F(gx_n, gy_n, gz_n), \tag{16}$$

$$g(gy_{n+1}) = g(F(y_n, x_n, y_n)) = F(gy_n, gx_n, gy_n), \tag{17}$$

and

$$g(gz_{n+1}) = g(F(z_n, y_n, x_n)) = F(gz_n, gy_n, gx_n). \tag{18}$$

From the G_b-continuity of F and 16, 17 and 18 and Lemma 1.25, $\{g(gx_{n+1})\}$ is G_b-convergent to $F(a, b, c)$, $\{g(gy_{n+1})\}$ is G_b-convergent to $F(b, a, b)$ and $\{g(gz_{n+1})\}$ is G_b-convergent to $F(c, b, a)$. From 13, 14 and 15 and uniqueness of the limit, we have $F(a, b, c) = ga$, $F(b, a, b) = gb$ and $F(c, b, a) = gc$, that is, g and F have a tripled coincidence point.

In what follows suppose that the assumption (b) holds. Following the proof of the previous step, there exist $u, v, w \in X$ such that

$$\lim_{n\to\infty} G(gx_n, gx_n, gu) = 0, \tag{19}$$

$$\lim_{n\to\infty} G(gy_n, gy_n, gv) = 0 \tag{20}$$

and

$$\lim_{n\to\infty} G(gz_n, gz_n, gw) = 0 \tag{21}$$

as $(g(X), G)$ is G_b-complete.

Now, we prove that $F(u,v,w)=gu$, $F(v,u,v)=gv$ and $F(w,v,u)=gw$.

Regularity of X yields that $gx_n \preceq gu$, $gy_n \succeq gv$ and $gz_n \preceq gw$ for all $n\in\mathbb{N}$. If for some n, $gx_n=gu$, $gy_n=gv$ and $gz_n=gw$, then

$$gu=gx_n \preceq gx_{n+1} \preceq gu,$$

$$gv=gy_n \succeq gy_{n+1} \succeq gv$$

and

$$gw=gz_n \preceq gz_{n+1} \preceq gw$$

which implies that (x_n,y_n,z_n) is a tripled coincidence point of F and g. Now, assume that, for all n, $(x_n,y_n,z_n)\neq(x,y,z)$. Thus, for each n,

$$\max\{G(x,x,x_n),G(y,y,y_n),G(z,z,z_n)\}>0. \tag{22}$$

From regularity of X and using 1, we have,

$$\begin{aligned}&sG(F(x_n,y_n,z_n),F(u,v,w),F(u,v,w))\\&\leqslant \psi(\max\{G(gx_n,gu,gu),G(gy_n,gv,gv),G(gz_n,gw,gw)\}).\end{aligned} \tag{23}$$

As $\{gx_n\}$ is G_b-convergent to gu, from Lemma 1.25, we have, $\lim_{n\to\infty} G(gx_n,gu,gu)=0$. Analogously, $\lim_{n\to\infty} G(gy_n,gv,gv)=\lim_{n\to\infty} G(gz_n,gw,gw)=0$.

Also, from 22 and 23, using the fact that $\phi(t)<t$ for all $t>0$, we have

$$\lim_{n\to\infty} sG(F(x_n,y_n,z_n),F(u,v,w),F(u,v,w))=0,$$

or, equivalently,

$$\lim_{n\to\infty} G(gx_{n+1},F(u,v,w),F(u,v,w))=0. \tag{24}$$

Similarly,

$$\lim_{n\to\infty} G(gy_{n+1},F(v,u,v),F(v,u,v))=\lim_{n\to\infty} G(gz_{n+1},F(w,v,u),F(w,v,u))=0. \tag{25}$$

On the other hand,

$$\begin{aligned}G(gu,F(u,v,w),F(u,v,w) &\leqslant sG(gu,gx_{n+1},gx_{n+1})\\&\quad+sG(gx_{n+1},F(u,v,w),F(u,v,w)).\end{aligned} \tag{26}$$

Taking limit when $n\to\infty$ and using 19 and 24, we get,

$$\begin{aligned}G(gu,F(u,v,w),F(u,v,w)) &\leqslant \lim_{n\to\infty} sG(gu,gx_{n+1},gx_{n+1})\\&\quad+\lim_{n\to\infty} sG(gx_{n+1},F(u,v,w),F(u,v,w)=0,\end{aligned} \tag{27}$$

that is, $gu=F(u,v,w)$.

Analogously, we can show that $gv = F(v, u, v)$ and $gw = F(w, v, u)$.

Thus, we have proved that g and F have a tripled coincidence point. This completes the proof of the theorem. ■

Remark 2.4 Taking $s = 1$ in the above theorem we obtain Theorem 2.1 of [5].

Corollary 2.5 Let $(X, \preceq)$ be a partially ordered set and (X, G) be a complete G_b-metric space with $s \geqslant 1$. Let $F : X^3 \to X$ be a mapping with the mixed monotone property such that,

$$(sG(F(x,y,z), F(u,v,w), F(r,s,t)\big) \leqslant \phi\Big(\frac{G(x,u,r) + G(y,v,s) + G(z,w,t)}{3}\Big) \tag{28}$$

for every $x, y, z, u, v, w, r, s, t \in X$ with $x \preceq u \preceq r$, $y \succeq v \succeq s$ and $z \preceq w \preceq t$, or, $r \preceq u \preceq x$, $s \succeq v \succeq y$ and $t \preceq w \preceq z$.

Also, suppose,
(a) Either F is G_b-continuous, or,
(b) (X, G) is regular.

If there exists $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then F has a tripled fixed point in X.

Proof. If F satisfies (28), then by taking $g(x) = I_X$, then F satisfies

$$sG(F(x,y,z), F(u,v,w), F(r,s,t)) \leqslant \phi(\max\{G(gx,gu,gr), G(gy,gv,gs), G(gz,gw,gt)\}).$$

So, the result follows from Theorem 2.3. ■

In Theorems 2.3, if we take $\phi(t) = kt$ for all $t \in [0, \infty)$, where $k \in [0, 1)$, we obtain the following result.

Corollary 2.6 Let $(X, \preceq)$ be a partially ordered set and (X, G) be a complete G_b-metric space with $s \geqslant 1$. Let $F : X^3 \to X$ be a mapping having the mixed monotone property and,

$$G(F(x,y,z), F(u,v,w), F(r,s,t)) \leqslant \frac{k}{s}\max\{G(x,u,r), G(y,v,s), G(z,w,t)\}, \tag{29}$$

for every $x, y, z, u, v, w, r, s, t \in X$ with $x \preceq u \preceq r$, $y \succeq v \succeq s$ and $z \preceq w \preceq t$, or, $r \preceq u \preceq x$, $s \succeq v \succeq y$ and $t \preceq w \preceq z$. Also, suppose,
(a) Either F is G_b-continuous, or,
(b) (X, G) is regular.

If there exists $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then F has a tripled fixed point in X.

Corollary 2.7 Let $(X, \preceq)$ be a partially ordered set and (X, G) be a complete G_b-metric space with $s \geqslant 1$. Let $F : X^3 \to X$ be a mapping with the mixed monotone property such that,

$$G(F(x,y,z), F(u,v,w), F(r,s,t)) \leqslant \frac{k}{3s}(G(x,u,r) + G(y,v,s) + G(z,w,t)) \tag{30}$$

for every $x, y, z, u, v, w, r, s, t \in X$ with $x \preceq u \preceq r$, $y \succeq v \succeq s$ and $z \preceq w \preceq t$, or, $r \preceq u \preceq x$, $s \succeq v \succeq y$ and $t \preceq w \preceq z$.

Also, suppose,
(a) Either F is G_b-continuous, or,
(b) (X, G) is regular.
If there exists $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then F has a tripled fixed point in X.

Proof. If F satisfies (30), of above corollary then F satisfies (29)therefore Fhas a fixed point in X . ■

Note that if $(X, \preceq)$ be a partially ordered set, then we can endow X^3 with the following partial order relation:

$$(x, y, z) \preceq (u, v, w) \Longleftrightarrow x \preceq u \ , \ y \succeq v, z \preceq w,$$

for all $(x, y, z), (u, v, w) \in X^3$. (see [7]).

In the following theorem, we give a sufficient condition for the uniqueness of the common tripled fixed point (Also, see e.g. [4], [5], [6] [11]).

Theorem 2.8 In addition to the hypotheses of Theorems 2.3 suppose that for every (x, y, z) and $(x^*, y^*, z^*) \in X \times X \times X$, there exists $(u, v, w) \in X^3$, such that $(F(u, v, w), F(v, u, v), F(w, v, u))$ is comparable with $(F(x, y, z), F(y, x, y), F(z, y, x))$ and $(F(x^*, y^*, z^*), F(y^*, x^*, y^*), F(z^*, y^*, x^*))$. Then, F and g have a unique common tripled fixed point.

Proof. From Theorems 2.3 the set of tripled coincidence points of F and g is non-empty. We shall show that if (x, y, z) and (x^*, y^*, z^*) be tripled coincidence points, that is,

$$g(x) = F(x, y, z), \ g(y) = F(y, x, y), \ g(z) = F(z, y, x)$$

and

$$g(x^*) = F(x^*, y^*, z^*), \ g(y^*) = F(y^*, x^*, y^*), \ g(z^*) = F(z^*, y^*, x^*)$$

then $gx = gx^*$ and $gy = gy^*$ and $gz = gz^*$.

Choose an element $(u, v, w) \in X^3$ such that $(F(u, v, w), F(v, u, v), F(w, v, u))$ is comparable with

$$(F(x, y, z), F(y, x, y), F(z, y, x))$$

and

$$(F(x^*, y^*, z^*), F(y^*, x^*, y^*), F(z^*, y^*, x^*)).$$

Let $u_0 = u$, $v_0 = v$, $w_0 = w$ and choose $u_1, v_1, w_1 \in X$ so that $gu_1 = F(u_0, v_0, w_0)$ and $gv_1 = F(v_0, u_0, v_0)$ and $gw_1 = F(w_0, v_0, u_0)$. Then, similarly as in the proof of Theorem 2.3, we can inductively define sequences $\{gu_n\}$, $\{gv_n\}$ and $\{gw_n\}$ such that $gu_{n+1} = F(u_n, v_n, w_n)$, $gv_{n+1} = F(v_n, u_n, v_n)$ and $gw_{n+1} = F(w_n, v_n, u_n)$. Since $(gx, gy, gz) = (F(x, y, z), F(y, x, y), F(w, y, x))$ and $(F(u, v, w), F(v, u, v), F(w, v, u)) = (gu_1, gv_1, gw_1)$ are comparable, we may assume that $(gx, gy, gz) \preceq (gu_1, gv_1, gw_1)$. Then $gx \preceq gu_1$, $gy \succeq gv_1$ and $gz \preceq gw_1$. Using the mathematical induction, it is easy to prove that $gx \preceq gu_n$, $gy \succeq gv_n$ and $gz \preceq gw_n$, for all $n \geqslant 0$.

Let $\gamma_n = \max\{G(gx, gx, gu_n), G(gy, , gy, gv_n), G(gz, , gz, gw_n)\}$. We will show that $\lim_{n\to\infty} \gamma_n = 0$. First, assume that $\gamma_n = 0$, for an $n \geqslant 1$.

Applying (2), as $gx \preceq gu_n$, $gy \succeq gv_n$ and $gz \succeq gw_n$ one obtains that

$$\begin{aligned}
&s\max\{G(gx, gx, gu_{n+1}), G(gy, gy, gv_{n+1}), G(gz, gz, gw_{n+1})\}\\
&= s\max\{G(F(x, y, z), F(x, y, z), F(u_n, v_n, w_n)), G(F(y, x, y), F(y, x, y), F(v_n, u_n, v_n)),\\
&G(F(w, y, x), F(w, y, x), F(w_n, v_n, u_n))\})\\
&\leqslant \phi\big(\max\{G(gx, gx, gu_n), G(gy, gy, gv_n), G(gz, gz, gw_n)\}\big)\\
&= \phi(\gamma_n) = \phi(0) = 0.
\end{aligned} \tag{31}$$

So, we deduce that $\gamma_{n+1} = 0$. Repeating this process, we can show that $\gamma_m = 0$, for all $m \geqslant n$. So, $\lim_{n\to\infty} \gamma_n = 0$.

Now, let $\gamma_n \neq 0$, for all n and let $\gamma_n < \gamma_{n+1}$, for some n.

From (31)

$$\begin{aligned}
s\gamma_n &= s\max\{G(gx, gx, gu_n), G(gy, , gy, gv_n), G(gz, , gz, gw_n)\}\\
&\leqslant s\gamma_{n+1}\\
&= s\max\{G(gx, gx, gu_{n+1}), G(gy, gy, gv_{n+1}), G(gz, gz, gw_{n+1})\}\\
&\leqslant \phi\big(\max\{G(gx, gx, gu_n), G(gy, gy, gv_n), G(gz, gz, gw_n)\}\big)\\
&= \phi(\gamma_n)\\
&\leqslant \phi(s\gamma_n) < s\gamma_n,
\end{aligned}$$

which is a contradiction.

Hence, if we proceed as in Theorem 2.3, we can show that

$$\lim_{n\to\infty} \max\{G(gx, gu_n, gu_n), G(gy, gv_n, gv_n), G(gz, gw_n, gw_n)\} = 0,$$

that is, $\{gu_n\}$, $\{gv_n\}$ and $\{gw_n\}$ are G_b-convergent to gx, gy and gz, respectively.

Similarly, we can show that

$$\lim_{n\to\infty} \max\{G(gx^*, gu_n, gu_n), G(gy^*, gv_n, gv_n), G(gz^*, gw_n, gw_n)\} = 0.$$

that is, $\{gu_n\}$, $\{gv_n\}$ and $\{gw_n\}$ are G_b-convergent to gx^*, gy^* and gz^*, respectively. Finally, since the limit is unique, $gx = gx^*$ and $gy = gy^*$ and $gz = gz^*$.

Since $gx = F(x, y, z)$, $gy = F(y, x, y)$ and $gz = F(z, y, x)$, by commutativity of F and g, we have $g(gx) = g(F(x, y, z)) = F(gx, gy, gz)$, $g(gy) = g(F(y, x, y)) = F(gy, gx, gy)$ and $g(gz) = g(F(z, y, x)) = F(gz, gy, gx)$. Let $gx = a$, $gy = b$ and $g(z) = c$. Then, $ga = F(a, b, c)$, $gb = F(b, a, b)$ and $gc = F(c, b, a)$. Thus (a, b, c) is another tripled coincidence point of F and g. Then, $a = gx = ga$, $b = gy = gb$ and $c = gz = gc$. Therefore, (a, b, c) is a tripled common fixed point of F and g.

To prove the uniqueness, assume that (p, q, r) is another tripled common fixed point of F and g. Then, $p = gp = F(p, q, r)$, $q = gq = F(q, p, q)$ and $r = gr = F(r, p, q)$. Since (p, q, r) is a tripled coincidence point of F and g, we have $gp = gx$, $gq = gy$ and $gr = gz$. Thus, $p = gp = ga = a$, $q = gq = gb = b$ and $r = gr = gc = c$. Hence, the tripled common fixed point is unique. ■

Recently, many papers are devoted to present different results to ensure the existence and uniqueness of coupled, tripled, quadrupled and multidimensional fixed points in various spaces. But, some other authors are proving that these results can be reduced to their corresponding unidimensional versions (for instance, see([30], [26], [16], [2], [27], [28]). Also, very recently, some relations between n-tuple fixed point theorems and fixed point results in various spaces are proved ([32]).

3. Examples

The following example support our results.

***Example* 3.1** Let $X = (-\infty, \infty)$ be endowed with the usual ordering and the complete G_b-metric

$$G(x, y, z) = (|x - y| + |y - z| + |z - x|)^2,$$

where, $s = 2$.

Define $F : X^3 \to X$ as

$$F(x, y, z) = \frac{x - 2y + 4z}{48},$$

for all $x, y, z \in X$ and $g : X \to X$ with $g(x) = x$ for all $x \in X$.

Let $\phi : [0, \infty) \to [0, \infty)$ be defined by $\phi(t) = \frac{t}{2}$.

Now, we have,

$$\begin{aligned}
& s(G(F(x, y, z), F(u, v, w), F(r, s, t))) \\
& = 2(\frac{1}{48}[|(x - 2y + 4z) - (u - 2v + 4w)|] + \frac{1}{48}[|(u - 2v + 4w) - (r - 2s + 4t)|] \\
& + \frac{1}{48}[|(r - 2s + 4t) - (x - 2y + 4z)|])^2 \\
& \leqslant 2(\frac{1}{24}|\frac{x}{2} - \frac{u}{2}| + \frac{1}{12}|\frac{y}{2} - \frac{v}{2}| + \frac{1}{6}|\frac{z}{2} - \frac{w}{2}| + \frac{1}{24}|\frac{u}{2} - \frac{r}{2}| + \frac{1}{12}|\frac{v}{2} - \frac{s}{2}| + \frac{1}{6}|\frac{w}{2} - \frac{t}{2}| \\
& + \frac{1}{24}|\frac{r}{2} - \frac{x}{2}| + \frac{1}{12}|\frac{s}{2} - \frac{y}{2}| + \frac{1}{6}|\frac{t}{2} - \frac{z}{2}|)^2 \\
& = 2(\frac{1}{24}[|\frac{x}{2} - \frac{u}{2}| + |\frac{u}{2} - \frac{r}{2}| + |\frac{r}{2} - \frac{x}{2}|] + \frac{1}{12}[|\frac{y}{2} - \frac{v}{2}| + |\frac{v}{2} - \frac{s}{2}| + |\frac{s}{2} - \frac{y}{2}|] \\
& + \frac{1}{6}[|\frac{z}{2} - \frac{w}{2}| + |\frac{w}{2} - \frac{t}{2}| + |\frac{t}{2} - \frac{z}{2}|])^2 \\
& \leqslant 2(\frac{1}{3}([|\frac{x}{2} - \frac{u}{2}| + |\frac{u}{2} - \frac{r}{2}| + |\frac{r}{2} - \frac{x}{2}|]^2 + [|\frac{y}{2} - \frac{v}{2}| + |\frac{v}{2} - \frac{s}{2}| + |\frac{s}{2} - \frac{y}{2}|]^2 \\
& \qquad + [|\frac{z}{2} - \frac{w}{2}| + |\frac{w}{2} - \frac{t}{2}| + |\frac{t}{2} - \frac{z}{2}|]^2)) \\
& \leqslant \frac{1}{2}\max\{[|x - u| + |u - r| + |r - x|]^2, [|y - v| + |v - s| + |s - y|]^2, \\
& \qquad [|z - w| + |w - t| + |t - z|]^2\} \\
& = \phi(\max\{G(gx, gu, gr), G(gy, gv, gs), G(gz, gw, gt)\}).
\end{aligned}$$

Hence, all of the conditions of Theorem 2.3 are satisfied. Moreover, $(0, 0, 0)$ is the unique common tripled fixed point of F and g.

The following example has been constructed according to example 2.12 of [1].

***Example* 3.2** Let $X = \{(x, 0, x)\} \cup \{(0, x, 0)\} \subset R^3$, where $x \in [0, \infty]$ with the order $\preceq$ defined as:

$$(x_1, y_1, z_1) \preceq (x_2, y_2, z_2) \Longleftrightarrow x_1 \leqslant x_2,\ y_1 \leqslant y_2 \text{ and } z_1 \leqslant z_2.$$

Let b-metric d be given as

$$d(x, y) = \max\{|x_1 - x_2|^2, |y_1 - y_2|^2, |z_1 - z_2|^2\}$$

and

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}.$$

(X, G) is, clearly, a G_b-complete G_b-metric space.

Let $g : X \to X$ and $F : X^3 \to X$ be defined as follows:

$$F(x, y, z) = x$$

and

$$g((x, 0, x)) = (0, x, 0) \text{ and } g((0, x, 0)) = (x, 0, x).$$

Let $\phi : [0, \infty) \to [0, \infty)$ be as in the above example.

According to the order of X and the definition of g we see that for any element $x \in X$, $g(x)$ is comparable only with itself.

By a careful computation it is easy to see that all of the conditions of Theorem 2.3 are satisfied. Finally, Theorem 2.3 guarantees the existence of a unique common tripled fixed point for F and g, i.e., the point $((0, 0, 0), (0, 0, 0), (0, 0, 0))$.

***Example* 3.3** Let $X = \{0, 1, 2, 3\}$ be endowed with the usual order. Let

$$A = \{(2, 0, 0), (0, 2, 0), (0, 0, 2)\}$$

and

$$B = \{(2, 2, 0), (2, 0, 2), (0, 2, 2)\}.$$

Define $G : X^3 \to \mathbb{R}^+$ by

$$G(x, y, z) = \begin{cases} 1, & \text{if}(x, y, z) \in A \\ 3, & \text{if}(x, y, z) \in B \\ 4, & \text{if}(x, y, z) \in X^3 - A \cup B \\ 0, & \text{if } x = y = z \end{cases}$$

It is easy to see that (X, G) is a nonsymmetric G_b-metric space.

Also, (X, G) is regular. Indeed, for each $\{x_n\}$ in X such that $G(x_n, x, x) \to 0$ for an $x \in X$, then there is a $k \in \mathbb{N}$ such that for each $n \geqslant k$, $x_n = x$.

Define the mappings F and g by

$$F = \begin{pmatrix} (0,0,0) & (0,0,1) & (0,0,2) & (1,0,0) & (1,0,1) & (1,0,2) & (2,0,0) & (2,0,1) & (2,0,2) \\ 0 & 2 & 2 & 0 & 2 & 2 & 0 & 2 & 2 \\ (0,1,0) & (0,1,1) & (0,1,2) & (1,1,0) & (1,1,1) & (1,1,2) & (2,1,0) & (2,1,1) & (2,1,2) \\ 0 & 2 & 2 & 0 & 2 & 2 & 0 & 2 & 2 \\ (0,2,0) & (0,2,1) & (0,2,2) & (1,2,0) & (1,2,1) & (1,2,2) & (2,2,0) & (2,2,1) & (2,2,2) \\ 0 & 2 & 2 & 0 & 2 & 2 & 0 & 2 & 2 \end{pmatrix}$$

and

$$g = \begin{pmatrix} 0\ 1\ 2 \\ 0\ 2\ 2 \end{pmatrix}.$$

We see that, $F(X^3) \subseteq gX$.

Define $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{t}{2}$.

Introduce an order $\preceq$ on $Y = \{0, 1, 2\}$ by

$$\preceq =: \{(0,0), (1,1), (2,2)\}.$$

It remains to check the condition for for every $x, y, z, u, v, w, r, s, t \in X$ with $gx \preceq gu \preceq gr$, $gy \succeq gv \succeq gs$ and $gz \preceq gw \preceq gt$, or, $gr \preceq gu \preceq gx$, $gs \succeq gv \succeq gy$ and $gt \preceq gw \preceq gz$

If $x = y = u = v = w = r = s = t = 0$, then $G(gx, gu, gr) = G(gy, gv, gs) = G(gz, gw, gt) = 0$ and $G(F(x, y, z), F(u, v, w), F(r, s, t)) = 0$. Hence, 2.3 reduces to $0 \leqslant 0$ and holds true. In all other possible cases

$((x, y, z), (u, v, w), (r, s, t) \in \{(0, 0, 0)(1, 1, 1), (2, 2, 2)\})$, one can see that 2.3 holds trivially.

Thus, all the conditions of Theorem 2.3 are fulfilled and F and g have a coincidence point $(0 = F(0, 0, 0) = g0)$. This is also their common coupled fixed point.

References

[1] M. Abbas, M. Ali Khan and S. Radenović, *Common coupled fixed point theorems in cone metric spaces for w-compatible mappings*, *Applied Math. Comput.*, **217**, (2010), 195–202.

[2] R.P. Agarwal, E. Karapinar, Remarks on some coupled fixed point theorems in G-metric spaces. Fixed Point Theory Appl, 2013, 2013:2.

[3] A. Aghajani, M. Abbas and J.R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces*, Accepted in Filomat, 2013.

[4] H. Aydi, E. Karapınar and W. Shatanawi, *Tripled coincidence point results for generalized contractions in ordered generalized metric spaces*, *Fixed Point Theory Appl.*, 2012, **2012**:101.

[5] H. Aydi, E. Karapınar and W. Shatanawi, *Tripled fixed point results in generalized metric space*, *J. Applied Math.*, Volume 2012, Article ID 314279, 10 pages, doi:10.1155/2012/314279.

[6] V. Berinde, *Coupled fixed point theorems for contractive mixed monotone mappings in partially ordered metric spaces*, *Nonlinear Anal.*, 75, (2012), 3218–3228.

[7] V. Berinde and M. Borcut. Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. *Nonlinear Anal.* **74** (2011), 4889–4897.

[8] M. Borcut, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces. *Appl. Math. Comput.* **218** (2012), 7339–7346.

[9] M. Borcut and V. Berinde. Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces. *Appl. Math. Comput.* **218** (2012), 5929–5936.

[10] M. Boriceanu, Strict fixed point theorems for multivalued operators in b-metric spaces, Int. J Modern Math. 4 (2) (2009), 285–301.

[11] Y.J. Cho, B.E. Rhoades, R. Saadati, B. Samet and W. Shatanawi, *Nonlinear coupled fixed point theorems in ordered generalized metric spaces with integral type*, *Fixed Point Theory Appl.*, 2012, 2012:8.

[12] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Sem. Mat. Univ. Modena, 46 (1998) 263-276.

[13] H. Huang, S. Xu, Fixed point theorems of contractive mappings in cone b-metric spaces and applications, Fixed Point Theory Appl, 2013:112.

[14] L.-G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, Journal of Mathematical Analysis and Applications, vol. 332, no. 2, pp. 1468-1476, 2007.

[15] N. Hussain, D. Dorić, Z. Kadelburg and S. Radenović, Suzuki-type fixed point results in metric type spaces, *Fixed Point Theory Appl.*, (2012), doi:10.1186/1687-1812-2012-126.

[16] M. Jleli, B. Samet, Remarks on G-metric spaces and fixed point theorems, Fixed Point Theory Appl. 2012, 2012:210.

[17] B.S. Choudhury, E. Karapınar and A. Kundu. Tripled coincidence point theorems for nonlinear contractions in partially ordered metric spaces. *Int. Journal of Math. and Math. Sciences*, to appear.

[18] M.A. Khamsi, Remarks on cone metric spaces and fixed point theorems of contractive mappings, Fixed Point Theory Appl. 2010, Article ID 315398, 7 pages.

[19] J. Matkowski, *Fixed point theorems for mapping with a contractive iterate at a point*, *Proceedings of the American Mathematical Society*, **V**ol. 62, No. 2, pp. 344–348, 1977.

[20] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, *J. Nonlinear Convex Anal.*, **7**(2), (2006), 289–297.

[21] V. Parvaneh, J.R. Roshan and S. Radenović, *Existence of tripled coincidence point in ordered b-metric spaces and application to a system of integral equations, Fixed Point Theory Appl.,*

[22] H. Rahimi, G. Soleimani Rad, P. Kumam, Coupled common fixed point theorems under weak contractions in cone metric type spaces, Thai. J. of Math. 12(1)(2014) 1-14.

[23] S.Radenovic, Z. Kadelburg, Quasi-contractions on symmetric and cone symmetric spaces. Banach J. Math. Anal. 5 (1) (2011) 38–50.

[24] S. Radenovic, Remarks on some recent coupled coincidence point results in symmetric G-metric spaces, Journal of Operators, Volume 2013, Article ID 290525, 8 pages.

[25] S. Radenovic, Remarks on some coupled coincidence point results in partially ordered metric spaces, Arab. J. Math. Sci., in press.

[26] S. Radenović, S. Pantelić, P. Salimi and J. Vujaković, *A note on some tripled coincidence point results in G-metric spaces*, *International J. of Math. Sci. Engg. Appls.*, Vol. 6, No. VI, (2012), pp. 23–38.

[27] H. Rahimi, G.Soleimani Rad, Fixed point theory in various spaces, Lambert Academic Publishing, Germany, 2013. 3.

[28] A. Rodlan, J. Martinez-Moreno, C. Rodlan, E. Karapinar, Some remarks on multidimensional fixed point theorems, Fixed Point Theory. 15 (2) (2014) 545-558.

[29] B. Rzepecki, On fixed point theorems of Maia type, Publications de lInstitut Mathematique, vol. 28(42), pp. 179-186, 1980.

[30] B. Samet, E. Karapinar, H. Aydi, V.C. Rajic, Discussion on some coupled fixed point theorems, Fixed Point Theory Appl. 2013, 2013:50.

[31] G. Soleimani Rad, H. Aydi, P. Kumam, H. Rahimi, Common tripled fixed point results in cone metric type spaces, Rend. Circ. Mat. Palermo, 2014, DOI 10.1007/s12215-014-0158-6.

[32] G. Soleimani Rad, S. Shukla, H. Rahimi, Some relations between n-tuple fixed point and fixed point results, Revista de la Real Academia de Ciencias Exactas,Fisicas y Naturales. Serie A. Matematicas. (RACSAM). 2014, In press, DOI. 10.1007/s13398-014-0196-0.

[33] H. Rahimi, P. Vetro, G. Soleimani Rad, Some common fixed point results for weakly compatible mappings in cone metric type space, Miskolc Math. Notes. 14 (1) (2013) 233-243.