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# Derivations in semiprime rings and Banach algebras

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**Abstract.** Let $R$ be a 2-torsion free semiprime ring with extended centroid $C$, $U$ the Utumi quotient ring of $R$ and $m, n > 0$ are fixed integers. We show that if $R$ admits derivation $d$ such that $b[[d(x), x]_n, [y, d(y)]_m] = 0$ for all $x, y \in R$ where $0 \neq b \in R$, then there exists a central idempotent element $e$ of $U$ such that $eU$ is commutative ring and $d$ induce a zero derivation on $(1 - e)U$. We also obtain some related result in case $R$ is a non-commutative Banach algebra and $d$ continuous or spectrally bounded.



**Keywords:** prime ring, semiprime ring, derivation, Utumi quotient ring, Banach Algebra.

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## 1. Introduction

In all that follows, unless stated otherwise, $R$ will be an associative ring, $Z(R)$ the center of $R$, $Q$ its Martindale quotient ring and $U$ its Utumi quotient ring. The center of $U$, denoted by $C$, is called the extended centroid of $R$ (we refer the reader to [1] for these objects). By a Banach algebra we shall mean complex normed algebra $A$ whose underlying vector space is Banach algebra. The radical Jacobson $\mathrm{rad}(A)$ of $A$ is the intersection of all primitive ideals. If the Jacobson radical reduces to the zero element, $A$ is called semi-simple.

An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. Also if $(x_i)_{i\in\mathbb{N}}$ is a squence of elements of $R$ and $k$ is a positive integer, we define $[x_1, \ldots, x_{k+1}]$ inductively as follows:

$$[x_1, x_2] = x_1x_2 - x_2x_1 \qquad , \qquad [x_1, \ldots, x_k, x_{k+1}] = [[x_1, \ldots, x_k], x_{k+1}].$$

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Let us introduce the background of our investigation. The classical result of Singer and Werner in [16] says that any continuous derivation on commutative Banach algebra has the range in the Jacobson radical of the algebra. Singer and Werner also formulated the conjecture that the continuity assumption can be removed. In [17], Thomas verified has conjecture. Of course the the same result of Singer and Werner does not hold in non-commutative Banach algebras. Hence in this context a very interesting question is how to obtain non-commutative version of Singer-Werner theorem. A first answer to this problem has been obtained by Sinclair in [15]. He proved that every continuous derivation of Banach algebra leaves primitive ideals of algebra invariant. Since then many authors obtained more information about derivations satisfying certain suitable conditions in Banach algebras. In [11], Mathieu and Murphy proved the result that if $d$ is a continuous derivation on an arbitrary Banach algebra such that $[d(x), x] \in Z(A)$ for all $x \in A$, then $d$ maps into the radical. Recently in [14], Park proved that if $d$ is a linear continuous derivation of a non-commutative Banach algebra $R$ such that $[[d(x), x], d(x)] \in \mathrm{rad}(A)$ for all $x \in A$, then $d(A) \subseteq \mathrm{rad}(A)$.

In the present article, our main purpose is to give genaralization of the above results from the commutator type to the Engle condition. More precisely, we here continue this line of investigation by examining what happens a semiprime ring $R$ (or an algebra $A$) satisfying the differential identity $b[[d(x), x]_n, [y, d(y)]_m] = 0$.
The following result is useful tools needed in the proof of main results.

**Lemma 1.1** *(see [4, Theorem 2])*. Let $R$ be a prime ring and $I$ a non-zero ideal of $R$. Then $I$, $R$ and $Q$ satisfy the same generalized polynomial identities with coefficient in $Q$.

**Theorem 1.2** *( Kharchenko Theorem [7])*. Let $R$ be a prime ring, $d$ a nonzero derivation of $R$ and $I$ a nonzero ideal of $R$. If $I$ satisfies the differential identity

$$f(r_1, r_2, \ldots, r_n, d(r_1), d(r_2), \ldots, d(r_n)) = 0,$$

for any $r_1, r_2, \ldots, r_n \in I$, then one of the following holds:

(i) $I$ satisfies the generalized polynomial identity

$$f(r_1, r_2, \ldots, r_n, x_1, x_2, \ldots, x_n) = 0,$$

(ii) $d$ is $Q$-inner, that is, for some $q \in Q$, $d(x) = [q, x]$ and $I$ satisfies the generalized polynomial identity

$$f(r_1, r_2, \ldots, r_n, [q, r_1], [q, r_2], \ldots, [q, r_n]) = 0.$$

## 2. Main Results

We prove the following result regarding the semiprime rings.

**Theorem 2.1** Let $R$ be a 2-torsion free semiprime ring, $0 \neq b \in R$ and $n, m > 0$ are fixed integers. If $R$ admits the derivation $d$ such that $b[[d(x), x]_n, [y, d(y)]_m] = 0$ for all $x, y \in R$, then there exists idempotent $e \in U$ such that $eU$ is a commutative ring and $d$ induce a zero derivation on $(1 - e)U$.

We prove the following results regarding the continuous derivations on non-commutative Banach algebras.

**Theorem 2.2** Let $R$ be a non-commutative Banach algebra, $d$ a continuous derivation of $R$ and $0 \neq b \in R$. If $b[[d(x), x]_n, [y, d(y)]_m] \in \mathrm{rad}(A)$ for all $x, y \in A$, where $m, n > 0$, then $d(A) \subseteq \mathrm{rad}(A)$.

## 3. Proof of main results

We establish the following technical result required in the proof of Theorem 2.1.

**Theorem 3.1** Let $R$ be a prime ring of $\mathrm{char}(R) \neq 2$, $m, n > 0$ and $0 \neq b \in R$. If $R$ admits the derivation $d$ such that $b[[d(x), x]_n, [y, d(y)]_m] = 0$ for all $x, y \in R$, then $R$ is commutative or $d = 0$.

**Proof.** We devide the proof in two cases.

*case (i): d is not a Q-inner derivation.*
Applying Theorem 1.2, for any $x, y, z, s \in R$, we obtain $b[[z, x]_n, [s, y]_m] = 0$. This is a polynomial identity and hence there exists a field $F$ such that $R \subseteq M_k(F)$ with $k > 1$ and $R, M_k(F)$ satisfy the same polynomial identity [8]. Now putting $z = e_{ij}, x = e_{ii}, s = e_{ji}, y = e_{ii}$ for any $i \neq j$, we have

$$0 = b[[z, x]_n, [s, y]_m] = b(-1)^n(e_{ii} + (-1)e_{jj})$$

implies $b = 0$, which is a contradiction.

*case (ii): d is a Q-inner derivation.*
Thus there exists an element $a \in U$ such that $d(x) = [a, x]$ for all $x \in R$. By Lemma 1.1, $U$ and $R$ satisfy the same generalized polynomial identities , hence for any $x, y \in Q$ we have $b[[a, x]_{n+1}, [y, [a, y]]_m] = 0$. Also since $U$ remains prime by the primeness of $R$, replacing $R$ by $Q$ we may assume that $b \in R$ and the extended centroid of $R$ is just the center of $R$. Note that $R$ is a centrally closed prime $C$-algebra in the present situation [5]. If $R$ is commutative, we have nothing to prove. So, let $R$ be non-commutative. Therefore $R$ satisfies a nontrivial (GPI). Since $R$ is a centrally closed prime $C$-algebra, by Martindale's Theorem [10], $R$ is a strongly primitive ring. Let ${}_RV$ be a faithful irreducible left $R$-module with commuting ring $D = \mathrm{End}({}_RV)$. By the Density Theorem, $R$ acts densely on $V_D$. For given any $v \in V$ we claim that $v$ and $av$ are $D$-dependent. Assume first that $bv \neq 0$. Suppose on the contrary that $v$ and $av$ are $D$-independent.
If $a^2v \in \mathrm{span}\{v, av\}$, then $a^2v = v\alpha + av\beta$ for some $\alpha, \beta \in D$. By density of $R$ in $\mathrm{End}(V_D)$ there exist two elements $x$ and $y$ in $R$ such that $xv = v$, $xav = 0$, $yv = 0$ and $yav = v$. Then $0 = b[[a, x]_{n+1}, [y, [a, y]]_m]v = (-2)^m bv$.
If $a^2v \notin \mathrm{span}\{v, av\}$, then $\{v, av, a^2v\}$ are all $D$-independent. Then by density of $R$ in $\mathrm{End}(V_D)$ there exist two elements $x$ and $y$ in $R$ such that $xv = v$, $xav = 0$, $xa^2v = 0$, $yv = 0$, $yav = 0$ and $ya^2v = 0$. Therefore again we have

$$0 = b[[a, x]_{n+1}, [y, [a, y]]_m]v = (-2)^m bv.$$

Now since $\mathrm{char}R \neq 2$, we get $bv = 0$ a contradiction. Thus $v$ and $av$ are D-dependent as claimed. Assume next that $bv = 0$. Since $b \neq 0$, we have $bw \neq 0$ for some $w \in V$. Then $b(v+w) = bw \neq 0$. Applying the first situation we have $aw = w\alpha$ and $a(v+w) = (v+w)\beta$. for some $\alpha, \beta \in D$. But $v$ and $w$ are clearly $D$-independent, and so there exist two elements $x$ and $y$ in $R$ such that $xw = w, xv = 0, yw = v, yv = 0$. Then

$$0 = b[[a, x]_{n+1}, [y, [a, y]]_m] = (-1)^{(n+1)} 2^m a(\beta - \alpha)^2 w,$$

which implies $\alpha = \beta$ and hence $av = v\alpha$ as claimed. From the above we have proved that $av = v\alpha(v)$ for all $v \in V$, where $\alpha(v) \in D$ depends on $v \in V$. In fact, it is easy to check that $\alpha(v)$ is independent of the choice of $v \in V$. That is, there exist $\delta \in D$ such that $av = v\delta$ for all $v \in V$. We claim $\delta \in Z(D)$, the center of $D$. Indeed, if $\beta \in D$, then $a(v\beta) = (v\beta)\delta = v(\beta\delta)$, and the other hand

$$a(v\beta) = (av)\beta = (v\delta)\beta = v(\delta\beta).$$

Therefore $v(\beta\delta - \delta\beta) = 0$ so $\beta\delta = \delta\beta$, which implies $\delta \in Z(D)$. Thus $a \in Z(R)$ and hence $d = 0$, as we wanted. ∎

In that follows, let $R$ be a semiprime ring, $U$ Utumi quotient ring of $R$. We note that $U$ is orthogonally complete. We refer the reader to [1, Chapter 3] for the definitions and the related properties of this objects. By using the method of orthogonal completion, initiated by Beidar see [1, Chapter 3] we can easily generalize Theorem 3.1 to Theorem 2.2.
For the proof of Theorem 2.1 we need the following two results, which can be found in [1].

**Lemma 3.2** [1, Proposition 2.5.1] Any derivation $d$ of a semiprime ring $R$ can be extended uniquely to a derivation of $U$ (we shall let $d$ also denote its extension to $U$).

**Lemma 3.3** [1, Theorem 3.2.18]. Let $R$ be an orthogonally complete $\Omega$-$\Delta$-ring with extended centroid $C$, $\Psi_i(x_1, x_2, \ldots, x_n)$ Horn formulas of signature $\Omega$-$\Delta$, $i = 1, 2, \ldots$ and $\Phi(y_1, y_2, \ldots, y_m)$ a Hereditary first order formula such that $\neg\Phi$ is a Horn formula. Further, let $\vec{a} = (a_1, a_2, \ldots, a_n) \in R^{(n)}$, $\vec{c} = (c_1, c_2, \ldots, c_m) \in R^{(m)}$. Suppose $R \models \Phi(\vec{c})$ and for every $M \in \mathrm{spec}(B)$ there exists a natural number $i = i(M) > 0$ such that $R_M \models \Phi(\phi_M(\vec{c})) \Longrightarrow \Psi_i(\phi_M(\vec{a}))$, where $\phi_M : R \to R_M = R/RM$ is the canonical projection. Then there exists a natural number $k > 0$ and pairwise orthogonal idempotents $e_1, e_2, \ldots, e_k \in B$ such that $e_1 + e_2 + \ldots + e_k = 1$ and $e_iR \models \Psi_i(e_i\vec{a})$ for all $e_i \neq 0$.

Now we can prove Theorem 2.1.

*Proof of Theorem 2.1.* By Lemma 3.2 the derivation $d$ can be extended uniquely to a derivation $d : U \to U$. According to [1, Remark 3.1.16] we know that $U$ is an orthogonally complete $\Omega$-$\Delta$-ring where $\Omega = \{o, +, -, \cdot, d\}$. Since $U$ and $R$ satisfy the same differential identities [9], we obtain that $b[[d(x), x]_n, [y, d(y)]_m] = 0$ for all $x, y \in U$ where $m, n$ are fixed positive integers. Consider the formulas:

$$\Phi = (\forall x)(\forall y)\|b[[d(x), x]_n, [y, d(y)]_m] = 0\|,$$

$$\Psi_1 = (\forall x)\|d(x) = 0\|,$$

$$\Psi_2 = (\forall x)(\forall y)\|xy = yx\|.$$

Using Theorem 3.1, we can easily check that all conditions of Lemma 3.3 are fulfilled. Hence there exist two orthogonal idempotent $e_1$ and $e_2$ such that $e_1 + e_2 = 1$, then $e_iU \models \Psi_i$, $i = 1, 2$. The proof is complete. □
The following results are useful tools needed in the proof of Theorem 2.2.

***Remark 1*** *(see [15]). Any continuous derivation of Banach algebra leaves the primitive ideals invariant.*

***Remark 2*** *(see [16]). Any continuous linear derivation on a commutative Banach algebra maps the algebra into its radical.*

***Remark 3*** *(see [6]). Any linear derivation on semi-simple Banach algebra is continuous.*

Now we can prove Theorem 2.2.

*Proof of Theorem 2.2.* Under the assumption that $d$ is continuous. As we have already remarked in Remark *1*, we may assume that for any primitive ideal $P$ of $A$, $d(P) \subseteq P$. Denote $\frac{A}{P} = \overline{A}$ for any primitive ideals $P$. Hence we may introduce the derivation $d_P : \overline{A} \to \overline{A}$ by $d_P(\overline{x}) = d_p(x + P) = d(x) + P$ for all $x \in A$ and $\overline{x} = x + P$. Moreover by $b[[d(x), x]_n, [y, d(y)]_m] \in \mathrm{rad}(A)$ for all $x, y \in A$, it follows

$$b[[d_p(\overline{x}), \overline{x}]_n, [\overline{y}, d_p(\overline{y})]_m] = \overline{0},$$

for all $\overline{x}, \overline{y} \in \overline{A}$. Since $\overline{A}$ is primitive, a fortiori it is prime. Thus by Theorem 3.1, we get that either $\overline{A}$ is commutative, i.e., $[A, A] \subseteq P$ or $d = \overline{0}$.

Now we assume that $P$ is primitive ideal such that $\overline{A}$ is commutative. By Remarks *2*, any continuous linear derivation on a commutative Banach algebra maps the algebra into the radical. Furthermore by a result of Remark *3*, any linear derivationon a semi-simple banach algebra is continuoue. We know that there are no non-zero linear continuous derivations on commutative semisimple Banach algebras. Therefore, $d = \overline{0}$ in $\overline{A}$. Here in any case we get $d(A) \subseteq P$ for all primitive ideal $P$ of $A$. Since $\mathrm{rad}(A)$ in the intersection of all primitive ideals, we get $d(A) \subseteq \mathrm{rad}(A)$, and we get the required conclusion. □

In the special case when $A$ is a semi-simple Banach algebra we have the following.

**Corollary 3.4** Let $A$ be a non-commutative semi-simple Banach algebra, $d$ a continuous derivation of $A$ and $0 \neq b \in A$. If $b[[d(x), x]_n, [y, d(y)]_m] = 0$ for all $x, y \in A$, where $m, n > 0$, then $d = 0$.

**Proof.** By Remark *3*, derivation $d$ is continuous. Now we use the fact that $\mathrm{rad}(A) = 0$, since $A$ is a semi-simple. ■

***Remark 4*** *The last result of this paper has the same flavor of Theorem 2.2. We replace assumption concerning the continuty of the derivation d by the one that $\delta$ is spectrally bounded. Here we denote by $G(A)$ the set of invertible elements of A. The spectrum of an element x is the subset given by $\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda e \notin G(A)\}$ where e denotes the unity of A. The spectral radius $r(x)$ of an element xis defined as $r(x) = sup\{|\lambda| : \lambda \in \sigma(x)\}$, provided that $\sigma(x)$ is not empty. A linear mapping $f : A \to A$ is called spectrally bounded if there exists a constant $\alpha \geqslant 0$ such that $r(f(x)) \leqslant \alpha r(x)$ for all $x \in A$. In [3], Bresar and Mathieu proved that on a unital Banach algebra every spectrally bounded derivation maps the algebra into the radical. Moreover they proved that, every spectrally bounded derivation leaves each primitive ideal invariant.*

Finally, we prove a result relating the spectrally bounded derivations on non-commutative Banach algebras

**Theorem 3.5** Let $A$ be a non-commutative Banach algebra, $\delta$ a spectrally bounded derivation of $A$ and $0 \neq b \in A$. If $b[[\delta(x), x]_n, [y, \delta(y)]_m] \in \mathrm{rad}(A)$ for all $x, y \in A$, where $m, n > 0$, then $[A, A] \subseteq \mathrm{rad}(A)$ and $\delta(A) \subseteq \mathrm{rad}(A)$.

**Proof.** By Remark *4*, $d$ leaves each primitive ideals invariant, it follows that for any primitive ideal $P$ of $A$ we may introduce the derivation $d_P : \overline{A} \to \overline{A}$ by $d_P(\bar{x}) = d(x) + P$

for all $x \in A$. Now as above, by $b[[\delta(x), x]_n, [y, \delta(y)]_m] \in \mathrm{rad}(A)$ for all $x, y \in A$, it follows $b[[d_p(\overline{x}), \overline{x}]_n, [\overline{y}, d_p(\overline{y})]_m] = \overline{0}$ for all $\overline{x}, \overline{y} \in \overline{A}$. By Theorem 3.1 one has $\overline{A}$ is commutative, i.e., $[A, A] \subseteq P$ or $\delta = \overline{0}$, i.e., $\delta(A) \subseteq P$. In the special $[A, A] \subseteq \mathrm{rad}(A)$ or $\delta(A) \subseteq \mathrm{rad}(A)$. ■

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