

The method of fundamental solutions for transient heat conduction in functionally graded materials: some special cases

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Abstract. In this paper, the Method of Fundamental Solutions (MFS) is extended to solve some special cases of the problem of transient heat conduction in functionally graded materials. First, the problem is transformed to a heat equation with constant coefficients using a suitable new transformation and then the MFS together with the Tikhonov regularization method is used to solve the resulting equation.

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1. Introduction

Heat conduction in many different kinds of materials, such as structured materials, functionally graded materials (FGMs), etc, appears in numerous applications in engineering and science. It is therefore important to have efficient procedures for calculating the heat flow in conducting FGMs. Due to its importance, there are many different approaches. In [32] an efficient and simple higher order shear and normal deformation theory is presented for FGM plates. A report on finding of an optimal layout of FGMs towards indentation resistance can be found in [29]. Also a stochastic perturbation-based finite element for buckling statistics of FGMs with uncertain material properties in thermal environments is investigated in [28]. Mainly, analytical methods have been proposed based on different expansion techniques and an early work in this direction is [30]; for some more recent

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results see [1, 4, 5, 8, 11, 14, 24, 31]. However, in more complex situations it becomes too cumbersome to try to express the solution analytically. Instead numerical approximations based, for example, on finite-difference, boundary element or finite element methods [22, 27] are utilized. In this study, we shall employ the so-called method of fundamental solutions (MFS) to approximate the temperature field in FGMs.

The MFS, first introduced by Kupradze and Aleksidze [20] approximates the solution of the problem by linear combinations of fundamental solutions of the governing differential operator. Therefore, it is an inherently meshless, integration-free technique for solving partial differential equations, so it has been used extensively for solving various types of partial differential equations. For instance, the solution for potential problems by Mathon and Johnston [23], the exterior Dirichlet problem in acoustics by Kress and Mohsen [18], the biharmonic problems by Karageorghis and Fairweather [17]. More recently, the MFS has been successfully applied to approximate the solutions of non-homogenous problems [2]. The details can be found in [7, 10]. Also in [25] the MFS is used to obtain approximate solutions of the inverse space-dependent heat source problem. In a recent work [15], as suggested by Kupradze [19], the authors proposed and investigated a MFS for transient heat conduction where the sources are placed outside the solution domain. In [16] the above mentioned work has been extended to the case of heat conduction in one-dimensional layered materials.

In this study, we extend the MFS for some special cases of heat conduction in one-dimensional FGMs, where the thermal diffusivity is a function of space-variable. The mathematical formulation is given in section 2. In section 3, the problem is transformed to a heat equation with constant coefficients using a new transformation. Also the initial and boundary conditions are expressed in terms of new variables. In section 4, considering the geometry of the new domain, the MFS is efficiently utilized to solve the resulting equations. In section 5, the Tikhonov regularization method is introduced for solving the ill-conditioned system of equations obtained in section 4. Finally, we perform numerical investigations of the proposed MFS in section 6. Three different examples are presented in which different boundary conditions are chosen. These examples show that accurate numerical approximations can be obtained with relatively few degrees of freedom.

2. Formulation of the problem

Under certain assumptions, such as that the conducting material is sufficiently large in two of its dimensions as compared to the third, it is reasonable to consider a one-dimensional model in the spatial direction. We assume that there is no heat generation within the material and therefore, we wish to find the temperature field u that satisfies the homogeneous one-dimensional heat equation, i.e.,

$$\frac{\partial u}{\partial t}(x, t) - \alpha(x) \frac{\partial^2 u}{\partial x^2}(x, t) = 0, \quad (x, t) \in (L_0, L_1) \times (0, T), \quad (1)$$

and the initial condition

$$u(x, 0) = \phi_0(x), \quad x \in (L_0, L_1). \quad (2)$$

Here, $\alpha(x) > 0$ is the thermal diffusivity and $k(x) > 0$ is the thermal conductivity which are related through $\alpha(x) = k(x)/C(x)$, where $C(x)$ is the heat capacity. Moreover, at the boundaries $\{L_0\} \times (0, T)$ and $\{L_1\} \times (0, T)$, either of the following boundary conditions

are specified:

Boundary temperature:

$$u(L_n, t) = g_n(t), \quad n = 0, 1, \quad (3)$$

Heat flux:

$$k(L_n) \frac{\partial u}{\partial x}(L_n, t) = q_n(t), \quad n = 0, 1, \quad (4)$$

Convection:

$$h(L_n)u(L_n, t) + k(L_n) \frac{\partial u}{\partial x}(L_n, t) = \xi_n(t), \quad n = 0, 1, \quad (5)$$

where $h(x)$ is the heat transfer coefficient.

3. Transformation of the problem

In this section, we introduce a new transformation to simplify the problem. Using the following change of variables:

$$\begin{cases} z(x, t) = -\frac{1}{c} \ln |cx + d| + ct, \\ w(z, t) = u(x(z, t), t), \end{cases} \quad (6)$$

we can transform the equation

$$u_t - (cx + d)^2 u_{xx} = 0, \quad (7)$$

into

$$w_t - w_{zz} = 0. \quad (8)$$

Moreover, the initial and boundary conditions (2), (3), (4) and (5) transform to the following conditions, respectively:

$$w(z, 0) = \phi_0\left(\frac{\exp(-cz) - d}{c}\right), \quad z \in \left(-\frac{1}{c} \ln |cL_1 + d|, -\frac{1}{c} \ln |cL_0 + d|\right), \quad (9)$$

$$w\left(-\frac{1}{c} \ln |cL_n + d| + ct, t\right) = g_n(t), \quad n = 0, 1, \quad (10)$$

$$\frac{-k(L_n)}{cL_n + d} \frac{\partial w}{\partial z}\left(-\frac{1}{c} \ln |cL_n + d| + ct, t\right) = q_n(t), \quad n = 0, 1, \quad (11)$$

$$h(L_n)w\left(-\frac{1}{c} \ln |cL_n + d| + ct, t\right) - \frac{k(L_n)}{cL_n + d} \frac{\partial w}{\partial z}\left(-\frac{1}{c} \ln |cL_n + d| + ct, t\right)$$

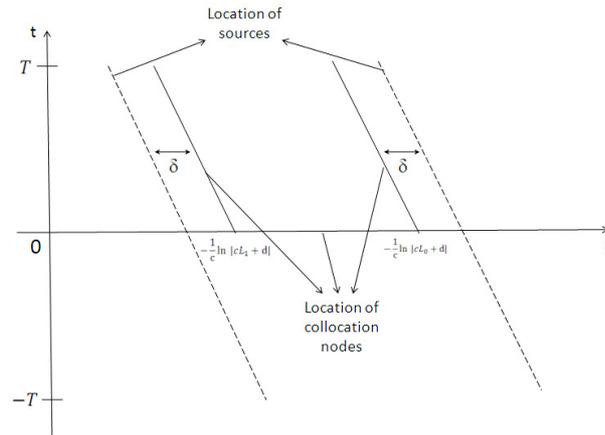


Figure 1. Geometry of the transformed domain and location of sources and collocation nodes.

$$= \xi_n(t), \quad n = 0, 1. \quad (12)$$

Note that the rectangular region $(L_0, L_1) \times (0, T)$ in xt plane maps into a parallelogram region in zt plane (see Fig. 1.).

In the resulting equation, the unknown function $w(z, t)$ can be obtained by the method of fundamental solutions. Finally, we can obtain $u(x, t)$ by transforming back into the original variables.

4. The method of fundamental solutions

The fundamental solution to the one-dimensional heat equation (8) is given by

$$G(z, t; y, \tau) = \frac{H(t - \tau)}{\sqrt{4\pi(t - \tau)}} e^{-(z-y)^2/(4(t-\tau))}, \quad (13)$$

where H is the Heaviside function which is introduced to emphasize that the fundamental solution is zero for $t \leq \tau$. It is straightforward to verify that, as a function of z and t , $G(z, t; y, \tau)$ satisfies (8) for any $(z, t) \neq (y, \tau)$. We shall approximate the solution to the heat equation (8) by a linear combination of fundamental solutions of the form

$$w(z, t) = \sum_{j=1}^M c_j G(z, t; y_j, \tau_j). \quad (14)$$

Here, the source points (y_j, τ_j) , for $j = 1, \dots, M$, are located outside the solution domain in the following way:

$$(y_j, \tau_j) = \begin{cases} (-\frac{1}{c} \ln |cL_1 + d| + c\tau_j - \delta, \tau_j), & j = 1, \dots, 2M_1, \\ (-\frac{1}{c} \ln |cL_0 + d| + c\tau_j + \delta, \tau_{j-2M_1}), & j = 2M_1 + 1, \dots, 4M_1, \end{cases} \quad (15)$$

where $M = 4M_1$ and $\tau_j = (2j - 1 - 2M_1)T/2M_1$ for $j = 1, \dots, 2M_1$, see Fig. 1. This selection of source points is motivated from [16]. We now discuss how to determine the coefficients c_j . We shall collocate the initial and boundary conditions at certain

collocation points. Put $t_i = Ti/p_t$ for $i = 0, \dots, p_t$ and $x_i = (L_0 + (L_1 - L_0)i/(1 + p_x))$ for $i = 1, \dots, p_x$ and $z_i = -\frac{1}{c} \ln |cx_i + d|$ for $i = 1, \dots, p_x$. In the case of the Dirichlet boundary condition (10), we collocate (9) and (10) as follows:

$$w(z_i, 0) = \phi_0\left(\frac{\exp(-cz_i) - d}{c}\right), \quad i = 1, \dots, p_x, \tag{16}$$

$$w\left(-\frac{1}{c} \ln |cL_n + d| + ct_i, t_i\right) = g_n(t_i), \quad i = 0, \dots, p_t, \quad n = 0, 1. \tag{17}$$

Eq. (17) can easily be adjusted to the other two boundary conditions (11) or (12). In total, the Eqs. (16) and (17) form a system of $2p_t + 2 + p_x$ equations in M unknowns. In order to obtain a unique solution we take $M_1 = p_t$ in (15) and $p_x = 2p_t - 2$. We can write (16) and (17) in a matrix form

$$Ac = \mathbf{h} \tag{18}$$

with the obvious notation. We point out that although the direct problem (1)-(3) is well-posed, the resulting MFS matrix A is ill-conditioned, see [3, 26]. Therefore, a straightforward inversion of the system of Eqs. (18) can produce unstable results. In order to stabilize the solution, we use Tikhonov's regularization method described in the next section for solving (18).

5. Tikhonov's regularization method

Most standard numerical methods cannot achieve good accuracy in solving the matrix equation (18) due to the bad condition number of the matrix A . In fact, the condition number of matrix A increases dramatically with respect to the total number of collocation points. Several regularization methods have been developed for solving these kinds of ill-conditioned problems [13]. In our computation we adapt the Tikhonov regularization [6] to solve the matrix equation (18). The Tikhonov regularized solution for Eq. (18) is defined as the solution of the following least squares problem:

$$\min_{\mathbf{c}} \{ \|\mathbf{Ac} - \tilde{\mathbf{h}}\|^2 + \alpha^2 \|\mathbf{c}\|^2 \}, \tag{19}$$

where $\|\cdot\|$ denotes the Euclidean norm and α is called the regularization parameter. We use the generalized cross-validation (GCV) criterion to choose the regularization parameter α . The GCV criterion is a very popular and successful method for choosing the regularization parameter [13]. The GCV method determines the optimal regularization parameter by minimizing the following GCV function:

$$G(\alpha) = \frac{\|A\tilde{\mathbf{c}} - \tilde{\mathbf{h}}\|^2}{(\text{trace}(I_M - AA^I))^2}, \tag{20}$$

where $A^I = (A^{tr}A + \alpha^2I)^{-1}A^{tr}$ is a matrix which produces the regularized solution $\tilde{\mathbf{c}}_\alpha$ when multiplied with the righthand side $\tilde{\mathbf{h}}$, i.e., $\tilde{\mathbf{c}}_\alpha = A^I\tilde{\mathbf{h}}$. In our computation, we used the Matlab code developed by Hansen [12] for solving the discrete ill-conditioned system of equations (18).

6. Numerical experiments

In this section, we apply the MFS with Tikhonov regularization outlined in previous sections to heat conduction in composite materials. We choose different boundary conditions in the various examples presented. To test the accuracy of the approximate solution, we use the root mean square error (RMS) and the relative root mean square error (RES) defined as

$$RMS(u) = \sqrt{\frac{1}{N_t} \sum_{i=1}^{N_t} (u(x_i, t_i) - u^*(x_i, t_i))^2}, \quad (21)$$

$$RES(u) = \frac{\sqrt{\sum_{i=1}^{N_t} (u(x_i, t_i) - u^*(x_i, t_i))^2}}{\sqrt{\sum_{i=1}^{N_t} u(x_i, t_i)^2}}, \quad (22)$$

where N_t is the total number of test points distributed in the domain $[L_0, L_1] \times [0, T]$. Unless otherwise specified, we take $N_t = 21^2$. $u(x_i, t_i)$ and $u^*(x_i, t_i)$ are, respectively, the exact and approximate values at these points.

Example 1. We shall construct approximations with Dirichlet boundary conditions, i.e., we consider (1)-(3), where the data are generated from the analytical temperature field

$$u(x, t) = \ln|x + 5| + x - t, \quad (23)$$

with the thermal diffusivity $\alpha(x) = (x + 5)^2$. Here we consider the domain $[L_0, L_1] \times [0, T] = [0, 1] \times [0, .5]$. RMS(u), RES(u) and Cond(A) for various values of δ have been shown in table 1 taking $p_t = 12$. RMS(u), RES(u) and Cond(A) for various values of p_t have been shown in table 2 taking $\delta = .7$. The accuracy of the numerical solution with respect to the parameter δ with $p_t = 12$ has been shown in Fig. 2.

Table 1. $RMS(u)$, $RES(u)$, Cond(A) for various values of δ , $T = .5$, $p_t = 12$ for Example 1.

δ	$RMS(u)$	$RES(u)$	$Cond(A)$
0.1	$6.4887E - 02$	$3.2378E - 02$	$1.4530E + 18$
0.2	$1.7628E - 02$	$8.7966E - 03$	$2.7170E + 18$
0.3	$1.1593E - 02$	$5.7852E - 03$	$2.5270E + 18$
0.4	$4.4758E - 03$	$2.2334E - 03$	$1.9700E + 18$
0.5	$2.1074E - 02$	$1.0515E - 02$	$4.5030E + 18$
0.6	$1.7659E - 03$	$8.8118E - 03$	$3.3308E + 19$
0.7	$7.1400E - 04$	$3.5628E - 04$	$5.5755E + 19$
0.8	$1.7589E - 04$	$8.7768E - 05$	$7.0282E + 19$

Table 2. $RMS(u)$, $RES(u)$, Cond(A) for various values of p_t , $T = .5$, $\delta = .7$ for

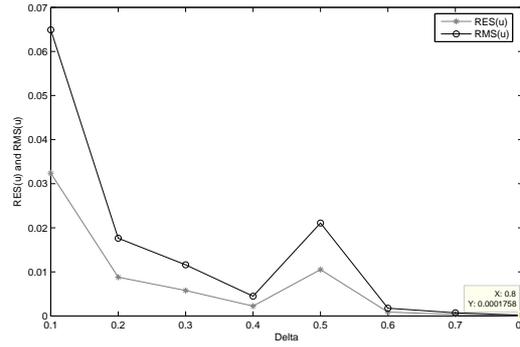


Figure 2. The accuracy of the numerical solution for Example 1 with $p_t = 12$, $T = .5$ with respect to the parameter δ .

Example 1.

Table 2

Pt	$RMS(u)$	$RES(u)$	$Cond(A)$
10	$1.7378E - 01$	$8.6718E - 02$	$4.6360E + 18$
12	$1.7659E - 03$	$8.8118E - 04$	$3.3308E + 19$
14	$1.2923E - 03$	$5.9499E - 04$	$1.8032E + 19$
16	$6.5184E - 04$	$3.2526E - 04$	$5.2106E + 19$

Tables 1 and 2 together with Fig. 2 show that the MFS is an accurate and reliable numerical technique for the solution of the problem of transient heat conduction in FGMs with Dirichlet boundary conditions. Also they show that:

- 1- Errors are generally decreasing with respect to both increasing δ and increasing number of source points.
- 2- The condition number generally increases with both increasing δ and increasing number of source points.

Example 2. We shall construct approximations with fluent boundary conditions, i.e., we consider (1)-(2) and (4) where the data are generated from the analytical temperature field

$$u(x, t) = x^2 e^{2t}, \tag{24}$$

with the thermal diffusivity $\alpha(x) = x^2$ and thermal conductivity $k(x) = \sin(x)$. Here we consider the domain $[L_0, L_1] \times [0, T] = [2, 3] \times [0, .5]$. $RMS(u)$, $RES(u)$ and $Cond(A)$ for various values of δ have been shown in table 3 taking $p_t = 12$. $RMS(u)$, $RES(u)$ and $Cond(A)$ for various values of p_t have been shown in table 4 taking $\delta = .8$. The accuracy of the numerical solution with respect to the parameter δ with $p_t = 12$ has been shown in Fig. 3.

Table 3. $RMS(u)$, $RES(u)$, $Cond(A)$ for various values of δ , $T = .5$, $p_t = 12$ for

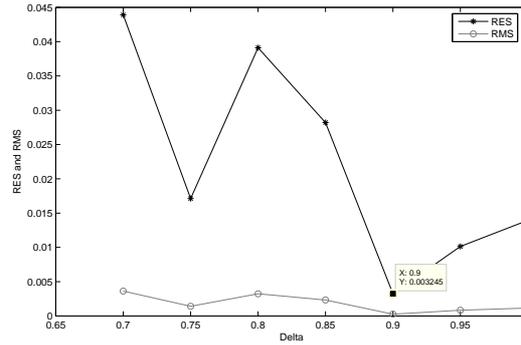


Figure 3. The accuracy of the numerical solution for Example 2 with $p_t = 12$, $T = .5$ with respect to the parameter δ .

Example 2.

Table 3

δ	$RMS(u)$	$RES(u)$	$Cond(A)$
0.70	$4.39060E - 02$	$3.6267E - 03$	$1.0160E + 19$
0.75	$1.71210E - 02$	$1.4142E - 03$	$1.5560E + 19$
0.80	$3.91050E - 02$	$3.2341E - 03$	$2.4710E + 20$
0.85	$2.81950E - 02$	$2.3284E - 03$	$6.3940E + 20$
0.90	$3.24500E - 03$	$2.6807E - 04$	$3.8930E + 20$
0.95	$1.01014E - 02$	$8.3762E - 04$	$9.5650E + 20$
1.00	$1.13930E - 02$	$1.1512E - 03$	$2.2687E + 21$

Table 4. $RMS(u)$, $RES(u)$, $Cond(A)$ for various values of p_t , $T = .5$, $\delta = .8$ for Example 2.

Table 4

Pt	$RMS(u)$	$RES(u)$	$Cond(A)$
06	$5.3868E - 01$	$4.4490E - 02$	$1.32600E + 11$
12	$3.5404E - 02$	$2.9241E - 03$	$7.87000E + 18$
14	$1.6869E - 02$	$1.3934E - 03$	$8.92200E + 19$
16	$4.5440E - 03$	$3.7539E - 04$	$3.02823E + 20$

Tables 3 and 4 together with Fig. 3 show that the MFS is an accurate and reliable numerical technique for the solution of the problem of transient heat conduction in FGMs with fluent boundary conditions. Also they show that:

- 1- Errors are decreasing with respect to increasing number of source points.
- 2- Errors are generally decreasing with respect to increasing δ but the behaviour is irregular.
- 3- The condition number generally increases with both increasing δ and increasing number of source points.

Example 3. We shall construct approximations with convective boundary conditions,

i.e., we consider (1)-(2) and (5) where the data are generated from the analytical temperature field

$$u(x, t) = \frac{e^{8t}}{x + .5}, \tag{25}$$

with the thermal diffusivity $\alpha(x) = (2x + 1)^2$, thermal conductivity $k(x) = x$ and heat transfer coefficient $h(x) = x^3$. Here we consider the domain $[L_0, L_1] \times [0, T] = [2, 3] \times [0, .5]$. RMS(u), RES(u) and Cond(A) for various values of δ have been shown in table 5 taking $p_t = 12$. RMS(u), RES(u) and Cond(A) for various values of p_t have been shown in table 6 taking $\delta = .75$. The accuracy of the numerical solution with respect to the parameter δ with $p_t = 12$ has been shown in Fig. 4.

Table 5. *RMS(u)*, *RES(u)*, *Cond(A)* for various values of δ , $T = .5$, $p_t = 12$ for Example 3.

δ	<i>RMS(u)</i>	<i>RES(u)</i>	<i>Cond(A)</i>
0.65	$5.3758E - 02$	$7.5750E - 03$	$4.8240E + 20$
0.70	$2.9440E - 02$	$4.1494E - 03$	$2.5560E + 20$
0.75	$1.6608E - 02$	$2.3401E - 03$	$2.6840E + 21$
0.80	$1.3731E - 02$	$1.9347E - 03$	$1.7820E + 22$
0.85	$1.3440E - 02$	$1.8947E - 03$	$3.9732E + 22$
0.90	$3.4460E - 02$	$4.8558E - 03$	$3.9479E + 22$

Table 6. *RMS(u)*, *RES(u)*, *Cond(A)* for various values of p_t , $T = .5$, $\delta = .75$ for Example 3.

Pt	<i>RMS(u)</i>	<i>RES(u)</i>	<i>Cond(A)</i>
10	$8.3078E - 03$	$5.8959E - 02$	$1.3970E + 20$
12	$2.3402E - 03$	$1.6608E - 02$	$2.6840E + 21$
14	$1.6933E - 03$	$1.2017E - 02$	$4.3440E + 21$
16	$1.8814E - 03$	$1.3351E - 02$	$1.3069E + 22$

Tables 5 and 6 together with Fig. 4 show that the MFS is an accurate and reliable numerical technique for the solution of the problem of transient heat conduction in FGMs with convective boundary conditions. Also they show that:

- 1- Errors are generally decreasing with respect to increasing number of source points.
- 2- The behaviour of errors with respect to δ is completely irregular.
- 3- The condition number generally increases with increasing δ .
- 4- The condition number increases with increasing number of source points.

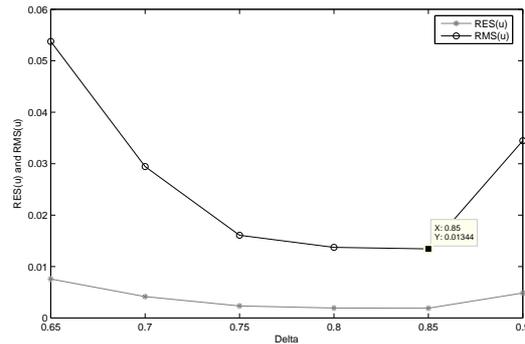


Figure 4. The accuracy of the numerical solution for Example 3 with $p_t = 12$, $T = .5$ with respect to the parameter δ .

7. Conclusions

In this paper, we have extended the MFS to solve some special cases of the problem of transient heat conduction in FGMs based on the Tikhonov regularization method with the GCV criterion. We successfully changed the equation to an equation with constant coefficients using a new transformation, and then applied the MFS technique to the resulting heat equation. Numerical results show that the MFS is an accurate and reliable numerical technique for the solution of the problem of transient heat conduction in FGMs. Also they show that the behaviour of errors with respect to δ is more irregular than the case of constant coefficient heat equation. Generally, errors are decreasing with respect to increasing number of source points.

There are several potential extensions of the present method. Firstly, the proposed scheme may be adapted to include wider classes of functions for $\alpha(x)$. Secondly, this method may be extended to higher dimensional problems.

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References

- [1] H. S. Carslaw, J. C. Jaeger, Conduction of heat in solids, 2nd ed. London: Oxford University Press; 1959.
- [2] C. S. Chen, The method of fundamental solutions for nonlinear thermal explosion, Commun. Numer. Methods Eng. 1995;11:675-81.
- [3] C. S. Chen, H. A. Cho, M. A. Golberg, Some comments on the ill-conditioning of the method of fundamental solutions. Eng. Anal. Boundary Elem. 2006;30:405-410.
- [4] F. de Monte, Transient heat conduction in one-dimensional composite slab. A natural approach. Int. J. Heat Mass Transf. 2000;43:3607-19.
- [5] F. de Monte, An analytical approach to the unsteady heat conduction processes in one-dimensional composite media. Int. J. Heat Mass Transf. 2002;26:1333-43.
- [6] H. W. Engl, M. Hanke, A. Neubauer, Regularization of inverse problems. Mathematics and its applications, vol. 357. Dordrecht: Kluwer Academic Publishers; 1996.
- [7] G. Fairweather, A. Karageorghis, The method of fundamental solutions for elliptic boundary value problems, Adv. Comput. Math. 1998;9(1-2):69-95.
- [8] Z. G. Feng, E. E. Michaelides, The use of modified Greens functions in unsteady heat transfer. Int. J. Heat Mass Transf. 1997;40:2997-3002.
- [9] A. Friedman, Partial differential equations of parabolic type. Englewood Cliffs, NJ: Prentice-Hall Inc.; 1964

- [10] M. A. Golberg, C. S. Chen, The method of fundamental solutions for potential, Helmholtz and diffusion problems. In: Boundary integral methods: numerical and mathematical aspects. Computational engineering, vol. 1. Boston, MA: WIT Press, Computational Mechanics Publications; 1999. p. 103-76.
- [11] A. Haji-Sheikh, J. V. Beck, Greens function partitioning in Galerkin-base integral solution of the diffusion equation. J. Heat Transf. 1990;112:28-34.
- [12] P. C. Hansen, Regularization tools: a Matlab package for analysis and solution of discrete ill-posed problems. Numer. Algorithms 1994;6:1-35.
- [13] P. C. Hansen, Rank-deficient and discrete ill-posed problems. Philadelphia: SIAM; 1998.
- [14] S. C. Huang, Y. P. Chang, Heat conduction in unsteady, periodic and steady states in laminated composites. J. Heat Transf. 1980;102:742-8.
- [15] B. T. Johansson, D. Lesnic, A method of fundamental solutions for transient heat conduction. Eng. Anal. Boundary Elem. 2008;32:697-703.
- [16] B. T. Johansson, D. Lesnic, A method of fundamental solutions for transient heat conduction in layered materials. Eng. Anal. Boundary Elem. 2009;33:1362-67.
- [17] A. Karageorghis, G. Fairweather, The method of fundamental solutions for the numerical solution of the biharmonic equation, J. Comput. Phys. 1987;69:434-59.
- [18] R. Kress, A. Mohsen, On the simulation source technique for exterior problems in acoustics, Math. Methods Appl. Sci. 1986;8:585-97.
- [19] V. D. Kupradze, A method for the approximate solution of limiting problems in mathematical physics. USSR Comput. Math. Math. Phys. 1964;4:199-205.
- [20] V. D. Kupradze, M. A. Aleksidze, The method of functional equations for the approximate solution of certain boundary value problems, USSR Comput. Math. Math. Phys. 1964;4:82-126.
- [21] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'ceva, Linear and quasilinear equations of parabolic type. Translational of mathematical monographs, vol. 23. Providence, RI: AMS; 1968.
- [22] D. Lesnic, The identification of piecewise homogeneous properties of rocks. In: Ingham DB, Wrobel LC, editors. Boundary integral formulations for inverse analysis. Southampton: Comput. Mech. Publ.; 1997. p. 237-57 [chapter 9].
- [23] R. Mathon, H. Johnston, The approximate solution of elliptic boundary-value problems by fundamental solutions, SIAM J. Numer. Anal. 1977;14(4):638-50.
- [24] M. D. Mikhailov, M. N. Özisik, M. D. Vulchanov, Diffusion in composite layers with automatic solution of the eigenvalue problem. Int. J. Heat Mass Transf. 1983;26:1131-41.
- [25] M. Nili Ahmadabadi, M. Arab, F. M. Maalek Ghaini, The method of fundamental solutions for the inverse space-dependent heat source problem. Eng. Anal. Boundary Elem. 2009;33:1231-35.
- [26] P. A. Ramachandran, Method of fundamental solutions: singular value decomposition analysis. Commun. Numer. Methods Eng. 2002;18:789-801.
- [27] J. N. Reddy, An introduction to the finite element method. New York: McGraw-Hill; 1984.
- [28] M. Talha, B. N. Singh, Stochastic perturbation-based finite element for buckling statistics of FGM plates with uncertain material properties in thermal environments, Composite Structures, Volume 108, February 2014, Pages 823-833, ISSN 0263-8223, <http://dx.doi.org/10.1016/j.compstruct.2013.10.013>.
- [29] Z. Tobias, K. Torsten, Functionally graded materials with a soft surface for improved indentation resistance: Layout and corresponding design principles, Computational Materials Science, Volume 86, 15 April 2014, Pages 88-92, ISSN 0927-0256, <http://dx.doi.org/10.1016/j.commatsci.2014.01.032>.
- [30] V. Vodicka, Wärmeleitung in geschichteten kugel -und zylinderkörpern. Schweiz Arch 1950;10:297-304.
- [31] Y. Yener, M. N. Özisik, On the solution of unsteady heat conduction in multi-region finite media with time-dependent heat transfer coefficient. In: Proceedings of the fifth international heat transfer conference, vol. 1, JSME, Tokyo; 1974. p. 188-92.
- [32] Zakaria Belabed, Mohammed Sid Ahmed Houari, Abdelouahed Tounsi, Mahmoud SR., Anwar Bg O., An efficient and simple higher order shear and normal deformation theory for functionally graded material (FGM) plates, Composites Part B: Engineering, Volume 60, April 2014, Pages 274-283, ISSN 1359-8368, <http://dx.doi.org/10.1016/j.compositesb.2013.12.057>.