Journal of
Linear and Topological Algebra
Vol. 04*, No.* 02*,* 2015*,* 87- 100

On the convergence of the homotopy analysis method to solve the system of partial differential equations

A. Fallahzadeh[a*], M. A. Fariborzi Araghi[b] and V. Fallahzadeh[c]

[a,b] *Department of Mathematics, Islamic Azad University, Central Tehran Branch, PO. Code 14168-94351, Iran;*
[c] *Department of Mathematics, Islamic Azad University, Arac Branch, Iran.*

Received 12 June 2015; Revised 5 September 2015; Accepted 20 October 2015.

Abstract. One of the efficient and powerful schemes to solve linear and nonlinear equations is homotopy analysis method (HAM). In this work, we obtain the approximate solution of a system of partial differential equations (PDEs) by means of HAM. For this purpose, we develop the concept of HAM for a system of PDEs as a matrix form. Then, we prove the convergence theorem and apply the proposed method to find the approximate solution of some systems of PDEs. Also, we show the region of convergence by plotting the H-surface.

Keywords: Homotopy analysis method, System of partial differential equations, H-surface.

2010 AMS Subject Classification: 65M12.

1. Introduction

Homotopy analysis method (HAM) was introduced by Liao in [8]. HAM is an efficient method for solving different kinds of partial differential equations. In recent years this method has been used to solve the various types of PDEs [1, 2, 5–7, 15]. The system of PDEs arise in mathematics, engineering and physical sciences. In [11], Sami Bataineh et al. used the HAM for solving the system of PDEs analytically, and in [3, 4], the homotopy perturbation method (HPM) was applied for solving the system of PDEs.

In this work, we introduce a development of the homotopy analysis method based on the matrix form of HAM and apply it to find the approximate solution of a given system of linear or nonlinear partial differential equations. Then, we prove the convergence of the

---

*Corresponding author.
E-mail address: amir_falah6@yahoo.com (A. Fallahzadeh).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

proposed method and apply the method to find the solution of some systems of PDEs. Also, we introduce the H-surface to determine the region of convergence.
In section 2, we present the main idea of the work and introduce the HAM via matrix form. In section 3, we prove the convergence theorem of this method for a general form of system of partial differential equations. Section 4 contains some linear and nonlinear systems of PDEs which are solved based on the HAM in matrix form and the regions of convergence are shown.

2. Main idea

In this section, we develop the idea of HAM to solve a system of linear or nonlinear differential equations. We consider the following system,

$$N[U(X,t)] = 0, \tag{1}$$

$$N = \begin{pmatrix} N_1[U(X,t)] \\ \vdots \\ N_n[U(X,t)] \end{pmatrix} \quad , \quad U(X,t) = \begin{pmatrix} U_1(X,t) \\ \vdots \\ U_n(X,t) \end{pmatrix}$$

where N is the matrix of nonlinear operators, $X = (x, y, z)$ is the vector of variables and U is the vector of unknown functions. At first, we construct the zero-order deformation system as follows.

$$(I - Q)L[\phi(X,t;Q) - U^{(0)}(X,t)] = QHN[\phi(X,t;Q)], \tag{2}$$

where I is the identity matrix, $L = \begin{pmatrix} L_1 & & 0 \\ & \ddots & \\ 0 & & L_n \end{pmatrix}$ is an auxiliary linear operator matrix, $H = \begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix}$ is an auxiliary parameter matrix, $\phi(X,t;Q)$ is the vector of unknown functions, $U^{(0)}(X,t)$ is the vector of initial guess and $Q = \begin{pmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_n \end{pmatrix}$, $0 \leqslant q_i \leqslant 1$, $1 \leqslant i \leqslant n$, is a diagonal matrix which denotes the embedding parameter matrix. It is obvious, when the q_i's, $1 \leqslant i \leqslant n$, increase from 0 to 1 or in other word, the embedding parameter matrix changes from $Q = \overline{0}$ to $Q = I$, the solution of system of equations (2) changes from $\phi(X,t;\overline{0}) = U^{(0)}(X,t)$ to $\phi(X,t;I) = U(X,t)$. Therefore, $\phi(X,t)$ varies from the initial guess $U^{(0)}(X,t)$ to the exact solution $U(X,t)$ of the system.

We consider $\phi(X,t;Q)$ in the following expansion in matrix form,

$$\phi(X,t;Q) = U^{(0)}(X,t) + \sum_{m=1}^{+\infty} Q^m U^{(m)}(X,t), \tag{3}$$

where

$$U^{(m)}(X,t)=\frac{1}{m!}\begin{pmatrix}\frac{\partial^m\phi_1(X,t,q_1)}{\partial q_1^m}|_{q_1=0}\\ \vdots\\ \frac{\partial^m\phi_n(X,t,q_n)}{\partial q_n^m}|_{q_n=0}.\end{pmatrix}. \tag{4}$$

The convergence of the vector series (3) depends upon the auxiliary parameter matrix H, if it is convergent at $Q=I$, we have

$$U(X,t)=U^{(0)}(X,t)+\sum_{m=1}^{+\infty}U^{(m)}(X,t). \tag{5}$$

Now, we define the vector,

$$\overrightarrow{U}_k=\{U^{(0)}(X,t),\ldots,U^{(k)}(X,t)\}, \tag{6}$$

where,

$$U^{(i)}=\begin{pmatrix}U_1^{(i)}(X,t)\\ \vdots\\ U_n^{(i)}(X,t)\end{pmatrix}\quad,\quad i=0,\ldots,k. \tag{7}$$

Differentiating the zero-order system (2) m times with respect to the diagonal elements of the embedding parameter matrix Q and setting $Q=\overline{0}$ and finally dividing them by $m!$, we have the so-called mth-order deformation system as follows,

$$L[U^{(m)}(X,t)-\chi_m U^{(m-1)}(X,t)]=HR_m(\overrightarrow{U}_{m-1}), \tag{8}$$

where,

$$R_m(\overrightarrow{U}_{m-1})=\frac{1}{(m-1)!}\begin{pmatrix}\frac{\partial^{m-1}N_1[\phi(X,t,Q)]}{\partial q_1^{m-1}}|_{Q=\overline{0}}\\ \vdots\\ \frac{\partial^{m-1}N_n[\phi(X,t,Q)]}{\partial q_n^{m-1}}|_{Q=\overline{0}}\end{pmatrix}\quad,\quad \chi_m=\begin{cases}\overline{0},\ m\leqslant 1,\\ I,\ m>1.\end{cases} \tag{9}$$

It should be emphasized that $U^{(m)}(X,t)$ for $m\geqslant 1$ is convergent by the linear system (8) with boundry conditions that comes from the original system.

3. Convergence of the HAM for system of PDEs

In this case, we consider the following general form of the system of partial differential equations,

$$ADS + BT + B'T' = C, \tag{10}$$

where A is the 3×3 coefficients matrix, D is the 3×3 differential operator matrix that

$$[D]_{ij} = \frac{\partial^{\alpha_{ij}^{(1)}}}{\partial x^{\alpha_{ij}^{(1)}}} + \frac{\partial^{\alpha_{ij}^{(2)}}}{\partial y^{\alpha_{ij}^{(2)}}} + \frac{\partial^{\alpha_{ij}^{(3)}}}{\partial z^{\alpha_{ij}^{(3)}}} + \frac{\partial^{\alpha_{ij}^{(4)}}}{\partial t^{\alpha_{ij}^{(4)}}}$$

S is unknowns vector $\big(u\ v\ w\big)^T$. B and B' are the 3×9 and 3×7 coefficients matrices of functions respectively and T and T' are the following vectors:

$$T = \big(u\ v\ w\ u^2\ v^2\ w^2\ uv\ uw\ vw\big)^T$$

$$T' = \big(uv^2\ uw^2\ vu^2\ vw^2\ wu^2\ wv^2\ uvw\big)^T$$

and $C = \big(c_1(x,y,z,t)\ c_2(x,y,z,t)\ c_3(x,y,z,t)\big)^T$ is the 3×1 vector of known functions.

In this part, we prove a theorem for convergence of the HAM for system of partial differential equations (10).

Theorem 3.1 If the series solution (5) of system (10) and also the series $\sum_{m=0}^{+\infty} DS_m$ where $S_m = \big(u_m\ v_m\ w_m\big)^T$ are convergent then the series (5) converges to the exact solution of the system (10).

Proof. Let:

$$S = \sum_{m=0}^{+\infty} S_m,$$

where

$$\lim_{m \to +\infty} S_m = \overrightarrow{0}. \tag{11}$$

We write

$$\sum_{m=1}^{n} [S_m - \chi_m S_{m-1}] = S_1 + (S_2 - S_1) + (S_3 - S_2) + \cdots + (S_n - S_{n-1}) = S_n.$$

Using (11), we have,

$$\sum_{m=1}^{+\infty} [u_m - \chi_m S_{m-1}] = \lim_{n \to +\infty} S_n = \overrightarrow{0}.$$

According to the definition of the operator L, we can write

$$\sum_{m=1}^{+\infty} L[S_m - \chi_m S_{m-1}] = L \sum_{m=1}^{+\infty} [S_m - \chi_m S_{m-1}] = \overrightarrow{0}.$$

From above expression and equation (8), we obtain

$$\sum_{m=1}^{+\infty} L[S_m - \chi_m S_{m-1}] = H \sum_{m=1}^{+\infty} [R_m(\overrightarrow{S}_{m-1})].$$

Since $H \neq \overline{0}$, we have

$$\sum_{m=1}^{+\infty} [R_m(\overrightarrow{S}_{m-1})] = \overrightarrow{0}. \qquad (12)$$

From (12), it holds

$$\begin{aligned}
&\sum_{m=1}^{+\infty} R_m(\overrightarrow{S}_{m-1}) = \sum_{m=1}^{+\infty} ADS_{m-1} \\
&\quad + B \sum_{m=1}^{+\infty} \begin{pmatrix} \left(u_{m-1}\ v_{m-1}\ w_{m-1}\right)^T \\ \sum_{i=0}^{m-1} \left(u_{m-1}u_{m-1-i}\ v_{m-1}v_{m-1-i}\ w_{m-1}w_{m-1-i}\ u_{m-1}v_{m-1-i}\ u_{m-1}w_{m-1-i}\ v_{m-1}w_{m-1-i}\right)^T \end{pmatrix} \\
&\quad + B' \sum_{m=1}^{+\infty} \sum_{i=0}^{m-1} \sum_{k=0}^{m-i-1} \begin{pmatrix} \left(u_i v_k v_{m-i-k-1}\ u_i w_k w_{m-i-k-1}\ v_i u_k u_{m-i-k-1}\right)^T \\ \left(v_i w_k w_{m-i-k-1}\ w_i u_k u_{m-i-k-1}\ w_i v_k v_{m-i-k-1}\ u_i v_k w_{m-i-k-1}\right)^T \end{pmatrix} \\
&\quad - (I - \chi_m)C = \overrightarrow{0}.
\end{aligned}$$

We consider the following element from previous matrix equation. The similar manipulations can be done for other elements.

$$
\begin{aligned}
&\sum_{m=1}^{+\infty}\sum_{i=0}^{m-1}\sum_{k=0}^{m-i-1} u_i v_k w_{m-i-k-1}\\
=&\sum_{i=0}^{+\infty}\sum_{m=i+1}^{m-1}\sum_{k=0}^{m-i-1} u_i v_k w_{m-i-k-1}\\
=&\sum_{i=0}^{+\infty} u_i \sum_{m=i+1}^{+\infty}\sum_{k=0}^{m-i-1} v_k w_{m-i-k-1}\\
=&\sum_{i=0}^{+\infty} u_i \sum_{m=1}^{+\infty}\sum_{k=0}^{m-1} v_k w_{m-k-1}\\
=&\sum_{i=0}^{+\infty} u_i \sum_{k=0}^{+\infty}\sum_{m=k+1}^{+\infty} v_k w_{m-k-1}\\
=&\sum_{i=0}^{+\infty} u_i \sum_{k=0}^{+\infty} v_k \sum_{m=0}^{+\infty} w_m.
\end{aligned}
$$

In addition, we have,

$$
\begin{aligned}
&\sum_{m=1}^{+\infty} ADS_{m-1} = AD\sum_{m=0}^{+\infty} S_m,\\
&\sum_{m=1}^{+\infty} (I-\chi_m)C = C.
\end{aligned}
$$

Therefore

$$ADS + BT + B'T' - C = \overrightarrow{0}. \tag{13}$$

■

4. Test Examples

In this part, we consider three sample systems of PDEs and apply the matrix form of HAM mentioned in previous section to solve these systems. The results in the tables have been provided by MAPLE.

***Example* 4.1** We consider the following linear system of PDEs:

$$
\begin{aligned}
u_{tt} + v_x + 2u = 0,\\
u_{xx} + v_t + 2u = 0,
\end{aligned}
\tag{14}
$$

with the initial conditions:

$$\begin{aligned} u(x,0) &= \sin(x), \\ u_t(x,0) &= \cos(x), \\ v(x,0) &= \cos(x). \end{aligned} \tag{15}$$

The exact solution of this system is:

$$S(x,t) = \begin{pmatrix} \sin(x+t) \\ \cos(x+t) \end{pmatrix}. \tag{16}$$

We see:

$$\begin{aligned} N[S(x,t)] &= \begin{pmatrix} N_1[S(x,t)] \\ N_2[S(x,t)] \end{pmatrix} = \begin{pmatrix} u_{tt} + v_x + 2u \\ u_{xx} + v_t + 2u \end{pmatrix}, \\ S(x,t) &= \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}. \end{aligned} \tag{17}$$

To solve the system (14) by means of HAM, we have:

$$N[\phi(x,t,Q)] = \begin{pmatrix} N_1[\phi(x,t,Q)] \\ N_2[\phi(x,t,Q)] \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \phi_1(x,t,q_1)}{\partial t^2} + \frac{\partial \phi_2(x,t,q_2)}{\partial x} + 2\phi_1(x,t,q_1) \\ \frac{\partial^2 \phi_1(x,t,q_1)}{\partial x^2} + \frac{\partial \phi_2(x,t,q_2)}{\partial t} + 2\phi_1(x,t,q_1) \end{pmatrix}, \tag{18}$$

where $Q = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}$ and the linear matrix operator

$$L[\phi(x,t,Q)] = \begin{pmatrix} \frac{\partial^2}{\partial t^2} & 0 \\ 0 & \frac{\partial}{\partial t} \end{pmatrix} \begin{pmatrix} \phi_1(x,t,q_1) \\ \phi_2(x,t,q_2) \end{pmatrix}, \tag{19}$$

with the property

$$L \begin{pmatrix} c_1(x) + c_2(x)t \\ c_3(x) \end{pmatrix} = 0, \tag{20}$$

where $c_1(x), c_2(x)$ and $c_3(x)$ are the integration constants. By using (8) under the initial conditions, we have:

$$R_m(\overrightarrow{u}_{m-1}) = \begin{pmatrix} \frac{\partial^2 u_{m-1}(x,t)}{\partial t^2} + \frac{\partial v_{m-1}(x,t)}{\partial x} + 2u_{m-1}(x,t) \\ \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} + \frac{\partial v_{m-1}(x,t)}{\partial t} + 2u_{m-1}(x,t) \end{pmatrix}. \tag{21}$$

The solution of the mth-order deformation system (8) for $m \geqslant 1$ becomes:

$$\begin{aligned} S_m(x,t) &= \chi_m S_{m-1}(x,t) + HL^{-1}R_m(\overrightarrow{S}_{m-1}) \\ &= \chi_m S_{m-1}(x,t) + \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \begin{pmatrix} \int\int dt & 0 \\ 0 & \int dt \end{pmatrix} \\ &\times \begin{pmatrix} \frac{\partial^2 u_{m-1}(x,t)}{\partial t^2} + \frac{\partial v_{m-1}(x,t)}{\partial x} + 2u_{m-1}(x,t) \\ \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} + \frac{\partial v_{m-1}(x,t)}{\partial t} + 2u_{m-1}(x,t) \end{pmatrix}. \end{aligned} \tag{22}$$

We choose the initial approximation as:

$$S_0(x,t) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} \sin(x) + t\cos(x) \\ \cos(x) \end{pmatrix}. \tag{23}$$

Applying (22) for $m \geqslant 1$, we have,

$$S_1 = \begin{pmatrix} \frac{1}{6}h_1 t^2(3\sin(x) + 2t\cos(x)) \\ \frac{1}{2}h_2 t(2\sin(x) + t\cos(x)) \end{pmatrix}$$

$$S_2 = \begin{pmatrix} \frac{1}{120}(60\sin(x) + 40t\cos(x) + 60h_1\sin(x) + 40h_1 t\cos(x) + \\ 20h_2 t\cos(x) - 5h_2 t^2\sin(x) + 10h_1 t^2\sin(x) + 4h_1 t^3\cos(x)) \\ \\ \frac{1}{12}h_2 t(12\sin(x) + 6t\cos(x) + 12h_2\sin(x) + 6h_2 t\cos(x) + \\ 2h_1 t^2\sin(x) + h_1 t^3\cos(x)) \end{pmatrix}$$

$$\vdots$$

In general, for $H = -I$ we have,

$$\begin{aligned} S_0 + S_1 + S_2 + \cdots &= \begin{pmatrix} (1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots)\sin(x) + (t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots)\cos(x) \\ (1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots)\cos(x) - (t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots)\sin(x) \end{pmatrix} \\ &= \begin{pmatrix} \sin(x+t) \\ \cos(x+t) \end{pmatrix}. \end{aligned}$$

We can see the convergence of this method at two points $(x_1, t_1) = (0.5, 1)$ and $(x_2, t_2) = (3, 1.5)$ for $H = -I$ in Table 1.

Table 1

	$(0.5, 1)$	$(3, 1.5)$
n=2	$\begin{pmatrix} 1.0202604 \\ 0.11240188 \end{pmatrix}$	$\begin{pmatrix} -1.1665823 \\ -0.42620403 \end{pmatrix}$
n=4	$\begin{pmatrix} 0.99747196 \\ 0.07079598 \end{pmatrix}$	$\begin{pmatrix} -0.97955171 \\ -0.21567996 \end{pmatrix}$
n=6	$\begin{pmatrix} 0.99749480 \\ 0.07073681 \end{pmatrix}$	$\begin{pmatrix} -0.97751894 \\ -0.21079367 \end{pmatrix}$
n=8	$\begin{pmatrix} 0.99749498 \\ 0.07073720 \end{pmatrix}$	$\begin{pmatrix} -0.97752994 \\ -0.21079568 \end{pmatrix}$
n=10	$\begin{pmatrix} 0.99749498 \\ 0.07073720 \end{pmatrix}$	$\begin{pmatrix} -0.97752999 \\ -0.21079578 \end{pmatrix}$

We present the H-surfaces of the example 4.1 which is a two-dimensional system to see the convergent region. In [8] the concept of h-curve was discussed to show the region of convergence in the HAM to solve a given linear or nonlinear equation, where h is the auxilary parameter. These curves convert to surfaces for a set of equations like a system of PDEs. By ploting the H-surface, it is easy to discover the valid region of the HAM and the optimal values of the entries of matrix H, which corresponds to the hyperplane parallel to the hyperplane that is made by the $h_1, \dots, h_n$.
In figures 1, 2 and 3, we plot the 5-approximation of u, v, v_t, v_x, u_{tt} and u_{xx} when $x = 0.5$ and $t = 1$. In this case, when $H = -I$ we obtain the Taylor series of the exact solution of the system. In this figures, we can see the region of convergence of the u, v, v_t, v_x, u_{tt} and u_{xx} is $[-2, 1] \times [-2, 1]$.

***Example* 4.2** We consider the following linear system of PDEs with variable coefficients:

$$\begin{aligned} u_t + v_{zz} - w_{xx} - u \quad &= 0, \\ u_{yy} + v_{tt} - e^x w_{xx} - v &= 0, \\ e^y u_{yy} + v_{xx} - w_{ttt} - w &= 0, \end{aligned} \tag{24}$$

with the initial conditions:

$$\begin{aligned} u(x, y, z, 0) &= y + z, \\ v(x, y, z, 0) &= y + x, \\ v_t(x, y, z, 0) &= y + x, \\ w(x, y, z, 0) &= x + y, \\ w_t(x, y, z, 0) &= x + y, \\ w_{tt}(x, y, z, 0) &= x + y. \end{aligned} \tag{25}$$

The exact solution of this system is:

$$S(x, y, z, t) = \begin{pmatrix} (y + z)e^t \\ (x + z)e^t \\ (x + y)e^t \end{pmatrix}. \tag{26}$$

Similar to the example 4.1, for $m \geqslant 1$ we have:

$$\begin{aligned} S_m(x, y, z, t) &= \chi_m S_{m-1}(x, y, z, t) + HL^{-1} R_m(\overrightarrow{S}_{m-1}) = \chi_m S_{m-1}(x, y, z, t) \\ &+ \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \times \begin{pmatrix} \int dt & 0 & 0 \\ 0 & \int\int dt & 0 \\ 0 & 0 & \int\int\int dt \end{pmatrix} \\ &\times \begin{pmatrix} \frac{\partial u_{m-1}(x,y,z,t)}{\partial t} + \frac{\partial^2 v_{m-1}(x,y,z,t)}{\partial z^2} - \frac{\partial^2 w_{m-1}(x,y,z,t)}{\partial x^2} - u_{m-1}(x, y, z, t) \\ \frac{\partial^2 u_{m-1}(x,y,z,t)}{\partial y^2} + \frac{\partial^2 v_{m-1}(x,y,z,t)}{\partial t^2} - e^x \frac{\partial^2 w_{m-1}(x,y,z,t)}{\partial x^2} - v_{m-1}(x, y, z, t) \\ e^y \frac{\partial^2 u_{m-1}(x,y,z,t)}{\partial y^2} + \frac{\partial^2 v_{m-1}(x,y,z,t)}{\partial x^2} - \frac{\partial^3 w_{m-1}(x,y,z,t)}{\partial t^3} - w_{m-1}(x, y, z, t) \end{pmatrix}. \end{aligned} \tag{27}$$

where

$$L = \begin{pmatrix} \frac{\partial}{\partial t} & 0 & 0 \\ 0 & \frac{\partial^2}{\partial t^2} & 0 \\ 0 & 0 & \frac{\partial^3}{\partial t^3} \end{pmatrix}, \tag{28}$$

with the property

$$L \begin{pmatrix} c_1(x,y,z) \\ c_2(x,y,z) + c_3(x,y,z)t \\ c_4(x,y,z) + c_5(x,y,z)t + c_6(x,y,z)t^2 \end{pmatrix} = 0, \tag{29}$$

where $c_1(x,y,z), \ldots, c_6(x,y,z)$ are integration constant.

We choose the initial approximation as:

$$S_0(x,y,z,t) = \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} y+z \\ (x+z)(1+t) \\ (x+y)(1+t+\frac{1}{2}t^2) \end{pmatrix}. \tag{30}$$

Applying (27) for $m \geqslant 1$ and $H = -I$ we have,

$$S_1 = \begin{pmatrix} t(y+z) \\ (\frac{t^2}{2!} + \frac{t^3}{3!})(x+z) \\ (\frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!})(x+y) \end{pmatrix}$$

$$S_2 = \begin{pmatrix} \frac{t^2}{2!}(y+z) \\ (\frac{t^4}{4!} + \frac{t^5}{5!})(x+z) \\ (\frac{t^6}{6!} + \frac{t^7}{7!} + \frac{t^8}{8!})(x+y) \end{pmatrix}$$

$$\vdots$$

In general, we have,

$$S_0 + S_1 + S_2 + \cdots = \begin{pmatrix} (1+t+\frac{t^2}{2}+\cdots)(x+z) \\ (1+t+\frac{t^2}{2}+\cdots)(y+z) \\ (1+t+\frac{t^2}{2}+\cdots)(x+z) \end{pmatrix} = \begin{pmatrix} e^t(y+z) \\ e^t(x+z) \\ e^t(x+y) \end{pmatrix}.$$

We can see the convergence of this method at two points $(x_1, y_1, z_1, t_1) = (1, 2, 3, 0.5)$ and $(x_2, y_2, z_2, t_2) = (2, 3, 1, 3)$ for $H = -I$ in Table 2.

Table 2

	$(1,2,3,0.5)$	$(2,3,1,3)$
n=2	$\begin{pmatrix} 8.1250000 \\ 6.5947917 \\ 4.9461639 \end{pmatrix}$	$\begin{pmatrix} 34.00000000 \\ 55.20000000 \\ 100.0457589 \end{pmatrix}$
n=4	$\begin{pmatrix} 8.2421875 \\ 6.5948851 \\ 4.9461639 \end{pmatrix}$	$\begin{pmatrix} 65.50000000 \\ 60.19017857 \\ 100.4276173 \end{pmatrix}$
n=6	$\begin{pmatrix} 8.2435981 \\ 6.5948851 \\ 4.9461639 \end{pmatrix}$	$\begin{pmatrix} 77.65000000 \\ 60.25640578 \\ 100.4276846 \end{pmatrix}$
n=8	$\begin{pmatrix} 8.2436064 \\ 6.5948851 \\ 4.9461639 \end{pmatrix}$	$\begin{pmatrix} 80.03660714 \\ 60.25661055 \\ 100.4276846 \end{pmatrix}$
n=10	$\begin{pmatrix} 8.2436064 \\ 6.5948851 \\ 4.9461639 \end{pmatrix}$	$\begin{pmatrix} 80.31866071 \\ 60.25661077 \\ 100.4276846 \end{pmatrix}$

Example **4.3** We consider the nonlinear system of ordinary differential equations as follows:

$$\begin{array}{c} u_{tt} - 2uv^2 = 0, \\ v_t - uv = 0, \end{array} \tag{31}$$

with the initial conditions:

$$\begin{array}{c} u(0) = 0, \\ u_t(0) = 1, \\ v(0) = 1. \end{array} \tag{32}$$

The exact solution of this system is:

$$S(t) = \begin{pmatrix} \tan(t) \\ \sec(t) \end{pmatrix}. \tag{33}$$

In this example, we have:

$$N[\phi(t,Q)] = \begin{pmatrix} N_1[\phi(t,Q)] \\ N_2[\phi(t,Q)] \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \phi_1(t,q_1)}{\partial t^2} - 2\phi_1(t,q_1)\phi_2^2(t,q_2) \\ \frac{\partial \phi_2(t,q_2)}{\partial t} - \phi_1(t,q_1)\phi_2(t,q_2) \end{pmatrix}, \tag{34}$$

where $Q = \left(\begin{smallmatrix} q_1 & 0 \\ 0 & q_2 \end{smallmatrix}\right)$ and the linear matrix operator

$$L[\phi(t,Q)] = \begin{pmatrix} \frac{\partial^2}{\partial t^2} & 0 \\ 0 & \frac{\partial}{\partial t} \end{pmatrix} \begin{pmatrix} \phi_1(x,t,q_1) \\ \phi_2(x,t,q_2) \end{pmatrix}, \tag{35}$$

with the property

$$L \begin{pmatrix} c_1 + c_2 t \\ c_3 \end{pmatrix} = 0, \tag{36}$$

where c_1, c_2 and c_3 are the integration constant. The solution of the mth-order deformation system (8) for $m \geqslant 1$ becomes:

$$S_m(t) = \chi_m S_{m-1}(t) + HL^{-1}R_m(\overrightarrow{S}_{m-1}) = \chi_m S_{m-1}(t) + \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \begin{pmatrix} \int\int dt\ dt & 0 \\ 0 & \int dt \end{pmatrix} \times$$
$$\begin{pmatrix} \frac{\partial^2 S_{m-1}(t)}{\partial t^2} - 2\sum_{m=1}^{+\infty}\sum_{i=0}^{m-1}\sum_{k=0}^{m-i-1} u_i(t)v_k(t)v_{m-i-k-1}(t) \\ \frac{\partial u_{m-1}(t)}{\partial t} - \sum_{m=1}^{+\infty}\sum_{i=0}^{m-1} u_i(t)v_{m-i-1}(t) \end{pmatrix}. \tag{37}$$

We choose the initial approximation as:

$$S_0(t) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix}. \tag{38}$$

Applying (37), we can see the results of the method at two points $t_1 = 0.5$ and $t_2 = 1$ for $H = -I$ in Table 3.

Table 3

	$t_1 = 0.5$	$t_2 = 1$
n=2	$\begin{pmatrix} 0.545833334 \\ 1.138020833 \end{pmatrix}$	$\begin{pmatrix} 1.46666667 \\ 1.708333333 \end{pmatrix}$
n=4	$\begin{pmatrix} 0.5462976742 \\ 1.139478798 \end{pmatrix}$	$\begin{pmatrix} 1.542504409 \\ 1.827405754 \end{pmatrix}$
n=6	$\begin{pmatrix} 0.5463024405 \\ 1.139493772 \end{pmatrix}$	$\begin{pmatrix} 1.554959773 \\ 1.846970484 \end{pmatrix}$
n=8	$\begin{pmatrix} 0.5463024894 \\ 1.139493926 \end{pmatrix}$	$\begin{pmatrix} 1.557005635 \\ 1.850184116 \end{pmatrix}$
n=10	$\begin{pmatrix} 0.5463024899 \\ 1.139493927 \end{pmatrix}$	$\begin{pmatrix} 1.557341679 \\ 1.850711974 \end{pmatrix}$

5. Conclusion

In this work, the homotopy analysis method was introduced in matrix form and applied to obtain the approximate solution of a linear or nonlinear system of partial differential equations. For this purpose, a convergence theorem was proved and some sample systems of PDEs were solved and the convergence of the HAM was discussed in each system. Also, the H-surface was introuced to illustrate the region of convergence in the HAM. Therefore, the HAM is able to solve the system of PDE via the matrix form and the convergence of method is guranteed.

Aknowledgements

The authors are thankful to the Islamic Azad University, central Tehran branch for their supports during this research and also the anonymous referee for careful reading and suggestion to improve the quality of this work.

References

[1] S. Abbasbandy, Homotopy analysis method for the Kawahara equation, Nonlinear Analysis: Real World Applications 11 (2010) 307-312.
[2] S. Abbasbandy, Solitary wave solutions to the modified form of CamassaHolm equation by means of the homotopy analysis method, Chaos, Solitons and Fractals 39 (2009) 428-435.

[3] J. Biazar, M. Eslami, A new homotopy perturbation method for solving system of partial differential equations, Computers and Mathematics with Applications 62 (2011) 225-234.

[4] J. Biazar, M. Eslami, H. Ghazvini, Homotopy perturbation method for system of partial differential equations, International Journal of Nonlinear Sciences and Numerical simulations 8 (3) (2007) 411-416.

[5] M.A. Fariborzi Araghi, A. Fallahzadeh, On the convergence of the Homotopy Analysis method for solving the Schrodinger Equation, Journal of Basic and Applied Scientific Research 2(6) (2012) 6076-6083.

[6] M.A. Fariborzi Araghi, A. Fallahzadeh, Explicit series solution of Boussinesq equation by homotopy analysis method, Journal of American Science, 8(11) (2012).

[7] T. Hayat, M. Khan, Homotopy solutions for a generalized second-grade fluid past a porous plate. Nonlinear Dyn 42 (2005) 395-405.

[8] S.J. Liao, Beyond pertubation: Introduction to the homotopy Analysis Method, Chapman and Hall/CRC Press, Boca Raton, (2003).

[9] S.J. Liao, Notes on the homotopy analysis method: some definitions and theorems, Communication in Nonlinear Science and Numnerical Simulation, 14 (2009) 983-997.

[10] P. Roul, P. Meyer, Numerical solution of system of nonlinear integro-differential equation by Homotopy perturbation method, Applied Mathematical Modelling 35 (2011) 4234-4242.

[11] A. Sami Bataineh, M.S.M. Noorani, I.Hashim, Approximation analytical solution of system of PDEs by homotopy analysis method, Computers and Mathematics with Applications 55 (2008) 2913-2923.

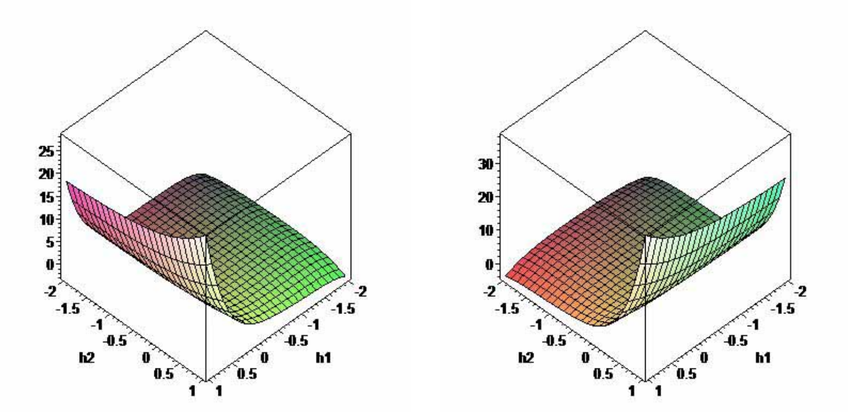
[12] F. Wang, Y. An, Nonnegative doubly periodic solution for nonlinear teleghraph system, J.math.Anal.Appl. 338 (2008) 91-100.

[13] A.M. Wazwaz, The variational iteration method for solving linear and nonlinear system of PDEs, Comput, Math, Appl 54 (2007) 895-902.

[14] W. Wu, Ch. Liou, Out put regulation of two-time-scale hyperbolic PDE systems, Journal of Process control 11 (2001) 637-647.

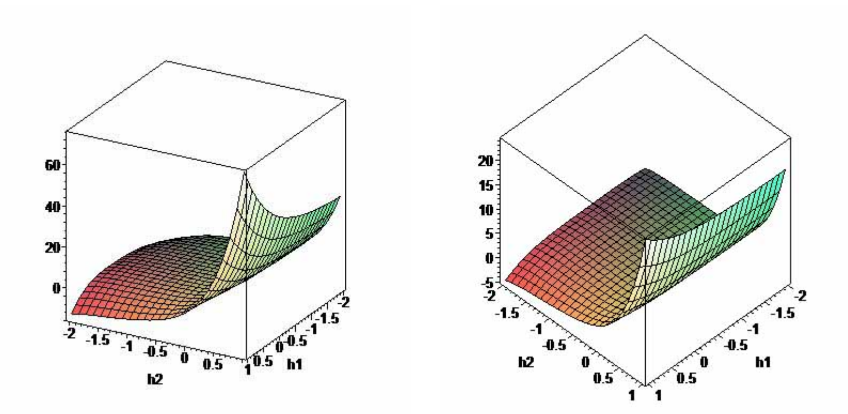
[15] W. Wu, S. Liao, Solving solitary waves with discontinuity by means of the homotopy analysis method. Chaos, Solitons & Fractals, 26 (2005) 177-185.

[16] E. Yusufoglu, An improvment to homotopy perturbation method for solving system of linear equations, Computers and Mathematic with Applications 58 (2009) 2231-2235.



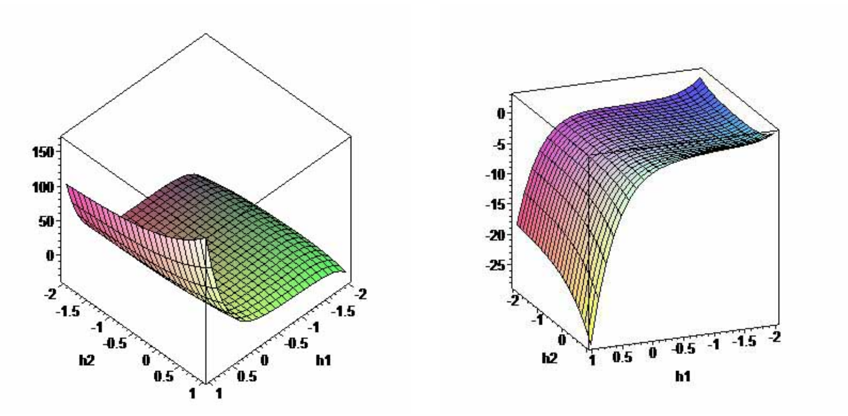
(a) H-surface of u (b) H-surface of v

Figure 1. H-Surfaces of u and v in example 4.1



(a) H-surface v_t (b) H-surface v_x

Figure 2. H-Surfaces of v_t and v_x in example 4.1



(a) H-surface u_{tt} (b) H-surface u_{xx}

Figure 3. H-Surfaces of u_{tt} and u_{xx} in example 4.1