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Compact composition operators on real Banach spaces of complex-valued bounded Lipschitz functions

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Abstract. We characterize compact composition operators on real Banach spaces of complex-valued bounded Lipschitz functions on metric spaces, not necessarily compact, with Lipschitz involutions and determine their spectra.

Keywords: Compact operator, composition operator, Lipschitz function, Lipschitz involution, spectrum of an operator.

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1. Introduction and Preliminaries

The symbol $\mathbb{K}$ denotes a field that can be either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathfrak{X}$ and $\mathcal{Y}$ be Banach spaces over $\mathbb{K}$. We denote by $BL_{\mathbb{K}}(\mathfrak{X}, \mathcal{Y})$ the Banach space of all bounded linear operators from $\mathfrak{X}$ into $\mathcal{Y}$ over $\mathbb{K}$ with the operator norm. Let us recall that $T \in BL_{\mathbb{K}}(\mathfrak{X}, \mathcal{Y})$ is compact if the closure of $T(E)$ is compact in $\mathcal{Y}$ whenever E is a bounded set in $\mathfrak{X}$.

It is known that if $\mathfrak{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are Banach spaces over $\mathbb{K}$ and $S \in BL_{\mathbb{K}}(\mathfrak{X}, \mathcal{Y})$ and $T \in BL_{\mathbb{K}}(\mathcal{Y}, \mathcal{Z})$, then $T \circ S$ is compact if S or T is compact.

Let $\mathfrak{X}$ be a Banach space over $\mathbb{K}$. Then $BL_{\mathbb{K}}(\mathfrak{X}, \mathfrak{X})$ is a unital Banach algebra over $\mathbb{K}$ when $ST = S \circ T$ for all $S, T \in BL_{\mathbb{K}}(\mathfrak{X}, \mathfrak{X})$. For $T \in BL_{\mathbb{K}}(\mathfrak{X}, \mathfrak{X})$, the spectrum of T is

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denoted by $\sigma(T)$ and defined by

$$\sigma(T) = \{\lambda \in \mathbb{K} : \lambda I_{\mathfrak{X}} - T \text{ is not invertible in } BL_{\mathbb{K}}(\mathfrak{X}, \mathfrak{X})\},$$

where $I_{\mathfrak{X}} : \mathfrak{X} \longrightarrow \mathfrak{X}$ is the identity operator on $\mathfrak{X}$.

Let X be a nonempty set, $V_{\mathbb{K}}(X)$ be a vector space over $\mathbb{K}$ of $\mathbb{K}$-valued functions on X and $\phi : X \longrightarrow X$ be a map such that $f \circ \phi \in V_{\mathbb{K}}(X)$ for all $f \in V_{\mathbb{K}}(X)$. Then $C_{\phi,V_{\mathbb{K}}(X)} : V_{\mathbb{K}}(X) \longrightarrow V_{\mathbb{K}}(X)$ defined by $C_{\phi,V_{\mathbb{K}}(X)}(f) = f \circ \phi$ is a linear operator on $V_{\mathbb{K}}(X)$ which is called the composition operator induced by ϕ on $V_{\mathbb{K}}(X)$.

Let X be a topological space. We denote by $C^b_{\mathbb{K}}(X)$ the set of all $\mathbb{K}$-valued bounded continuous functions on X. Then $C^b_{\mathbb{K}}(X)$ is a unital commutative Banach algebra over $\mathbb{K}$ under the pointwise operations and with the uniform norm

$$\| f \|_X = \sup\{|f(x)| : x \in X\} \quad (f \in C^b_{\mathbb{K}}(X)).$$

We denote by $C_{\mathbb{K}}(X)$ the algebra of all $\mathbb{K}$-valued continuous functions on X. Clearly, $C^b_{\mathbb{K}}(X) = C_{\mathbb{K}}(X)$ whenever X is compact. We write $C^b(X)$ and $C(X)$ instead of $C^b_{\mathbb{C}}(X)$and $C_{\mathbb{C}}(X)$, respectively.

Let (X, d) and (Y, ρ) be metric spaces. A map $\phi : X \longrightarrow Y$ is called a Lipschitz mapping from (X, d) into (Y, ρ) if there exists a constant $M \geqslant 0$ such that $\rho(\phi(x), \phi(y)) \leqslant Md(x, y)$ for all $x, y \in X$. A map $\phi : X \longrightarrow Y$ is called supercontractive from (X, d) into (Y, ρ) if

$$\lim_{d(x,y)\to 0} \frac{\rho(\phi(x), \phi(y))}{d(x, y)} = 0,$$

that is, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\dfrac{\rho(\phi(x), \phi(y))}{d(x, y)} < \varepsilon$ whenever $x, y \in X$ and $0 < d(x, y) < \delta$.

Let (X, d) be a metric space. A function $f : X \longrightarrow \mathbb{K}$ is called a $\mathbb{K}$-valued Lipschitz function on (X, d) if f is a Lipschitz mapping from (X, d) into the Euclidean metric space $\mathbb{K}$. For a $\mathbb{K}$-valued Lipschitz function f on (X, d), the Lipschitz number of f on (X, d) is denoted by $L_{(X,d)}(f)$ and defined by

$$L_{(X,d)}(f) = \sup\{\frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y\}.$$

We denote by $Lip_{\mathbb{K}}(X, d)$ the set of all $\mathbb{K}$-valued bounded Lipschitz functions on (X, d). Clearly, $Lip_{\mathbb{K}}(X, d)$ is a subalgebra of $C^b_{\mathbb{K}}(X)$ and $1_X \in Lip_{\mathbb{K}}(X, d)$, where 1_X is the constant function with value 1 on X. Moreover, $Lip_{\mathbb{K}}(X, d)$ with the norm

$$\|f\|_{X,L} = \max\{\|f\|_X, L_{(X,d)}(f)\}$$

is a Banach space and with the norm

$$\|f\|_{Lip(X,d)} = \|f\|_X + L_{(X,d)}(f)$$

is a unital commutative Banach algebra over $\mathbb{K}$. Since

$$\|f\|_{X,L} \leqslant \|f\|_{Lip(X,d)} \leqslant 2\|f\|_{X,L}$$

for all $f \in Lip_{\mathbb{K}}(X,d)$, we deduce that $\|\cdot\|_{X,L}$ and $\|\cdot\|_{Lip(X,d)}$ are equivalent norms on $Lip_{\mathbb{K}}(X,d)$. The set of all $f \in Lip_{\mathbb{K}}(X,d)$ for which f is supercontractive on (X,d), is denoted by $lip_{\mathbb{K}}(X,d)$. Clearly, $lip_{\mathbb{K}}(X,d)$ is a subalgebra of $Lip_{\mathbb{K}}(X,d)$ and $1_X \in lip_{\mathbb{K}}(X,d)$. Moreover, $lip_{\mathbb{K}}(X,d)$ is a closed set in $(Lip_{\mathbb{K}}(X,d), \|\cdot\|_{X,L})$ and $(Lip_{\mathbb{K}}(X,d), \|\cdot\|_{Lip(X,d)})$. So $(lip_{\mathbb{K}}(X,d), \|\cdot\|_{X,L})$ is a Banach space and $(lip_{\mathbb{K}}(X,d), \|\cdot\|_{Lip(X,d)})$ is a unital commutative Banach algebra over $\mathbb{K}$. We write $Lip(X,d)$ and $lip(X,d)$ instead of $Lip_{\mathbb{C}}(X,d)$ and $lip_{\mathbb{C}}(X,d)$, respectively. These algebras were first introduced by Sherbert in [8, 9]. Note that, if $\phi : X \longrightarrow X$ is a Lipschitz mapping then $f \circ \phi \in Lip_{\mathbb{K}}(X,d)$ ($f \circ \phi \in lip_{\mathbb{K}}(X,d)$, respectively) for all f in $Lip_{\mathbb{K}}(X,d)$ ($lip_{\mathbb{K}}(X,d)$, respectively).

Let (X,d) be a pointed metric space with the base point $e \in X$. We denote by $Lip_{0,\mathbb{K}}(X,d)$ the set of all $\mathbb{K}$-valued Lipschitz functions f on X such that $f(e) = 0$. Clearly, $Lip_{0,\mathbb{K}}(X,d)$ is a linear subspace of $C_{\mathbb{K}}(X)$. Moreover, $Lip_{0,\mathbb{K}}(X,d)$ with the norm $L_{(X,d)}(\cdot)$ is a Banach space over $\mathbb{K}$. Note that if $\phi : X \longrightarrow X$ is a base point preserving Lipschitz mapping, then $f \circ \phi \in Lip_{0,\mathbb{K}}(X,d)$ for all $f \in Lip_{0,\mathbb{K}}(X,d)$. We write $Lip_0(X,d)$ instead of $Lip_{0,\mathbb{C}}(X,d)$. For further general facts about Lipschitz spaces $Lip_{\mathbb{K}}(X,d)$, $lip_{\mathbb{K}}(X,d)$ and $Lip_{0,\mathbb{K}}(X,d)$, we refer to [10].

Kamowitz and Scheinberg [5] characterized compact endomorphisms of complex Lipschitz algebras on compact metric spaces and determined their spectra.

Jiménez-Vargas and Villegas-Vallecillos [4] characterized compact composition operators on Banach spaces of Lipschitz functions $Lip_{\mathbb{K}}(X,d)$ with the norm $\|\cdot\|_{X,L}$, $lip_{\mathbb{K}}(X,d)$ with the norm $\|\cdot\|_{X,L}$ and $Lip_{0,\mathbb{K}}(X,d)$ with the norm $L_{(X,d)}(\cdot)$ and determined the spectrum of compact composition operators on $Lip_{\mathbb{K}}(X,d)$ and $lip_{\mathbb{K}}(X,d)$, where (X,d) is a metric space, not necessarily compact.

Let X be a topological space. A self-map $\tau : X \longrightarrow X$ is called a topological involution on X if τ is continuous and $\tau(\tau(x)) = x$ for all $x \in X$.

Let X be a topological space and τ be a topological involution on X. The map $\sigma : C^b(X) \longrightarrow C^b(X)$ defined by $\sigma(f) = \overline{f} \circ \tau$ is an algebra involution on the complex algebra $C^b(X)$, which is called the algebra involution induced by τ on $C^b(X)$. Note that $\|\sigma(f)\|_X = \|f\|_X$ for all $f \in C^b(X)$. We now define

$$C^b(X,\tau) = \{f \in C^b(X) : \sigma(f) = f\}.$$

Then $C^b(X,\tau)$ is a unital self-adjoint uniformly closed real subalgebra of $C^b(X)$, $i_X \notin C^b(X,\tau)$ where i_X is the constant function with value i on X, $C^b(X) = C^b(X,\tau) \oplus i\,C^b(X,\tau)$ and

$$\max\{\|f\|_X, \|g\|_X\} \leqslant \|f + ig\|_X \leqslant 2\max\{\|f\|_X, \|g\|_X\},$$

for all $f, g \in C^b(X,\tau)$. Moreover, $C^b(X,\tau) = C^b_{\mathbb{R}}(X)$ if τ is the identity map on X. Note that if X is compact, then $C^b(X,\tau) = C(X,\tau)$, where $C(X,\tau) = \{f \in C(X) : \overline{f} \circ \tau = f\}$. Real Banach algebra $C(X,\tau)$ was defined explicitly by Kulkarni and Limaye in [6]. For further general facts about $C(X,\tau)$ and its real subalgebras, we refer to [7].

In this part we introduce real Lipschitz spaces $Lip(X,d,\tau)$, $lip(X,d,\tau)$ and $Lip_0(X,d,\tau)$.

Definition 1.1 Let (X,d) be a metric space. A self-map $\tau : X \longrightarrow X$ is called a Lipschitz involution on (X,d) if $\tau(\tau(x)) = x$ and τ is a Lipschitz mapping from (X,d) into (X,d).

Note that if τ is a Lipschitz involution on (X,d), then τ is a topological involution on

(X, d) and $C \geqslant 1$ whenever $d(\tau(x), \tau(y)) \leqslant Cd(x, y)$ for all $x, y \in X$.

Let (X, d) be a metric space, τ be a Lipschitz involution on (X, d) and σ be the algebra involution induced by τ on $C^b(X)$. We can easily show that $\sigma(Lip(X, d)) = Lip(X, d)$, $\sigma(lip(X, d)) = lip(X, d)$, $L_{(X,d)}(\sigma(f)) \leqslant CL_{(X,d)}(f)$ for all $f \in Lip(X, d)$ and $\|\sigma(f)\|_{X,L} \leqslant C\|f\|_{X,L}$ for all $f \in Lip(X, d)$, where $C \geqslant 1$ and $d(\tau(x), \tau(y)) \leqslant Cd(x, y)$ for all $x, y \in X$. We now define

$$Lip(X, d, \tau) := \{f \in Lip(X, d) : \sigma(f) = f\},$$
$$lip(X, d, \tau) := \{f \in lip(X, d) : \sigma(f) = f\}.$$

In fact, $Lip(X, d, \tau) = Lip(X, d) \cap C^b(X, \tau)$ and $lip(X, d, \tau) = lip(X, d) \cap C^b(X, \tau)$.

In the following result, we give some properties of $Lip(X, d, \tau)$ and $lip(X, d, \tau)$.

Theorem 1.2 Let (X, d) be a metric space and τ be a Lipschitz involution on (X, d). Suppose that $\mathcal{A} = Lip(X, d, \tau)$ and $\mathcal{B} = Lip(X, d)$ ($\mathcal{A} = lip(X, d, \tau)$ and $\mathcal{B} = lip(X, d)$, respectively). Then:

(i) $\mathcal{A}$ is a self-adjoint real subalgebra of $C^b(X, \tau)$ and $\mathcal{B}$, $1_X \in \mathcal{A}$ and $i_X \notin \mathcal{A}$.

(ii) $\mathcal{B} = \mathcal{A} \oplus i\,\mathcal{A}$.

(iii) For all $f, g \in \mathcal{A}$ we have

$$\max\{\|f\|_{X,L}, \|g\|_{X,L}\} \leqslant C\|f + ig\|_{X,L} \leqslant 2C \max\{\|f\|_{X,L}, \|g\|_{X,L}\},$$

where $C \geqslant 1$ and $d(\tau(x), \tau(y)) \leqslant Cd(x, y)$ for all $x, y \in X$.

(iv) $\mathcal{A}$ is closed in $(\mathcal{B}, \|\cdot\|_{X,L})$ and so $(\mathcal{A}, \|\cdot\|_{X,L})$ is a real Banach space.

(v) $f \circ \phi \in \mathcal{A}$ for all $f \in \mathcal{A}$ whenever $\phi : X \longrightarrow X$ is a Lipschitz mapping from (X, d) into (X, d) with $\phi \circ \tau = \tau \circ \phi$.

(vi) $\mathcal{A} = Lip_{\mathbb{R}}(X, d)(\mathcal{A} = lip_{\mathbb{R}}(X, d)$, respectively), if τ is the identity map on X.

Note that $lip(X, d, \tau)$ is a real subalgebra of $Lip(X, d, \tau)$ and a closed set in $(Lip(X, d, \tau), \|\cdot\|_{X,L})$.

Real Lipschitz algebras $Lip(X, d, \tau)$ and $lip(X, d, \tau)$ were first introduced in [1], whenever (X, d) is a compact metric space. In this case, Ebadian and Ostadbashi [3] characterized compact endomorphisms of real Lipschitz algebras $Lip(X, d, \tau)$ with the norm $\|\cdot\|_{Lip(X,d)}$ and determined their spectra.

Let (X, d) be a pointed metric space with a base point $e \in X$, τ be a base point-preserving Lipschitz involution on (X, d) and σ be the algebra involution induced by τ on $C^b(X)$. Then $L_{(X,d)}(\sigma(f)) \leqslant CL_{(X,d)}(f)$ for all $f \in Lip_0(X, d)$, where $C \geqslant 1$ and $d(\tau(x), \tau(y)) \leqslant Cd(x, y)$ for all $x, y \in X$. Therefore, $\sigma(Lip_0(X, d)) = Lip_0(X, d)$. We now define

$$Lip_0(X, d, \tau) = \{f \in Lip_0(X, d) : \sigma(f) = f\}.$$

In fact, $Lip_0(X, d, \tau) = Lip_0(X, d) \cap C(X, \tau)$.

In the following result, we give some properties of $Lip_0(X, d, \tau)$.

Theorem 1.3 Let (X, d) be a pointed metric space and τ be a base point preserving Lipschitz involution on (X, d). Then:

(i) $Lip_0(X, d, \tau)$ is a self-adjoint real subspace of $C^b(X, \tau)$ and $Lip_0(X, d)$, $1_X \notin Lip_0(X, d, \tau)$ and $i_X \notin Lip_0(X, d, \tau)$.

(ii) $Lip_0(X, d) = Lip_0(X, d, \tau) \oplus i\, Lip_0(X, d, \tau)$.

(iii) For all $f, g \in Lip_0(X, d, \tau)$ we have

$$\begin{aligned}\max\{L_{(X,d)}(f), L_{(X,d)}(g)\} &\leqslant CL_{(X,d)}(f + ig)\\ &\leqslant 2C \max\{L_{(X,d)}(f), L_{(X,d)}(g)\},\end{aligned}$$

where $C \geqslant 1$ and $d(\tau(x), \tau(y)) \leqslant Cd(x, y)$ for all $x, y \in X$.

(iv) $Lip_0(X, d, \tau)$ is closed in $(Lip_0(X, d), L_{(X,d)}(\cdot))$ and so $Lip_0(X, d, \tau)$ with the norm $L_{(X,d)}(\cdot)$ is a real Banach space.

(v) $f \circ \phi \in Lip_0(X, d, \tau)$ for all $f \in Lip_0(X, d, \tau)$, whenever $\phi : X \longrightarrow X$ is a base point preserving Lipschitz mapping from (X, d) into (X, d) with $\phi \circ \tau = \tau \circ \phi$.

(vi) $Lip_0(X, d, \tau) = Lip_{0,\mathbb{R}}(X, d)$, if τ is the identity map on X.

In Section 2, we characterize compact composition operators on real Lipschitz spaces $(Lip(X, d, \tau), \|\cdot\|_{X,L})$, $(lip(X, d, \tau), \|\cdot\|_{X,L})$ and $(Lip_0(X, d, \tau), L_{(X,d)}(\cdot))$ and in Section 3 we determine the spectrum of compact composition operators on real Lipschitz spaces $(Lip(X, d, \tau), \|\cdot\|_{X,L})$ and $(lip(X, d, \tau), \|\cdot\|_{X,L})$, whenever (X, d) is a metric space, not necessarily compact and τ is a Lipschitz involution on (X, d). In fact, we extend basic results of [3] and [4].

2. Compact composition operators

Let $\mathfrak{X}$ be a real linear space. The complexification of $\mathfrak{X}$ is the complex linear space $\mathfrak{X}_{\mathbb{C}} := \mathfrak{X} \oplus i\mathfrak{X}$ with addition and scalar multiplication defined by

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \qquad (x_1, y_1, x_2, y_2 \in \mathfrak{X}).$$

$$(\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\beta x + \alpha y) \qquad (\alpha, \beta \in \mathbb{R}, x, y \in \mathfrak{X}).$$

Let $(\mathfrak{X}, \|\cdot\|)$ be a real Banach space. By a modification of [2, Proposition I.13.3], there exists a norm $|||\cdot|||$ on $\mathfrak{X}_{\mathbb{C}}$ such that $|||x + i0||| = \|x\|$ for all $x \in X$, and

$$\max\{\|x\|, \|y\|\} \leqslant |||x + iy||| \leqslant 2 \max\{\|x\|, \|y\|\},$$

for all $x, y \in \mathfrak{X}$, and so $(\mathfrak{X}_{\mathbb{C}}, |||\cdot|||)$ is a complex Banach space.

Theorem 2.1 Let $(\mathfrak{X}, \|\cdot\|)$ be a real Banach space, $\mathfrak{X}_{\mathbb{C}}$ be the complexification of $\mathfrak{X}$ and $|||\cdot|||$ be a norm on $\mathfrak{X}_{\mathbb{C}}$ with $|||f||| = \|f\|$ for all $f \in \mathfrak{X}$ and C be a positive costant satisfying

$$\max\{\|f\|, \|g\|\} \leqslant C|||f + ig||| \leqslant 2C \max\{\|f\|, \|g\|\},$$

for all $f, g \in \mathfrak{X}$. Let $T \in BL_{\mathbb{R}}(\mathfrak{X}, \mathfrak{X})$ and $T' : \mathfrak{X}_{\mathbb{C}} \longrightarrow \mathfrak{X}_{\mathbb{C}}$ be the mapping defined by $T'(f + ig) = Tf + iTg \quad (f, g \in \mathfrak{X})$. Then:

(i) $T^{'} \in BL_{\mathbb{C}}(\mathfrak{X}_{\mathbb{C}}, \mathfrak{X}_{\mathbb{C}})$ and $\|T^{'}\| \leqslant 2C\|T\|$.
(ii) $T^{'}$ is compact if and only if T is compact.
(iii) $T^{'}$ is invertible in $BL_{\mathbb{C}}(\mathfrak{X}_{\mathbb{C}}, \mathfrak{X}_{\mathbb{C}})$ if and only if T is invertible in $BL_{\mathbb{R}}(\mathfrak{X}, \mathfrak{X})$.
(iv) $T^{'} = I_{\mathfrak{X}_{\mathbb{C}}}$ if and only if $T = I_{\mathfrak{X}}$.
(v) $\sigma(T^{'}) \cap \mathbb{R} = \sigma(T)$.

Proof. Clearly $T^{'}$ is a complex linear map from $\mathfrak{X}_{\mathbb{C}}$ into $\mathfrak{X}_{\mathbb{C}}$. Since

$$\begin{aligned} |||T^{'}(f+ig)||| &= |||Tf+iTg||| \leqslant |||Tf||| + |||Tg||| \\ &= \|Tf\| + \|Tg\| \leqslant \|T\|\|f\| + \|T\|\|g\| \\ &\leqslant 2\|T\| \max\{\|f\|, \|g\|\} \leqslant 2\|T\|C|||f+ig||| \end{aligned}$$

for all $f, g \in \mathfrak{X}$, we deduce that $T^{'} \in BL_{\mathbb{C}}(\mathfrak{X}_{\mathbb{C}}, \mathfrak{X}_{\mathbb{C}})$ and $|||T^{'}||| \leqslant 2C\|T\|$. Hence, (i) holds.

To prove (ii), we first assume that $T^{'}$ is compact. Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $(\mathfrak{X}, \|\cdot\|)$. Since $\|f_n\| = |||f_n|||$ for all $n \in \mathbb{N}$, we deduce that $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $(\mathfrak{X}_{\mathbb{C}}, |||\cdot|||)$. The compactness of $T^{'}$ implies that there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $\{T^{'}f_{n_k}\}_{k=1}^{\infty}$ is a Cauchy sequence in $(\mathfrak{X}_{\mathbb{C}}, |||\cdot|||)$. Since

$$\|Tf_{n_j} - Tf_{n_k}\| = |||T^{'}f_{n_j} - T^{'}f_{n_k}|||$$

for all $j, k \in \mathbb{N}$, we conclude that $\{Tf_{n_k}\}_{k=1}^{\infty}$ is a Cauchy sequence in $(\mathfrak{X}, \|\cdot\|)$. The completeness of $(\mathfrak{X}, \|\cdot\|)$ implies that $\{Tf_{n_k}\}_{k=1}^{\infty}$ is convergence in $(\mathfrak{X}, \|\cdot\|)$. Therefore, T is compact.

We now assume that T is compact. Let $\{h_n\}_{n=1}^{\infty}$ be a bounded sequence in $(\mathfrak{X}_{\mathbb{C}}, |||\cdot|||)$. Since $\mathfrak{X}_{\mathbb{C}} = \mathfrak{X} \oplus i\mathfrak{X}$, there exists unique elements $f_n, g_n \in \mathfrak{X}$ such that $h_n = f_n + ig_n$ for all $n \in \mathbb{N}$. Since

$$\max\{\|f_n\|, \|g_n\|\} \leqslant C\|f_n + ig_n\|$$

for all $n \in \mathbb{N}$, we deduce that $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are bounded sequences in $(\mathfrak{X}, \|\cdot\|)$. The compactness of T implies that there exist strictly increasing functions $p : \mathbb{N} \longrightarrow \mathbb{N}$ and $q : \mathbb{N} \longrightarrow \mathbb{N}$ and elements f and g in $\mathfrak{X}$ such that

$$\lim_{k\to\infty} \|f_{p(k)} - f\| = 0, \quad \lim_{k\to\infty} \|g_{q(k)} - g\| = 0.$$

For each $k \in \mathbb{N}$, set $n_k = q(p(k))$. Clearly, $\{f_{n_k}\}_{k=1}^{\infty}$ is a subsequence $\{f_n\}_{n=1}^{\infty}$, $\lim\limits_{k\to\infty} \|Tf_{n_k} - f\| = 0$, $\{g_{n_k}\}_{k=1}^{\infty}$ is a subsequence $\{g_n\}_{n=1}^{\infty}$ and $\lim\limits_{k\to\infty} \|Tg_{n_k} - g\| = 0$. Clearly $\{h_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{h_n\}_{n=1}^{\infty}$, $f + ig \in \mathfrak{X}_{\mathbb{C}}$ and

$$|||T^{'}h_{n_k} - (f+ig)||| \leqslant 2\max\{\|Tf_{n_k} - f\|, \|Tg_{n_k} - g\|\}$$

for all $k \in \mathbb{N}$. Thus, $\lim\limits_{k\to\infty} |||T^{'}h_{n_k} - (f+ig)||| = 0$. Therefore, $T^{'}$ is compact. Hence (ii)

holds.

To prove (iii), we first assume that $T^{'}$ is invertible in $BL_{\mathbb{C}}(\mathfrak{X}_{\mathbb{C}}, \mathfrak{X}_{\mathbb{C}})$. Then there exists $(T^{'})^{-1} \in BL_{\mathbb{C}}(\mathfrak{X}_{\mathbb{C}}, \mathfrak{X}_{\mathbb{C}})$ such that $T^{'} \circ (T^{'})^{-1} = (T^{'})^{-1} \circ T^{'} = I_{\mathfrak{X}_{\mathbb{C}}}$. We now define the maps $\Psi_1 : \mathfrak{X} \longrightarrow \mathfrak{X}_{\mathbb{C}}$ and $P_1 : \mathfrak{X}_{\mathbb{C}} \longrightarrow \mathfrak{X}$ by

$$\Psi_1(f) = f + i0 \quad (\forall f \in \mathfrak{X}) \quad \text{and} \quad P_1(f + ig) = f \quad (\forall f, g \in \mathfrak{X}).$$

We can easily show that

$$\Psi_1 \in BL_{\mathbb{R}}(\mathfrak{X}, \mathfrak{X}_{\mathbb{C}}), \quad \|\Psi_1\| \leqslant 2C, \quad P_1 \in BL_{\mathbb{R}}(\mathfrak{X}_{\mathbb{C}}, \mathfrak{X}) \text{ and } \|P_1\| \leqslant C.$$

Moreover, $\Psi_1 \circ T = T^{'} \circ \Psi_1$ and $T \circ P_1 = P_1 \circ T^{'}$. Now, we have

$$(P_1 \circ (T^{'})^{-1} \circ \Psi_1) \circ T = I_{\mathfrak{X}} = T \circ (P_1 \circ (T^{'})^{-1} \circ \Psi_1).$$

Therefore, T is invertible in $BL_{\mathbb{R}}(\mathfrak{X}, \mathfrak{X})$ and $T^{-1} = P_1 \circ (T^{'})^{-1} \circ \Psi_1$.

We now assume that T is invertible in $BL_{\mathbb{R}}(\mathfrak{X}, \mathfrak{X})$. Then there exists $T^{-1} \in BL_{\mathbb{R}}(\mathfrak{X}, \mathfrak{X})$ such that $T \circ T^{-1} = T^{-1} \circ T = I_{\mathfrak{X}}$. We now define the map $(T^{-1})^{'} : \mathfrak{X}_{\mathbb{C}} \longrightarrow \mathfrak{X}_{\mathbb{C}}$ by

$$(T^{-1})^{'}(f + ig) = T^{-1}f + iT^{-1}g \quad (\forall f, g \in \mathfrak{X}).$$

Then $(T^{-1})^{'} \in BL_{\mathbb{C}}(\mathfrak{X}_{\mathbb{C}}, \mathfrak{X}_{\mathbb{C}})$ and $\|(T^{-1})^{'}\| \leqslant 2C\|T^{-1}\|$. Moreover,

$$(T^{-1})^{'} \circ T^{'} = T^{'} \circ (T^{-1})^{'} = I_{\mathfrak{X}_{\mathbb{C}}}.$$

Therefore, $T^{'}$ is invertible in $BL_{\mathbb{C}}(\mathfrak{X}_{\mathbb{C}}, \mathfrak{X}_{\mathbb{C}})$ and $(T^{'})^{-1} = (T^{-1})^{'}$. Hence, (iii) holds.

The proof of (iv) is obvious. From (iii) and (iv), we deduce that (v) holds. ■

Compact composition operators on Lipschitz spaces $(Lip_{\mathbb{K}}(X, d), \|\cdot\|_{X,L})$ characterized in [4] as the following.

Theorem 2.2 (see [4, Theorem 1.1]). Let (X, d) be a metric space and let $\phi : X \longrightarrow X$ be a Lipschitz mapping from (X, d) into (X, d). Then the composition operator $C_{\phi, Lip_{\mathbb{K}}(X,d)} : Lip_{\mathbb{K}}(X, d) \longrightarrow Lip_{\mathbb{K}}(X, d)$ is compact if and only if ϕ is supercontrative and $\phi(X)$ is totally bounded in (X, d).

In the following result, we characterize compact composition operators on real lipschitz spaces $(Lip(X, d, \tau), \|\cdot\|_{X,L})$.

Theorem 2.3 Let (X, d) be a metric space, τ be a Lipschitz involution on (X, d) and $\phi : X \longrightarrow X$ be a Lipschitz mapping from (X, d) into (X, d) such that $\phi \circ \tau = \tau \circ \phi$. Then the composition operator $C_{\phi, Lip(X,d,\tau)} : Lip(X, d, \tau) \longrightarrow Lip(X, d, \tau)$ is compact if and only if ϕ is supercontractive and $\phi(X)$ is totally bounded in (X, d).

Proof. Since τ is a Lipschitz involution on (X, d), by Theorem 1.2, we deduce that

$Lip(X,d) = Lip(X,d,\tau) \oplus i\, Lip(X,d,\tau)$, there exists a constant $C \geqslant 1$ such that

$$\max\{\|f\|_{X,L}, \|g\|_{X,L}\} \leqslant C\|f+ig\|_{X,L} \leqslant 2C\max\{\|f\|_{X,L}, \|g\|_{X,L}\},$$

for all $f,g \in Lip(X,d,\tau)$ and $Lip(X,d,\tau)$ is a real Banach space. Hence, by Theorem 2.1, the compactness of $C_{\phi,Lip(X,d,\tau)} : Lip(X,d,\tau) \longrightarrow Lip(X,d,\tau)$ is equivalent to the compactness of $(C_{\phi,Lip(X,d,\tau)})^{'} : Lip(X,d) \longrightarrow Lip(X,d)$ which is defined by

$$(C_{\phi,Lip(X,d,\tau)})^{'}(f+ig) = C_{\phi,Lip(X,d,\tau)}(f) + iC_{\phi,Lip(X,d,\tau)}(g)$$

for all $f,g \in Lip(X,d,\tau)$.

Since

$$\begin{aligned}(C_{\phi,Lip(X,d,\tau)})^{'}(f+ig) &= (f\circ\phi) + i(g\circ\phi)\\ &= (f+ig)\circ\phi\\ &= C_{\phi,Lip(X,d)}(f+ig)\end{aligned}$$

for all $f,g \in Lip(X,d,\tau)$, we conclude that

$$(C_{\phi,Lip(X,d,\tau)})^{'} = C_{\phi,Lip(X,d)}.$$

Thus, the compactness of $C_{\phi,Lip(X,d,\tau)} : Lip(X,d,\tau) \longrightarrow Lip(X,d,\tau)$ is equivalent to the compactness of $C_{\phi,Lip(X,d)} : Lip(X,d) \longrightarrow Lip(X,d)$, and this is equivalent to ϕ is supercontractive from (X,d) into (X,d) and $\phi(X)$ is totally bounded in (X,d) by Theorem 2.2. Hence, the proof is complete. ■

Note that Theorem 2.3 is a generalization of Theorem 2.2, whenever $\mathbb{K} = \mathbb{R}$.

We now show that the class of real Lipschitz spaces $(Lip(Y,\rho,\tau), \|\cdot\|_{Y,L})$ is larger than the class of complex Lipschitz spaces $(Lip(X,d), \|\cdot\|_{X,L})$ regarded as real Lipschitz spaces (Theorem 2.4, below), and the class of compact composition operators on real Lipschitz spaces $(Lip(Y,\rho,\tau), \|\cdot\|_{Y,L})$ is larger than the class of compact composition operators on complex Lipschitz spaces $(Lip(X,d), \|\cdot\|_{X,L})$ (Theorem 2.5, below).

Theorem 2.4 Let (X,d) be a metric space. Suppose that $Y = X \times \{0,1\}$ and ρ is the metric on Y defined by

$$\rho((x_1,j_1),(x_2,j_2)) = \max\{d(x_1,x_2), |j_1-j_2|\}.$$

Let $\tau : Y \longrightarrow Y$ be the self-map on Y defined by

$$\tau(x,0) = (x,1) \quad (x\in X), \qquad \tau(x,1) = (x,0) \quad (x \in X).$$

Then:

(i) τ is a Lipschitz involution on (Y,ρ).

(ii) The map $\Lambda : Lip(X,d) \longrightarrow Lip(Y,\rho,\tau)$ defined by

$$(\Lambda f)(x,0) = f(x) \quad (f \in Lip(X,d), x \in X),$$
$$(\Lambda f)(x,1) = \overline{f(x)} \quad (f \in Lip(X,d), x \in X),$$

is an injective bounded real-linear operator from $(Lip(X,d), \|\cdot\|_{X,L})$ regarded as a real Banach space onto $(Lip(Y,\rho,\tau), \|\cdot\|_{Y,L})$, satisfying

$$\|f\|_{X,L} \leqslant \|\Lambda f\|_{Y,L} \leqslant 2\|f\|_{X,L}$$

for all $f \in Lip(X,d)$.

Proof. Clearly, $\tau(\tau(x,j)) = (x,j)$ for all $(x,j) \in Y$, and

$$\rho(\tau(x_1,j_1), \tau(x_2,j_2)) = \rho((x_1,j_1),(x_2,j_2))$$

for all $(x_1,j_1),(x_2,j_2) \in Y$. Hence, (i) holds.
It is easy to see that Λ is well-defined and a real-linear operator from $Lip(X,d)$, regarded a real Banach space, into $Lip(Y,\rho,\tau)$. Let $g \in Lip(Y,\rho,\tau)$. We define the function $f : X \longrightarrow \mathbb{C}$ by $f(x) = g(x,0)$. Then $f \in C^b(X)$, $\|f\|_X \leqslant \|g\|_Y$ and $L_{(X,d)}(f) \leqslant L_{(Y,\rho)}(g)$. Hence, $f \in Lip(X,d)$. Moreover,

$$\begin{aligned}(\Lambda f)(x,0) &= f(x) = g(x,0),\\ (\Lambda f)(x,1) &= \overline{f(x)} = \overline{g(x,0)} = (g\circ\tau)(x,0)\\ &= g(\tau(x,0)) = g(x,1)\end{aligned}$$

for all $x \in X$. Therefore, $\Lambda(f) = g$ and so Λ is onto.

Let $f \in Lip(X,d)$. Clearly, $\|f\|_X = \|\Lambda f\|_Y$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Then

$$\begin{aligned}|f(x_1) - f(x_2)| &= |(\Lambda f)(x_1,0) - (\Lambda f)(x_2,0)|\\ &\leqslant L_{(Y,\rho)}(\Lambda f)\rho((x_1,0),(x_2,0))\\ &= L_{(Y,\rho)}(\Lambda f) d(x_1,x_2).\end{aligned}$$

Hence, $L_{(X,d)}(f) \leqslant L_{(Y,\rho)}(\Lambda f)$. Therefore,

$$\|f\|_{X,L} \leqslant \|\Lambda f\|_{Y,L}.$$

Now, let $(x_1, j_1), (x_2, j_2) \in Y$ with $(x_1, j_1) \neq (x_2, j_2)$. If $j_1 = j_2$, then

$$\begin{aligned}|(\Lambda f)(x_1, j_1) - (\Lambda f)(x_2, j_2)| &= |f(x_1) - f(x_2)| \\ &\leqslant L_{(X,d)}(f) d(x_1, x_2) \\ &\leqslant 2\|f\|_{X,L} \rho((x_1, j_1), (x_2, j_2)),\end{aligned}$$

and if $j_1 \neq j_2$, then

$$\begin{aligned}|(\Lambda f)(x_1, j_1) - (\Lambda f)(x_2, j_2)| &= |f(x_1) - \overline{f(x_2)}| \\ &\leqslant 2\|f\|_X |j_1 - j_2| \\ &\leqslant 2\|f\|_{X,L} \rho((x_1, j_1), (x_2, j_2)).\end{aligned}$$

Thus,

$$L_{(Y,\rho)}(\Lambda f) \leqslant 2\|f\|_{X,L}.$$

On the other hand, we have

$$\|\Lambda f\|_Y = \|f\|_X \leqslant 2\|f\|_{X,L}.$$

Therefore,

$$\|\Lambda f\|_{Y,L} \leqslant 2\|f\|_{X,L}.$$

Hence, (ii) holds. ■

Theorem 2.5 Let (X, d) be a metric space, $Y = X \times \{0, 1\}$, ρ be the metric on Y defined by $\rho((x_1, j_1), (x_2, j_2)) = \max\{d(x_1, x_2), |j_1 - j_2|\}$ and τ be the Lipschitz involution on (Y, ρ) defined by

$$\tau(x, 0) = (x, 1), \qquad \tau(x, 1) = (x, 0), \quad (x \in X).$$

Let $\phi : X \longrightarrow X$ be a Lipschitz mapping from (X, d) into (X, d) and let $\psi : Y \longrightarrow Y$ be the self-map on Y defined by

$$\psi(x, 0) = (\phi(x), 0), \qquad \psi(x, 1) = (\phi(x), 1) \quad (x \in X),$$

Then:

(i) ψ is a Lipschitz mapping from (Y, ρ) into (Y, ρ) such that $\psi \circ \tau = \tau \circ \psi$.

(ii) The composition operator $C_{\phi, Lip(X,d)} : Lip(X, d) \longrightarrow Lip(X, d)$ is compact if and only if the composition operator $C_{\psi, Lip(Y,\rho,\tau)} : Lip(Y, \rho, \tau) \longrightarrow Lip(Y, \rho, \tau)$ is compact.

Proof. Clearly, (i) holds.
Let $\Lambda : Lip(X,d) \longrightarrow Lip(Y,\rho,\tau)$ defined by

$$(\Lambda f)(x,0) = f(x), \quad (\Lambda f)(x,1) = \overline{f(x)} \quad (x \in X).$$

By Theorem 2.4, Λ is an injective bounded real-linear operator from $Lip(X,d)$ with the norm $\|\cdot\|_{X,L}$ regarded as a real Banach space, onto the real Banach space $Lip(Y,\rho,\tau)$ with the norm $\|\cdot\|_{Y,L}$. We can easily show that

$$\Lambda \circ C_{\phi,Lip(X,d)} = C_{\psi,Lip(Y,\rho,\tau)} \circ \Lambda. \tag{1}$$

According to $\Lambda \in BL_{\mathbb{R}}(Lip(X,d), Lip(Y,\rho,\tau))$ and (1), we deduce that the operator $C_{\phi,Lip(X,d)} : Lip(X,d) \longrightarrow Lip(X,d)$ is compact if and only if $C_{\psi,Lip(Y,\rho,\tau)} : Lip(Y,\rho,\tau) \longrightarrow Lip(Y,\rho,\tau)$ is compact. Hence, (ii) holds. ∎

According to Theorems 2.4 and 2.5, it is clear that Theorem 2.3 is also a generalization of Theorem 2.2, whenever $\mathbb{K} = \mathbb{C}$.

In [4], Jiménez-Vargas and Villegas-Vallecillos obtained the analogous result for compact composition operators on little Lipschitz spaces $(lip_{\mathbb{K}}(X,d), \|\cdot\|_{X,L})$ that satisfy a kind of uniform separation property.

Definition 2.6 (see [4, Definition 1.1]). Let (X,d) be a metric space, not assumed to be compact. It is said that a linear subspace $\mathcal{M}$ of $Lip_{\mathbb{K}}(X,d)$ separates the points uniformly on bounded subsets of X if for each bounded set $K \subseteq X$, there exists a constant $a \geqslant 1$ (which may depend on K) such that for every $x, y \in K$, some $f \in \mathcal{M}$ satisfies $\|f\|_{X,L} \leqslant a$ and $|f(x) - f(y)| = d(x,y)$.

Note that $lip_{\mathbb{K}}(X,d)$ satisfies aforementioned uniform separation property when (X,d) is uniformly discrete (that is, $\inf\{d(x,y) : x \neq y\} > 0$), or when (X,d) is a totally disconnected metric space [10, Example 3.1.6].

Theorem 2.7 (see [4, Theorem 1.3]). Let (X,d) be a metric space and $\phi : X \longrightarrow X$ be a bounded Lipschitz mapping from (X,d) into (X,d). Assume that $lip_{\mathbb{K}}(X,d)$ separates points uniformly on bounded subsets of X. Then the composition operator $C_{\phi,lip_{\mathbb{K}}(X,d)} : lip_{\mathbb{K}}(X,d) \longrightarrow lip_{\mathbb{K}}(X,d)$ is compact if and only if ϕ is supercontractive and $\phi(X)$ is totally bounded in (X,d).

In the following result, we characterize compact composition operators on real little Lipschitz spaces $(lip(X,d,\tau), \|\cdot\|_{X,L})$ when $lip(X,d)$ satisfies aforementioned uniform separation property.

Theorem 2.8 Let (X,d) be a metric space, τ be a Lipschitz involution on (X,d) and $\phi : X \longrightarrow X$ be a Lipschitz mapping from (X,d) into (X,d) with $\phi \circ \tau = \tau \circ \phi$. Suppose that $lip(X,d)$ separates points uniformly on bounded subsets of X. Then the composition operator $C_{\phi,lip(X,d,\tau)} : lip(X,d,\tau) \longrightarrow lip(X,d,\tau)$ is compact if and only if ϕ is supercontractive and $\phi(X)$ is totally bounded in (X,d).

Proof. Since τ is a Lipschitz involution on (X, d), by Theorem 1.2, we deduce that $lip(X, d) = lip(X, d, \tau) \oplus i\, lip(X, d, \tau)$, there exists a constant $C \geqslant 1$ such that

$$\max\{\|f\|_{X,L}, \|g\|_{X,L}\} \leqslant C\|f + ig\|_{X,L} \leqslant 2C \max\{\|f\|_{X,L}, \|g\|_{X,L}\}$$

for all $f, g \in lip(X, d, \tau)$ and $lip(X, d, \tau)$ with the norm $\|\cdot\|_{X,L}$ is a real Banach space. Hence, by Theorem 2.1, the compactness of the operator $C_{\phi,lip(X,d,\tau)} : lip(X, d, \tau) \longrightarrow lip(X, d, \tau)$ is equivalent to the compactness of the operator $(C_{\phi,lip(X,d,\tau)})' : lip(X, d) \longrightarrow lip(X, d)$ which is defined by

$$(C_{\phi,lip(X,d,\tau)})'(f + ig) = C_{\phi,lip(X,d,\tau)}(f) + iC_{\phi,lip(X,d,\tau)}(g)$$

for all $f, g \in lip(X, d, \tau)$. It is easy to see that

$$(C_{\phi,lip(X,d,\tau)})' = C_{\phi,lip(X,d)}.$$

Since $lip(X, d)$ separates the points uniformly on bounded subsets of X and ϕ is a bounded Lipschitz mapping from (X, d) into (X, d), by Theorem 2.7, the compactness of $C_{\phi,lip(X,d)}$ is equivalent to ϕ is supercontractive and $\phi(X)$ is totally bounded in (X, d). Hence, the proof is complete. ■

Note that Theorem 2.8 is a generalization of [4, Theorem 1.3] whenever $\mathbb{K} = \mathbb{R}$.

We now show that the class of real little Lipschitz space $lip(Y, \rho, \tau)$ with the norm $\|\cdot\|_{Y,L}$ is larger than the class of complex little Lipschitz spaces $lip(X, d)$ with the norm $\|\cdot\|_{X,L}$ regarded as real Lipschitz spaces (Theorem 2.9, below) and the class of compact composition operators on $(lip(Y, \rho, \tau), \|\cdot\|_{Y,L})$ is larger than the class of compact composition operators on $(lip(X, d), \|\cdot\|_{X,L})$ (Theorem 2.10, below).

Theorem 2.9 Let (X, d) be a metric space, $Y = X \times \{0, 1\}$, ρ be the metric on Y defined by $\rho((x_1, j_1),(x_2, j_2)) = \max\{d(x_1, x_2), |j_1 - j_2|\}$ and τ be the Lipschitz involution on (Y, ρ) defined by

$$\tau(x, 0) = (x, 1), \qquad \tau(x, 1) = (x, 0) \quad (x \in X).$$

Then the map $\Gamma : lip(X, d) \longrightarrow lip(Y, \rho, \tau)$ defined by

$$(\Gamma f)(x, 0) = f(x), \quad (\Gamma f)(x, 1) = \overline{f(x)} \quad (f \in lip(X, d), x \in X),$$

is an injective real-linear operator from $(lip(X, d), \|\cdot\|_{X,L})$ regarded as a real Banach space onto $(lip(Y, \rho, \tau), \|\cdot\|_{Y,L})$ satisfying

$$\|f\|_{X,L} \leqslant \|\Gamma f\|_{Y,L} \leqslant 2\|f\|_{X,L},$$

for all $f \in lip(X, d)$.

Proof. Let $\Lambda : Lip(X,d) \longrightarrow Lip(Y,\rho,\tau)$ defined by

$$(\Lambda f)(x,0) = f(x), \quad (\Lambda f)(x,1) = \overline{f(x)} \quad (f \in Lip(X,d), x \in X).$$

By Theorem 2.4, Λ is an injective bounded real-linear operator from $Lip(X,d)$ with the norm $\|\cdot\|_{X,L}$ regarded as a real Banach space onto $Lip(Y,\rho,\tau)$ with the norm $\|\cdot\|_{Y,L}$ satisfying

$$\|f\|_{X,L} \leqslant \|\Lambda f\|_{Y,L} \leqslant 2\|f\|_{X,L}$$

for all $f, g \in Lip(X,d)$. We claim that

$$\Lambda(lip(X,d)) = lip(Y,\rho,\tau). \tag{2}$$

Let $f \in lip(X,d)$. Then $f \in Lip(X,d)$ and so $\Lambda f \in Lip(Y,\rho,\tau)$. Let $\varepsilon > 0$ be given. There exists $\delta_0 > 0$ such that $\dfrac{|f(x_1) - f(x_2)|}{d(x_1,x_2)} < \varepsilon$, whenever $x_1, x_2 \in X$ and $0 < d(x_1,x_2) < \delta_0$. Set $\delta = \min\{\delta_0, 1/2\}$. If $(x_1,j_1),(x_2,j_2) \in Y$ with $0 < \rho((x_1,j_1),(x_2,j_2))$, then $0 < d(x_1,x_2) < \delta_0$ and $j_1 = j_2$, so we have

$$\frac{|(\Lambda f)(x_1,j_1) - (\Lambda f)(x_2,j_2)|}{\rho((x_1,j_1),(x_2,j_2))} = \frac{|f(x_1) - f(x_2)|}{d(x_1,x_2)} < \varepsilon.$$

Thus, $\Lambda f \in lip(Y,\rho,\tau)$.

Now, let $g \in lip(Y,\rho,\tau)$. Then $g \in Lip(Y,\rho,\tau)$ and so there exists $f \in Lip(X,d)$ such that $\Lambda f = g$. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $\dfrac{|g(x_1,j_1) - g(x_2,j_2)|}{\rho((x_1,j_1),(x_2,j_2))} < \varepsilon$, whenever $(x_1,j_1),(x_2,j_2) \in Y$ and $0 < \rho((x_1,j_1),(x_2,j_2)) < \delta$. If $x_1, x_2 \in X$ with $0 < d(x_1,x_2) < \delta$, then $(x_1,0),(x_2,0) \in Y$ with $0 < \rho((x_1,0),(x_2,0)) < \delta$, and so

$$\frac{|f(x_1) - f(x_2)|}{d(x_1,x_2)} = \frac{|(\Lambda f)(x_1,0) - (\Lambda f)(x_2,0)|}{\rho((x_1,0),(x_2,0))} < \varepsilon.$$

Thus, $f \in lip(X,d)$ implies that $g \in \Lambda(lip(X,d))$. Hence, our claim is justified.

From (2) and definitions of Γ and Λ, we conclude that Γ is well- defined and $\Gamma = \Lambda|_{lip(X,d)}$. According to (2) and the above mentioned properties of Λ, we conclude that Γ satisfies the required conditions. ∎

Theorem 2.10 Let (X,d) be a metric space, $Y = X \times \{0,1\}$, ρ be the metric on Y defined by $\rho((x_1,j_1),(x_2,j_2)) = \max\{d(x_1,x_2), |j_1 - j_2|\}$ and τ be the Lipschitz involution on (Y,ρ) defined by

$$\tau(x,0) = (x,1), \qquad \tau(x,1) = (x,0) \quad (x \in X).$$

Let ϕ be a bounded Lipschitz mapping from (X, d) into (X, d) and the map $\psi : Y \longrightarrow Y$ defined by

$$\psi(x, 0) = (\phi(x), 0), \qquad \psi(x, 1) = (\phi(x), 1) \quad (x \in X),$$

Then:

(i) ψ is a bounded Lipschitz mapping from (Y, ρ) into (Y, ρ) such that $\psi \circ \tau = \tau \circ \psi$.
(ii) The composition operator $C_{\phi,lip(X,d)} : lip(X, d) \longrightarrow lip(X, d)$ is compact if and only if the composition operator $C_{\psi,lip(Y,\rho,\tau)} : lip(Y, \rho, \tau) \longrightarrow lip(Y, \rho, \tau)$ is compact.

Proof. By part (i) of Theorem 2.5, ψ is a Lipschitz mapping from (Y, ρ) onto (Y, ρ) such that $\psi \circ \tau = \tau \circ \psi$. Since ϕ is bounded, there exists $x_1 \in X$ and $\delta_1 > 0$ such that

$$\phi(X) \subseteq \{x \in X : d(x, x_1) < \delta_1\}.$$

We assume that $y_1 = (x_1, 0)$ and $\gamma_1 = 1 + \delta_1$. It is easy to see that

$$\psi(Y) \subseteq \{y \in Y : \rho(y, y_1) < \gamma_1\}.$$

Therefore, ψ is bounded. Hence, (i) holds.
Let $\Gamma : lip(X, d) \longrightarrow lip(Y, \rho, \tau)$ defined by

$$(\Gamma f)(x, 0) = f(x), \quad (\Gamma f)(x, 1) = \overline{f(x)} \quad (f \in lip(X, d), x \in X).$$

By Theorem 2.9, Γ is an injective bounded real linear operator from $lip(X, d)$ with the norm $\| \cdot \|_{X,L}$ regarded as a real Banach space onto real Banach space $lip(Y, \rho, \tau)$ with the norm $\| \cdot \|_{Y,L}$. We can easily show that

$$\Gamma \circ C_{\phi,lip(X,d)} = C_{\psi,lip(Y,\rho,\tau)} \circ \Gamma. \tag{3}$$

According to $\Gamma \in BL_{\mathbb{R}}(lip(X, d), lip(Y, \rho, \tau))$ and (2.3) , we deduce that the operator $C_{\phi,lip(X,d)} : lip(X, d) \longrightarrow lip(X, d)$ is compact if and only if $C_{\psi,lip(Y,\rho,\tau)} : lip(Y, \rho, \tau) \longrightarrow lip(Y, \rho, \tau)$ is compact. Hence, (iii) holds. ■

According to Theorems 2.9 and 2.10, it is clear that Theorem 2.8 is also a generalization of [4, Theorem 1.3], whenever $\mathbb{K} = \mathbb{C}$.

The following result is concerning the compactness of composition operators on Lipschitz spaces $Lip_{0,\mathbb{K}}(X, d)$ obtained by Jiménez-Vargas and Villegas-Vallecillos [4].

Theorem 2.11 (see [4, Theorem 1.2]). Let (X, d) be a base pointed metric space and $\phi : X \longrightarrow X$ be a base point preserving Lipschitz mapping from (X, d) into (X, d). Then the composition operator $C_{\phi,Lip_{0,\mathbb{K}}(X,d)} : Lip_{0,\mathbb{K}}(X, d) \longrightarrow Lip_{0,\mathbb{K}}(X, d)$ is compact if and only if ϕ supercontractive and $\phi(X)$ is totally bounded in (X, d).

In the following result, we characterize compact composition operators on real Lipschitz spaces $Lip_0(X,d,\tau)$.

Theorem 2.12 Let (X,d) be a base pointed metric space, τ be a base point preserving Lipschitz involution on (X,d) and $\phi : X \longrightarrow X$ be a base point preserving Lipschitz mapping from (X,d) into (X,d) satisfying $\phi \circ \tau = \tau \circ \phi$. Then the composition operator $C_{\phi,Lip_0(X,d,\tau)} : Lip_0(X,d,\tau) \longrightarrow Lip_0(X,d,\tau)$ is compact if and only if ϕ is supercontractive and $\phi(X)$ is totally bounded in (X,d).

Proof. Since τ is a Lipschitz involution on (X,d), by Theorem 1.3, we deduce that $Lip_0(X,d) = Lip_0(X,d,\tau) \oplus i\, Lip_0(X,d,\tau)$, there exists a constant $C \geqslant 1$ such that

$$\begin{aligned}\max\{L_{(X,d)}(f), L_{(X,d)}(g)\} &\leqslant CL_{(X,d)}(f+ig)\\ &\leqslant 2C\max\{L_{(X,d)}(f), L_{(X,d)}(g)\}\end{aligned}$$

for all $f,g \in Lip_0(X,d,\tau)$, and $(Lip_0(X,d,\tau), L_{(X,d)}(\cdot))$ is a real Banach space. Hence, by Theorem 2.1, the compactness of $C_{\phi,Lip_0(X,d,\tau)} : Lip_0(X,d,\tau) \longrightarrow Lip_0(X,d,\tau)$ is equivalent to the compactness of $(C_{\phi,Lip_0(X,d,\tau)})' : Lip_0(X,d) \longrightarrow Lip_0(X,d)$ which is defined by

$$(C_{\phi,Lip_0(X,d,\tau)})'(f+ig) = C_{\phi,Lip_0(X,d,\tau)}(f) + iC_{\phi,Lip_0(X,d,\tau)}(g)$$

for all $f,g \in Lip_0(X,d,\tau)$. It is easy to see that

$$(C_{\phi,Lip_0(X,d,\tau)})' = C_{\phi,Lip_0(X,d)}.$$

Therefore, the compactness of $C_{\phi,Lip(X,d,\tau)} : Lip(X,d,\tau) \longrightarrow Lip(X,d,\tau)$ is equivalent to the compactness of $C_{\phi,Lip_0(X,d)} : Lip_0(X,d) \longrightarrow Lip_0(X,d)$ and this is equivalent to ϕ is supercontractive and $\phi(X)$ is totally bounded in (X,d). Hence, the proof is complete. ■

Note that Theorem 2.12 is a generalization of Theorem 2.11 whenever $\mathbb{K} = \mathbb{R}$.

3. Spectra of compact composition operators

We recall that if Y is a nonempty set, $n \in \mathbb{N}$ and $\psi : Y \longrightarrow Y$ is a self-map of Y, then a point $y_0 \in Y$ is called a fixed point of ψ of order n if $\psi(y_0) = y_0$ whenever $n = 1$, and $\psi^n(y_0) = y_0$ and $\psi^k(y_0) \neq y_0$ for all $k \in \{1,...,n-1\}$ whenever $n \geqslant 2$.

Let (X,d) be a metric space and the metric space $(\tilde{X},\tilde{d})$ be the completion of (X,d). It is known [10, Proposition 1.7.1] that if (Y,ρ) is a complete metric space, then every Lipschitz mapping $\phi : X \longrightarrow Y$ from (X,d) into (Y,ρ) has a unique Lipschitz extension $\tilde{\phi} : \tilde{X} \longrightarrow Y$ from $(\tilde{X},\tilde{d})$ into (Y,ρ), and

$$\sup\{\frac{\rho(\tilde{\phi}(\tilde{x}),\tilde{\phi}(\tilde{y}))}{\tilde{d}(\tilde{x},\tilde{y})} : \tilde{x},\tilde{y} \in \tilde{X}, \tilde{x} \neq \tilde{y}\} = \sup\{\frac{\rho(\phi(x),\phi(y))}{d(x,y)} : x,y \in X, x \neq y\}.$$

Jiménez-Vargas and Villegas-Vallecillos [4] determined spectra of compact composition operators on Lipschitz spaces $(Lip_{\mathbb{K}}(X,d), \|\cdot\|_{X,L})$ and little Lipschitz spaces $(lip_{\mathbb{K}}(X,d), \|\cdot\|_{X,L})$ as the following.

Theorem 3.1 (see [4, Theorem 1.4]). Let (X,d) be a metric space, $\phi : X \longrightarrow X$ is a Lipschitz mapping from (X,d) into (X,d), $\tilde{\phi} : \tilde{X} \longrightarrow \tilde{X}$ its extension to the completion $(\tilde{X},\tilde{d})$ of (X,d) and A the set of all $n \in \mathbb{N}$ such that $\tilde{\phi}$ has a fixed point of order n.

(i) If $C_{\phi,Lip_{\mathbb{K}}(X,d)} : Lip_{\mathbb{K}}(X,d) \longrightarrow Lip_{\mathbb{K}}(X,d)$ is a compact operator, then A is finite and

$$\sigma(C_{\phi,Lip_{\mathbb{K}}(X,d)}) \setminus \{0\} = \bigcup_{n\in A} \{\lambda \in \mathbb{K} : \lambda^n = 1\}.$$

Moreover, if X is infinite and connected in (X,d), then

$$\sigma(C_{\phi,Lip_{\mathbb{K}}(X,d)}) = \{0,1\}.$$

(ii) Assume that ϕ is bounded and $lip_{\mathbb{K}}(X,d)$ separates points uniformly on bounded subsets of X. If $C_{\phi,lip_{\mathbb{K}}(X,d)} : lip_{\mathbb{K}}(X,d) \longrightarrow lip_{\mathbb{K}}(X,d)$ is compact, then A is finite and

$$\sigma(C_{\phi,lip_{\mathbb{K}}(X,d)}) \setminus \{0\} = \bigcup_{n\in A} \{\lambda \in \mathbb{K} : \lambda^n = 1\}.$$

Moreover, if X is infinite and connected in (X,d), then

$$\sigma(C_{\phi,lip_{\mathbb{K}}(X,d)}) = \{0,1\}.$$

In the following theorem, we determine spectra of compact composition operators on $Lip(X,d,\tau)$ and $lip(X,d,\tau)$.

Theorem 3.2 Let (X,d) be a metric space, τ a topological involution on X, $\phi : X \longrightarrow X$ a Lipschitz mapping from (X,d) into (X,d) with $\phi \circ \tau = \tau \circ \phi$, $\tilde{\phi}$ the unique Lipschitz extension to completion $(\tilde{X},\tilde{d})$ of (X,d) and A the set of all $n \in \mathbb{N}$ such that $\tilde{\phi}$ has a fixed point of order n.

(i) If $C_{\phi,Lip(X,d,\tau)} : Lip(X,d,\tau) \longrightarrow Lip(X,d,\tau)$ is a compact operator, then A is finite and

$$\sigma(C_{\phi,Lip(X,d,\tau)}) \setminus \{0\} = \bigcup_{n\in A} \{\lambda \in \mathbb{R} : \lambda^n = 1\}.$$

Moreover, if X is infinite and connected in (X,d), then

$$\sigma(C_{\phi,Lip(X,d,\tau)}) = \{0,1\}.$$

(ii) Assume that ϕ is bounded and $lip(X,d)$ separates points uniformly on bounded subsets of X. If $C_{\phi,lip(X,d,\tau)} : lip(X,d,\tau) \longrightarrow lip(X,d,\tau)$ is compact, then A is finite and

$$\sigma(C_{\phi,lip(X,d,\tau)}) \setminus \{0\} = \bigcup_{n\in A} \{\lambda \in \mathbb{R} : \lambda^n = 1\}.$$

Moreover, if X is infinite and connected in (X,d), then

$$\sigma(C_{\phi,lip(X,d,\tau)}) = \{0,1\}.$$

Proof. Let $\mathcal{A} = Lip(X,d,\tau)$ and $\mathcal{B} = Lip(X,d)$ ($\mathcal{A} = lip(X,d,\tau)$ and $\mathcal{B} = lip(X,d)$, respectively). Suppose that ϕ is bounded and $\mathcal{B}$ separates points uniformly on bounded subsets of X whenever $\mathcal{A} = lip(X,d,\tau)$.

Let $C_{\phi,\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ be a compact operator. Since τ is a topological involution on (X,d), by Theorem 1.2, $\mathcal{B} = \mathcal{A} \oplus i\,\mathcal{A}$ and there exists a constant $C \geqslant 1$ such that

$$\max\{\|f\|_{X,L}, \|g\|_{X,L}\} \leqslant C\|f+ig\|_{X,L} \leqslant 2C\max\{\|f\|_{X,L}, \|g\|_{X,L}\},$$

for all $f,g \in \mathcal{A}$. By Theorem 2.1, $(C_{\phi,\mathcal{A}})' : \mathcal{B} \longrightarrow \mathcal{B}$ is a compact operator and

$$\sigma(C_{\phi,\mathcal{A}}) = \mathbb{R} \cap \sigma((C_{\phi,\mathcal{A}})').$$

By the argument given in the proofs of Theorem 2.3 for $\mathcal{A} = Lip(X,d,\tau)$ and Theorem 2.8 for $\mathcal{A} = lip(X,d,\tau)$, we have

$$(C_{\phi,\mathcal{A}})' = C_{\phi,\mathcal{B}}.$$

Therefore,

$$\sigma(C_{\phi,\mathcal{A}}) = \mathbb{R} \cap \sigma(C_{\phi,\mathcal{B}}). \tag{4}$$

On the other hand, by Theorem 3.1, we have

$$\sigma(C_{\phi,\mathcal{B}}) \setminus \{0\} = \bigcup_{n\in A} \{\lambda \in \mathbb{C} : \lambda^n = 1\}. \tag{5}$$

From (4)and (5), we conclude that

$$\sigma(C_{\phi,\mathcal{A}}) \setminus \{0\} = \bigcup_{n\in A} \{\lambda \in \mathbb{R} : \lambda^n = 1\}.$$

Moreover, if X is infinite and connected in (X,d), then

$$\sigma(C_{\phi,\mathcal{B}}) = \{0,1\}.$$

So by (4), we have

$$\sigma(C_{\phi,\mathcal{A}}) = \{0, 1\}.$$

Hence, the proof is complete. ∎

In the following example which is a modification of [4, Example 1.1], we determine the spectrum of the compact composition operator $C_{\phi,Lip(X,d,\tau)}$ on $Lip(X, d, \tau)$, where τ is a suitable Lipschitz involution on (X, d).

***Example* 3.3** Take the sets $Z = [-1, -1/2] \cup [1/2, 1]$ and $Y = [-1/2, -1/4] \cup [1/4, 1/2]$ endowed, respectively, with the metrics

$$d_Z(x, y) = |x - y|, \quad (\forall x, y \in Z); \quad d_Y(x, y) = \sqrt{|x - y|}, \quad (\forall x, y \in Y).$$

Let $X = Y \cup Z$ and let $d : X \times X \longrightarrow \mathbb{R}$ the distance on X given by

$$d(x,y) = \begin{cases} d_Z(x,y) & if\ x, y \in Z; \\ d_Y(x,y) & if\ x, y \in Y; \\ d_Z(x,-1/2) + d_Y(-1/2,y) & if\ x \in [-1,-1/2], y \in Y; \\ d_Z(y,-1/2) + d_Y(-1/2,x) & if\ y \in [-1,-1/2], x \in Y; \\ d_Z(x,1/2) + d_Y(1/2,y) & if\ x \in [1/2,1], y \in Y; \\ d_Z(y,1/2) + d_Y(1/2,x) & if\ y \in [1/2,1], x \in Y. \end{cases}$$

Notice that (X, d) is compact since the topology generated by d is the usual topology of X. Define the map $\tau : X \longrightarrow X$ by $\tau(x) = -x$. It is easy to see that

$$d(\tau(x), \tau(y)) = d(x, y),$$

for all $x, y \in X$, and so τ is a Lipschitz involution on (X, d). Consider now $\phi : X \longrightarrow X$ defined by

$$\phi(x) = \begin{cases} -2x & if\ x \in Y, \\ 1 & if\ x \in [-1,-1/2], \\ -1 & if\ x \in [1/2, 1]. \end{cases}$$

It is not hard to check that ϕ is Lipschitz mapping from (X, d) into (X, d) and $\phi \circ \tau = \tau \circ \phi$. Thus, $C_{\phi,Lip(X,d,\tau)} : Lip(X, d, \tau) \longrightarrow Lip(X, d, \tau)$ is compact by Theorem 2.3. It is easy to see that -1 and 1 are fixed point of ϕ of order 2 and if $x \in X \setminus \{-1, 1\}$, then x is not fixed point of ϕ of order n for all $n \in \mathbb{N}$. Since (X, d) is a compact metric space, we deduce that $(\tilde{X}, \tilde{d}) = (X, d)$ and $\tilde{\phi} = \phi$. Thus, $A = \{2\}$ and so, by Theorem 3.2, we have

$$\sigma(C_{\phi,Lip(X,d,\tau)}) \setminus \{0\} = \bigcup_{n \in A} \{\lambda \in \mathbb{R} : \lambda^2 = 1\} = \{-1, 1\}. \tag{6}$$

On the other hand, $0 \in \sigma(C_{\phi,Lip(X,d)})$ since X is infinite. Thus, $0 \in \mathbb{R} \cap \sigma(C_{\phi,Lip(X,d)})$. By the argument given in the proof of Theorem 3.2, we conclude that $0 \in \sigma(C_{\phi,Lip(X,d,\tau)})$. Now, from (6) we have

$$\sigma(C_{\phi,Lip(X,d,\tau)}) = \{-1, 0, 1\}.$$

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