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On strongly J-clean rings associated with polynomial identity $g(x) = 0$

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Abstract. In this paper, we introduce the new notion of strongly J-clean rings associated with polynomial identity $g(x) = 0$, as a generalization of strongly J-clean rings. We denote strongly J-clean rings associated with polynomial identity $g(x) = 0$ by strongly $g(x)$-J-clean rings. Next, we investigate some properties of strongly $g(x)$-J-clean.

Keywords: strongly $g(x)$-clean rings, strongly $g(x)$-J-clean rings, strongly J-clean rings, rings generated by units.

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1. Introduction

Throughout, all rings are associative rings with identity. We denote the set of all invetrible elements in R by $U(R)$, and the Jacobson radical is denote by $J(R)$. Chen says a ring R is strongly clean if for each element $a \in R$, $a = e + u$ with $u \in U(R)$, $e^2 = e \in R$ and $eu = ue$ [6]. A ring R is strongly J-clean provided that there exist an idempotent $e \in R$ and an element $w \in J(R)$ such that $a = e + w$ and $ew = we$ [2]. Let $C(R)$ denote the center of a ring R and $g(x)$ be a polynomial in $C(R)[x]$. Camillo and simón [1] say R is $g(x)$-clean if for every element $r \in R$, $r = s + u$ with $g(s) = 0$ and $u \in U(R)$. If V is a countable dimensional vector space over a division ring D, Camillo and Simón proved that $End({}_DV)$ is $g(x)$-clean if $g(x)$ has two distinct roots in $C(D)$ [1]. Nicholson and Zhou generalized Camillo and Simón's result by proving that $End({}_RM)$ is $g(x)$-clean if ${}_RM$ is a semisimple R-module and $g(x) \in (x - a)(x - b)C(R)[x]$, where $a, b \in C(R)$ and $b, b - a \in U(R)$ [7]. [3, 8] Completely determined the relation between clean ring and

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$g(x)$-clean ring idependently. In [4] determined the relation between strongly $g(x)$-clean rings and strongly clean rings. In this paper, we prove that the relation between strongly J-clean ring and strongly $g(x)$-J-clean ring and some general properties of strongly $g(x)$-J-clean rings are given. Throughout the paper, $\mathbb{T}_n(R)$ denote the upper triangular matrix ring of order n over R, $\mathbb{N}$ denote the set of all positive integers, and $\mathbb{Z}$ represents the ring of integers.

2. Main results

Definition 2.1 Let $g(x) \in C(R)[x]$ be a fixed polynomial. An element $r \in R$ is strongly $g(x)$-J-clean if $r = s + w$ where $g(s) = 0$, $w \in J(R)$ and $sw = ws$. R is strongly $g(x)$-J-clean ring if every element of R is strongly $g(x)$-J-clean.

Note that strongly J-clean rings are exactly strongly $(x^2 - x)$-J-clean rings.

Proposition 2.2 Let R be a ring and $g(x) \in C(R)[x]$ be a fixed polynomial. Then every strongly $g(x)$-J-clean element is strongly $g(x)$-clean.

Proof. Let $a \in R$ be strongly $g(x)$-J-clean. Then $a = s + w$ with $g(s) = 0$, $w \in J(R)$ and $sw = ws$. R is a ring with identity so $a+1 \in R$. Hence, $a+1 = s'+w'$ whit $g(s') = 0$, $w' \in J(R)$ and $s'w' = w's'$. So, $a = s' + u'$ where $u' \in U(R)$. Thus, $a \in R$ is strongly $g(x)$-clean. ■

Theorem 2.3 Let R be a strongly $g(x)$-clean ring and $\dfrac{R}{J(R)}$ is boolean, then R is strongly $g(x)$-J-clean ring.

Proof. Since R is strongly $g(x)$-clean for $a \in R$, $a+1 = s+u$ where $g(s) = 0$, $u \in U(R)$ and $su = us$. Since $\dfrac{R}{J(R)}$ is boolean, $a = s + w$ where $w := u - 1 \in J(R)$, as required. ■

Theorem 2.4 Let R be a ring and $g(x) \in (x - a)(x - b)C(R)[x]$ where $a, b \in C(R)$. Then the following hold:

(1) R is strongly $(x - a)(x - b)$-J-clean if and only if R is strongly J-clean and $b - a \in U(R)$.
(2) If R is strongly J-clean and $b - a \in U(R)$, then R is strongly $g(x)$-J-clean.

Proof. (1). $\Rightarrow$ *Let* a $\in R$. Since a is strongly $(x - a)(x - b)$-J-clean, $a + 1 = s + w$ where $w \in J(R)$, s is a root of $(x - a)(x - b)$ and $sw = ws$. Since $(s - a) \in U(R)$ so $(s - b) = 0$. Hence $(b - a) \in U(R)$. Let $r \in R$. Since R is strongly $(x - a)(x - b)$-J-clean, $r(b - a) + a = s + w$ where $(s - a)(s - b) = 0$, $sw = ws$ and $w \in J(R)$. Thus, $r = \dfrac{s-a}{b-a} + \dfrac{w}{b-a}$ where $\dfrac{w}{b-a} \in J(R)$ and $(\dfrac{s-a}{b-a})^2 = \dfrac{(s-a)(s-b+b-a)}{(b-a)^2} = \dfrac{(s-a)}{(b-a)}$ and $\dfrac{s-a}{b-a}.\dfrac{w}{b-a} = \dfrac{w}{b-a}.\dfrac{s-a}{b-a}$. So R is strongly J-clean.

$\Leftarrow$ *Let* r $\in R$. Since R is strongly J-clean and $b - a \in U(R)$, $\dfrac{r-a}{b-a} = e + w$ where e is an idempotent, $w \in J(R)$ and $ew = we$. Then $r = [e(b - a) + a] + w(b - a)$, where $w(b - a) \in J(R)$, $e(b - a) + a$ is a root of $(x - a)(x - b)$ and $[e(b-a)+a]w(b-a) = w(b-a)[e(b-a)+a]$. Hence, R is strongly $(x-a)(x-b)$-J-clean.

(2). This follows from (1). ■

Corollary 2.5 For a ring R, R is strongly J-clean if and only if R is strongly (x^2+x)-J-clean.

Proof. By Theorem 2.4, let $a = 0$ and $b = -1$. ∎

Theorem 2.6 Let R be strongly $g(x)$-J-clean ring and strongly $h(x)$-J-clean ring where $h(x), g(x) \in C(R)[x]$. Then R is $h(x)g(x)$-J-clean.

Proof. The proof is clear. ∎

***Example* 2.7** Let $R = \mathbb{Z}_2$ be strongly $x(x-1)^2$-J-clean, but R is not $(x-1)^2$-J-clean.

Proof. The proof is clear. ∎

2.1 *Some proportise of strongly $g(x)$-J-clean rings*

Let R and S be rings and $\theta : C(R) \to C(S)$ be a ring homomorphism with $\theta(1) = 1$, For $g(x) = \Sigma a_i x^i \in C(R)[x]$, let $\theta'(g(x)) := \Sigma\theta(a_i)x^i \in C(S)[x]$. Then θ induces a map θ' from $C(R)[x]$ to $C(S)[x]$. If $g(x)$ is a polynomial with coefficients in $\mathbb{Z}$, Then $\theta'(g(x)) = g(x)$.

Proposition 2.8 Let $\theta : R \longrightarrow S$ be a ring epimorphism. If R is strongly $g(x)$-J-clean, then S is strongly $\theta'(g(x))$-J-clean.

Proof. Let $g(x) = a_0 + a_1x + ... + a_nx^n \in C(R)[x]$. Then $\theta'(g(x)) = \theta(a_0) + \theta(a_1)x + ... + \theta(a_n)x^n \in C(S)[x]$. For any $s \in S$, there exist $r \in R$ such that $\theta(r) = s$. Since R is strongly $g(x)$-J-clean, there exist $t \in R$ and $w \in J(R)$ such that $r = t + w$ whit $g(t) = 0$, and $tw = wt$. Then $s = \theta(r) = \theta(t) + \theta(w)$ with $\theta(w) \in J(S)$, $\theta'(g(x))|_{x=\theta(t)} = 0$, and $\theta(t)\theta(u) = \theta(u)\theta(t)$. So S is strongly $\theta'(g(x))$-J-clean. ∎

Corollary 2.9 Let $g(x) \in \mathbb{Z}[x]$ and $\{R_i\}_{i\in I}$ be a family of rings. Then $\Pi_{i\in I}R_i$ is strongly $g(x)$-J-clean if and only if R_i is strongly $g(x)$-J-clean for any $i \in I$.

Proof. This follows from definition and proposition 2.8. ∎

Corollary 2.10 Let R be a ring and $g(x) \in C(R)([x])$. If the formal power series ring $R[[t]]$ is strongly $g(x)$-J-clean, then R is strongly $g(x)$-J-clean.

Proof. This is because $\theta : R[[t]] \to R$, with $\theta(f) = a_0$ is a ring epimorphism where $f = \Sigma_{i\geqslant 0}a_it^i \in R[[t]]$. ∎

Corollary 2.11 Let R be a ring, $g(x) \in C(R)[x]$, and $1 < n \in \mathbb{N}$. If $\mathbb{T}_n(R)$ is strongly $g(x)$-J-clean, then R is strongly $g(x)$-J-clean.

Proof. $A = (a_{ij}) \in \mathbb{T}_n(R)$ with $a_{ij} = 0$ and $1 \leqslant j < i \leqslant n$. Note that $\theta : \mathbb{T}_n(R) \to R$ with $\theta(A) = a_{11}$ is a ring epimorphism. ∎

Theorem 2.12 Let R be a strongly $(x-a)(x-b)$-J-clean ring with $a, b \in C(R)$. Then for any $e^2 = e \in R$, eRe is strongly $(x-ea)(x-eb)$-J-clean. In particular if

$g(x) \in (x-ea)(x-eb)C(R)[x]$ and R is strongly $(x-a)(x-b)$-J-clean with $a, b \in C(R)$, then eRe is strongly $g(x)$-J-clean.

Proof. By theorem 2.4, R is strongly $(x-a)(x-b)$-J-clean if and only if R is strongly J-clean and $(b-a) \in U(R)$. If R is strongly J-clean, then eRe is strongly J-clean by [2, theorem 3.5]. Again by theorem 2.4, eRe is strongly $(x-ea)(x-eb)$-J-clean. ■

For any $n \in \mathbb{N}$, $U_n(R)$ denote the set of elements of R which can be written as a sum of no more n units of R [5]. A ring R is called genereted by its unit if $R = \cup_{n=1}^{\infty} U_n(R)$. We use strong $g(x)$-J-cleanness to characterize some rings in which every element can be wrriten as the sum of unit and a root of 1 which commute.

Theorem 2.13 Let R be a ring and $n \in \mathbb{N}$. Then the following are equivalent:
(1) R is strongly $(x^2 - 2^n x)$-J-clean.
(2) R is strongly $(x^2 + 2^n x)$-J-clean.
(3) R is strongly $(x^2 - 1)$-J-clean.
(4) R is strongly J-clean and $2 \in U(R)$.
(5) $R = U_2(R)$ and for any $a \in R$, a can be expressed as $a = u + v$ with some $u, v \in U(R)$ and $uv = vu$ and $v^2 = 1$ and $\frac{R}{J(R)}$ is boolean.

Proof. 1⇒ 4. *To prove* $2 \in U(R)$. Suppose $2 \notin U(R)$, then $\bar{R} = \frac{R}{2^n R} \neq 0$. Since R is strongly $(x^2 - 2^n x)$-J-clean and $2^n - 1 \in R$, $2^n - 1 = s + w$ where $w \in J(R)$, $s^2 - 2^n s = 0$ and $sw = ws$. Hence, $2^n = s + u$ where s is root of $(x^2 - 2^n x)$, $u \in U(R)$ and $su = us$. $\overline{0} = \overline{2^n} = \overline{s} + \overline{u}$ implied that $\overline{s} = -\overline{u} \in U(\bar{R})$. But $(\overline{s})^2 = \overline{s^2} = \overline{2^n s} = \overline{0}$, a contradiction. So $2 \in U(R)$. By theorem 2.4, R is strongly J-clean.

4⇒ 1. *By*(1) *of theorem* 2.4, R *is strongly* $(\mathrm{x}^2 - 2^n x)$-J-clean.

Similarly, we can prove 2 ⇔ 4.

4⇒ 5. *Let* a $\in R$. By 1⇔ 4. *Let* n=1. *Then* 2 -a=s+w *where* $\mathrm{s}^2 = 2s$, $w \in J(R)$ and $sw = ws$. Then $a = (1-s) + (1-w)$ with $1 - w \in U(R)$, $(1-s)^2 = 1$ and $(1-w)(1-s) = (1-s)(1-w)$. So $R = U_2(R)$ and by [2, Theorem 2.3], $\frac{R}{J(R)}$ is boolean.

5⇒ 4. *Let* a $\in R$. By (5). $2 - a = u + v$ where $u \in U(R)$, $v^2 = 1$ and $uv = vu$. Thus, $a = (1-u)+(1-v)$ with $1-u \in J(R)$, $(1-v)^2 = 2(1-v)$ and $(1-u)(1-v) = (1-v)(1-u)$. By "1⇔ 3", *and* n=1 *we prove that* (5) *implies* (4).

3⇒ 5. *If* R *is strongly* $(\mathrm{x}^2 - 1)$-J-clean, then for any $r \in R$, there exist $w \in J(R)$ such that $1 - r = w + s$ with $s^2 = 1$, $sw = ws$. So $r = 1 - w + (-s)$. Hence, $R = U_2(R)$ and by 2.4 and [2, theorem 2.3], $\frac{R}{J(R)}$ is boolean.

5⇒ 3. *Let* a $\in R$. Then $a + 1$ can be expressed as $a + 1 = u + v$ with $u, v \in U(R)$, $v^2 = 1$ and $uv = vu$. So v is the root of $x^2 - 1$ and since $\frac{R}{J(R)}$ is boolean, $1 - u \in J(R)$. Hence, R is strongly $(x^2 - 1)$-J-clean. ■

Remark 1 *Let* $m, k \in \mathbb{N}$*, similar to 2.13, it can be proved that, for a ring* R *and a fixed integer* $n > 0$*, the following are equivalent:*

(1) R is strongly $(x^2 - n^m x)$-J-clean.
(2) R is strongly $(x^2 + n^k x)$-J-clean.
(3) R is strongly J-clean and $n \in U(R)$.

Proposition 2.14 Let R be a ring with $c, d \in C(R)$ and $d \in U(R)$. If R is strongly (x^2+cx+d)-J-clean, then $R = U_2(R)$. In particular, if R is strongly (x^2+x+1)-J-clean, then $R = U_2(R)$ is strongly $(x^4 - x)$-J-clean with every element is the sum of a unit and a cubic root of 1 which commute with each other.

Proof. Let R be strongly $(x^2 + cx + d)$-J-clean and $r \in R$. Then $r + 1 = s + w$ where $w \in J(R)$, $s^2 + cs + d = 0$ and $ws = sw$. So $r = s + w - 1$ where $w - 1, s \in U(R)$.
Let $r \in R$. Then $r = s + w$ where $w \in J(R)$, $s^2 + s + 1 = 0$ and $sw = ws$ so $s^4 - s = 0$. Thus, R is strongly $(x^4 - x)$-J-clean. Moreover, every element in strongly $(x^2 + x + 1)$-J-clean ring R can be written as the sum of a unit and a cubic root of 1 which commute with each other. ■

Lemma 2.15 [3] Let $a \in R$. The following are equivalent for $n \in \mathbb{N}$:

(1) $a = a(ua)^n$ for some $u \in U(R)$.
(2) $a = \nu e$ for some $e^{n+1} = e$ and some $\nu \in U(R)$.
(3) $a = f\omega$ for some $f^{n+1} = f$ and some $\omega \in U(R)$.

Proposition 2.16 Let R be an strongly $(x^n - x)$-J-clean ring where $n \geqslant 2$ and $a \in R$. Then either (i) $a = u + \nu$ where $u \in U(R)$, $\nu^{n-1} = 1$, and $u\nu = \nu u$ or (ii) both aR and Ra contain non-trivial idempotents.

Proof. Since R is strongly $(x^n - x)$-J-clean, $a + 1 = s + w$ where $w \in J(R)$, $s^n = s$ and $sw = ws$. Then $w - 1 = u \in U(R)$, $s^{n-1}a = s^{n-1}s + s^{n-1}u = s + s^{n-1}u$. So $(1-s^{n-1})a = (1-s^{n-1})u$. Since $1-s^{n-1}$ is an idempotent. Then by 2.15, $(1-s^{n-1})u = \nu g$ where $\nu \in U(R)$ and $g^2 = g \in R$. So $g = \nu^{-1}(1 - s^{n-1})a \in Ra$. Suppose (i) dose not hold. Then $1 - s^{n-1} \neq 0$ this implies $g \neq 0$. Thus, Ra contains a non-trivial idempotent. Similary, aR contains a non-trivial idempotent. ■

Proposition 2.17 Let R be a ring with $n \in \mathbb{N}$. Then R is strongly $(ax^{2n} - bx)$-J-clean if and only if R is strongly $(ax^{2n} + bx)$-J-clean.

Proof. $\Rightarrow$ *Suppose* R *is strongly* $(\mathrm{a}x^{2n} - bx)$-J-clean. Then for any $r \in R$, $-r = s + w$ with $w \in J(R)$, $as^{2n} - bs = 0$ and $sw = ws$. So $r = (-s) + (-w)$ where $(-w) \in J(R)$, $a(-s)^{2n} + b(-s) = 0$. Hence, R is strongly $(ax^{2n} + bx)$-J-clean.

$\Leftarrow$ *Suppose* R *is strongly* $(\mathrm{a}x^{2n} + bx)$-J-clean. Let $r \in R$. Then there exist s and w such that $-r = s+w$, $sw = ws$, $as^{2n}+bs = 0$ and $w \in J(R)$. So $r = (-s)+(-w)$ satisfies $a(-s)^{2n}-b(-s) = 0$ and $(-s)(-w) = (-w)(-s)$. Hence, R is strongly $(ax^{2n}-bx)$-J-clean. ■

Theorem 2.18 Let R be a ring, R' be the ring of $n \times n$ diagonal matrices on R, and $(x - a)(x - b) \in C(R)[x]$. If R is strongly $(x - a)(x - b)$-J-clean, then R' is strongly $(x - aI)(x - bI)$-J-clean, where I is $n \times n$ identity matrix.

Proof. Consider $R' = diag(r_1, ..., r_n)$ where $r_1, ..., r_n \in R$. R is strongly $(x - a)(x - b)$-J-clean, so $r_i = s_i + j_i$ for $1 \leqslant i \leqslant n$ where $(s_i - a)(s_i - b) = 0$, $j_i \in J(R)$ and $s_i j_i = j_i s_i$. Write

$$\begin{bmatrix} r_1 & 0 & \dots & 0 & 0 \\ 0 & r_2 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & r_n \end{bmatrix} = \begin{bmatrix} s_1 & 0 & \dots & 0 & 0 \\ 0 & s_2 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & s_n \end{bmatrix} + \begin{bmatrix} j_1 & 0 & \dots & 0 & 0 \\ 0 & j_2 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & j_n \end{bmatrix}.$$

It is easy to show that $\begin{bmatrix} s_1 & 0 & \dots & 0 \\ 0 & s_2 & \dots & 0 \\ . & . & \dots & \\ . & . & \dots & \\ 0 & 0 & \dots & s_n \end{bmatrix}$ is the root of $(x-aI)(x-bI)$,

$$\begin{bmatrix} j_1 & 0 & \dots & 0 & 0 \\ 0 & j_2 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & j_n \end{bmatrix} \in J(diag(r_1, ..., r_n))$$

and $\begin{bmatrix} j_1 & 0 & \dots & 0 & 0 \\ 0 & j_2 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & j_n \end{bmatrix}$ commute with $\begin{bmatrix} s_1 & 0 & \dots & 0 & 0 \\ 0 & s_2 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & s_n \end{bmatrix}$. So $diag(r_1, ..., r_n)$ is strongly $(x-aI)(x-bI)$-J-clean. ■

***Example* 2.19** Let R be a ring and $(x-a)(x-b) \in C(R)[x]$. If R is strongly $(x-a)(x-b)$-J-clean then for any $c \in R$, then $A = \begin{bmatrix} a+1 & c \\ 0 & b+1 \end{bmatrix} \in T_n(R)$ is $(x-aI)(x-bI)$-J-clean.

Proof. The proof is clear. ■

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