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Existence and uniqueness of solution of Schrödinger equation in extended Colombeau algebra

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Abstract. In this paper, we establish the existence and uniqueness result of the linear Schrödinger equation with Marchaud fractional derivative in Colombeau generalized algebra. The purpose of introducing Marchaud fractional derivative is regularizing it in Colombeau sense.

Keywords: Colombeau algebra, Marchaud fractional differentiation, Schrödinger equation.

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1. Introduction

Fractional calculus has been emerging as a very interesting tool for an increasing number of scientific fields, namely, in the areas of electromagnetism, control engineering, and signal processing. Riemann-Liouville, Caputo, Grünwald-Letnikov, Hadamard, Marchaud, Riesz are some of the known definitions. Various classes of fractional differential equations have been investigated with the aid of the theory of Colombeau. Existence and uniqueness some of equation was shown via regularized fractional derivative in Colombeau algebra (cf. [6]).

This work concerns the study of existence and uniqueness to equation with Marchaud fractional differentiation in extended Colombeau algebra. We consider Marchaud fractional differentiation for indicating to existence and uniqueness Schrödinger equation in extended Colombeau algebra. The reason for introducing fractional derivatives into

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algebra of generalized functions was the possibility of solving nonlinear problems with singularities and derivatives of arbitrary real order in it. We use an algebra of generalized functions which will be an extension of the Colombeau algebra in a sense of extension of fractional derivatives. Colombeau algebras (usually denoted by the letter $\mathcal{G}$) are differential (quotient) algebras with unit, and were introduced by J. F. Colombeau (cf.[1],[2],[3]) as a nonlinear extension of distribution theory to deal with nonlinearities and singularities in PDE theory. These algebras contain the space of distributions $\mathcal{D}'$ as a subspace with an embedding realized through convolution with a suitable mollifier. Elements of these algebras are classes of nets of smooth functions. The fractional calculus by application of distributed order PDEs in Colombeau algebra was considered by [5].

The paper is organized as follows. After the introduction some basic preliminaries such as notation and definitions of the used objects are given. Also the spaces of Colombeau generalized functions are introduced. In addition, imbedding the Marchaud fractional derivative into the extended Colombeau algebra of generalized functions is shown. Finally, the existence-uniqueness result for a linear Schrödinger equation is proven.

2. Preliminaries

2.1 *Colombeau algebra*

First the definitions of some generalized function algebras of Colombeau type are mentioned which are as follows.
The elements of Colombeau algebras $\mathcal{G}$ are equivalence classes of regularizations, i.e., sequences of smooth functions satisfying asymptotic conditions in the regularization parameter ϵ. Therefore, for any set X, the family of sequences $(u_\epsilon)_\epsilon \in (0,1]$ of elements of a set X will be denoted by $X^{(0,1]}$; such sequences will also be called nets and simply written as u_ϵ.
Let Ω be an open subset of $\mathbb{R}^d$. The algebra of generalized functions on Ω equals $\mathcal{G}(\Omega)$, is defined $\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$, where

$$\mathcal{E}_M(\Omega) = \{(u_\epsilon)_\epsilon \in (C^\infty(\Omega))^{(0,1]} |\ \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}_0^n$$

$$\exists N \in \mathbb{N} \text{ s.t. } \sup_{x\in K} |\partial^\alpha u_\epsilon(x)| = O(\epsilon^{-N}),\ \epsilon \to 0\},$$

$$\mathcal{N}(\Omega) = \{(u_\epsilon)_\epsilon \in (C^\infty(\Omega))^{(0,1]} |\ \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}_0^n$$

$$\forall s \in \mathbb{N} \text{ s.t. } \sup_{x\in K} |\partial^\alpha u_\epsilon(x)| = O(\epsilon^{s}),\ \epsilon \to 0\}.$$

Element of $\mathcal{E}_M(\Omega)$ and $\mathcal{N}(\Omega)$ are called moderate, negligible functions, respectively. Families $(r_\epsilon)_\epsilon$ of complex numbers such as $|r_\epsilon| = O(\epsilon^{-p})$ as $\epsilon \to 0$ for some $p \geqslant 0$ are called moderate, in which $|r_\epsilon| = O(\epsilon^q)$ for every $q \geqslant 0$ are termed negligible. The ring $\tilde{\mathbb{R}}$ of Colombeau generalized numbers is obtained by factoring moderate families of complex numbers with respect to negligible families.
The definition of extended Colombeau algebras of generalized functions on open subset of Ω is in a sense of extension of the entire derivatives to the fractional ones. Let $\mathcal{E}^e(\Omega)$ be an algebra of all sequences $(u_\epsilon)_{\epsilon>0}$ of real valued smooth functions $u_\epsilon \in C^\infty(\Omega)$. The definition of extended Colombeau algebra is based on the ratio of spatial variable x. Moreover for a fractional derivative in the Marchaud sense is used. An interval $\Omega = (-\infty, \infty)$, and for PDEs the derivative (w.r.) to spatial variable x in the domain

$\Omega = ((0,T] \times \mathbb{R})$ is considered. The Colombeau algebra generalized functions is the set $\mathcal{G}^e_{L^\infty}(\Omega) = \mathcal{E}^e_{M,L^\infty}(\Omega)/\mathcal{N}^e_{L^\infty}(\Omega)$, where

$$\mathcal{E}^e_{M,L^\infty}(\Omega) = \{(u_\epsilon)_\epsilon \in \mathcal{E}^e(\Omega)|\ \forall\alpha \in \mathbb{R}_+ \ \cup\ \{0\}, \exists N \geqslant 0,$$

$$\text{s.t. } \|D^\alpha u_\epsilon(x)\|_{L^\infty(\Omega)} = O(\epsilon^{-N})\ as\ \epsilon \to 0\},$$

$$\mathcal{N}^e_{L^\infty}(\Omega) = \{(u_\epsilon)_\epsilon \in \mathcal{E}^e(\Omega)|\ \forall\alpha \in \mathbb{R}_+ \ \cup\ \{0\}, \forall s \geqslant 0,$$

$$\text{s.t. } \|D^\alpha u_\epsilon(x)\|_{L^\infty(\Omega)} = O(\epsilon^{s})\ as\ \epsilon \to 0\}.$$

Imbedding the fractional derivatives (w.r.) to the spatial variable is given by the convolution of the Marchaud derivative with the delta sequence:
$i_{frac} : \nu \to [\tilde{D}^\alpha(\nu_\epsilon)_{\epsilon>0}] = [D^\alpha(\nu_\epsilon * \phi_\epsilon(x))_{\epsilon>0}]$, where

$$\phi_\epsilon(x) = \frac{1}{\epsilon}\phi(\frac{x}{\epsilon}),\ \phi(x) \in C_0^\infty(\mathbb{R}), \phi(x) \geqslant 0, \int \phi(x)dx = 1,$$

$$\int x^\alpha \phi(x)dx = 0, \forall\alpha \in \mathbb{N}, |\alpha| > 0.$$

3. Imbedding of the Marchaud fractional differentiation into extended Colombeau algebra of generalized functions

Let $f_\epsilon(x)$ represents a Colombeau generalized function $f(x) \in \mathcal{G}^e(\mathbb{R})$. The Marchaud fractional derivative for $0 < \gamma < 1$ is defined by:

$$D^\gamma f_\epsilon(x) = \frac{\gamma}{\Gamma(1-\gamma)} \int_0^\infty \frac{f_\epsilon(x) - f_\epsilon(x-t)}{t^{1+\gamma}} dt.$$

We use the regularization for $0 < \gamma < 1$,

$$\tilde{D}^\gamma f_\epsilon(x) = \frac{\gamma}{\Gamma(1-\gamma)} \int_0^\infty \int_0^\infty (f_\epsilon(x) - f_\epsilon(x-t)) t^{-1-\gamma} \phi_\epsilon(t-h) dt dh.$$

The convolution form is given by:

$$\tilde{D}^\gamma f_\epsilon(x) = \frac{\gamma}{\Gamma(1-\gamma)} \int_0^\infty (f_\epsilon(x) - f_\epsilon(x-t)) t^{-1-\gamma} * \phi_\epsilon(t) dt.$$

We indicate that $|\tilde{D}^\gamma f_\epsilon(x) - D^\gamma f_\epsilon(x)| \approx 0$.

$$\sup_{x\in\mathbb{R}} |\tilde{D}^\gamma f_\epsilon(x) - D^\gamma f_\epsilon(x)| = \frac{\gamma}{\Gamma(1-\gamma)} \sup_{x\in\mathbb{R}} |\tilde{D}^\gamma f_\epsilon(x) - D^\gamma f_\epsilon(x)|$$

$$= \frac{\gamma}{\Gamma(1-\gamma)} \sup_{x\in\mathbb{R}} \int_0^\infty (f_\epsilon(x) - f_\epsilon(x-t))t^{-1-\gamma}|\phi_\epsilon(t) - \delta(t)| \longrightarrow 0,$$

as $\epsilon \longrightarrow 0$. Since $\lim_{\epsilon\longrightarrow 0} |\phi_\epsilon(t) - \delta(t)| \longrightarrow 0$, then $\tilde{D}^\gamma f_\epsilon(x) \approx D^\gamma f_\epsilon(x)$.
Using the fact that $\phi_\epsilon(t)$ has the compact support on $[0,x]$, and define $\forall x, g_x(t) = f_\epsilon(x) - f_\epsilon(x-t)$, where $g_x(t)$ has the compact support on $[0,x]$, so by Hölder inequalities, have the following calculations:

$$\sup_{x\in\mathbb{R}} |\tilde{D}^\gamma f_\epsilon(x)| \leqslant \frac{\gamma}{\Gamma(1-\gamma)} \int_0^\infty (f_\epsilon(x) - f_\epsilon(x-t))t^{-1-\gamma} * \phi_\epsilon(t)dt$$

$$= \frac{\gamma}{\Gamma(1-\gamma)} \int_0^\infty (f_\epsilon(x) - f_\epsilon(x-t)) \int_{-\infty}^\infty (t-h)^{-1-\gamma}\phi_\epsilon(h)dhdt$$

$$= \frac{\gamma}{\Gamma(1-\gamma)} \int_0^\infty (f_\epsilon(x) - f_\epsilon(x-t)) \int_{-\infty}^\infty (t-\epsilon p)^{-1-\gamma}\phi(p)dpdt$$

$$\leqslant \frac{\gamma}{\Gamma(1-\gamma)} \int_0^\infty (f_\epsilon(x) - f_\epsilon(x-t)) \sup_{p\in[0,x]} \phi(p) \int_0^x (t-\epsilon p)^{-1-\gamma}dpdt$$

$$\leqslant \frac{\gamma}{\Gamma(1-\gamma)} \int_0^\infty (f_\epsilon(x) - f_\epsilon(x-t)) \sup_{p\in[0,x]} \phi(p)\frac{1}{\epsilon} \int_{t-\epsilon x}^t (k)^{-1-\gamma}dkdt$$

$$\leqslant \frac{\gamma}{\Gamma(1-\gamma)} \sup_{t\in\mathbb{R}}(f_\epsilon(x) - f_\epsilon(x-t)) \sup_{p\in[0,x]} \phi(p) \int_0^x \frac{1}{\epsilon} \int_{t-\epsilon x}^t (k)^{-1-\gamma}dkdt$$

$$\leqslant \frac{\gamma}{\Gamma(1-\gamma)} \sup_{t\in\mathbb{R}}(f_\epsilon(x) - f_\epsilon(x-t)) \sup_{p\in[0,x]} \phi(p) \int_0^x \frac{1}{\epsilon}\frac{1}{-\gamma}((t)^{-\gamma} - (t-\epsilon x)^{-\gamma})dt$$

$$\leqslant \frac{\gamma}{\Gamma(1-\gamma)} \sup_{t\in\mathbb{R}}(f_\epsilon(x) - f_\epsilon(x-t)) \sup_{p\in[0,x]} \phi(p)\frac{1}{\epsilon^2}\frac{1}{-\gamma(1-\gamma)} \times$$

$$((t)^{-\gamma+1} - (t-\epsilon x)^{-\gamma+1})\Big|_0^x$$

$$= \frac{\gamma}{\Gamma(1-\gamma)} \sup_{t\in\mathbb{R}}(f_\epsilon(x) - f_\epsilon(x-t)) \sup_{p\in[0,x]} \phi(p)\frac{1}{\epsilon^2}\frac{1}{-\gamma(1-\gamma)}C_\gamma\epsilon^{-\gamma+1}X^{-\gamma+1}$$

$$\leqslant \frac{1}{-\gamma(1-\gamma)\Gamma(1-\gamma)} \sup(f_\epsilon(x) - f_\epsilon(x-t))C_{\gamma,\phi}\epsilon^{-\gamma+1}X^{-\gamma+1}$$

$$\leqslant C_{\gamma,\phi}\epsilon^{-N}X^{-\gamma+1}.$$

since $x < X, X > 0$ and $f_\epsilon(x)$ is of the moderate class. Thus,

$$\sup_{x\in\mathbb{R}}|\tilde{D}^\gamma f_\epsilon(x)| \leqslant C_{\gamma,\phi}\epsilon^{-N}X^{-\gamma+1},\ 0<\gamma<1.$$

In order to prove moderateness for higher derivatives a similar calculation is applied.

3.1 *Imbedding of the linear Schrödinger equation into extended Colombeau algebra of generalized functions*

We consider the existence and uniqueness result for a linear Schrödinger equation and an equation driven by the fractional derivative of the delta distribution in the extended algebra of generalized functions.
We consider the problem

$$\frac{1}{i}\partial_t u(t,x) = (\Delta - V(x))u(t,x), \quad u(0,x) = u_0(x) = \delta(x),\ V(x) = \delta(x).$$

The following regularization for delta distribution will be used:

$$u_{0\epsilon}(x) = |\ln\epsilon|^{an}\phi(x.|\ln\epsilon|), \quad V_\epsilon(x) = |\ln\epsilon|^{cn}\phi(x.|\ln\epsilon|),\ 0<a,c<1,$$

where $\phi(x)\in C_0^\infty(\mathbb{R}^n), \quad \phi(x)\geqslant 0, \quad \int\phi(x)dx = 1.$
Fractional integral of the delta sequence [4]

$$J^\alpha\phi_\epsilon(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-z)^{\alpha-1}\phi_\epsilon(z)dz, \quad t>0,\ \alpha\in\mathbb{R},$$

where $\phi_\epsilon(x) = |\ln\epsilon|\phi(x\cdot|\ln\epsilon|)$ has the following bounds in L^1-norm:

$$\| J^\alpha\phi_\epsilon(t) \|_{L^1} \leqslant \begin{cases} C & \alpha>0, \\ C(\ln|\ln\epsilon|)^m & \alpha\leqslant 0,\ m>-\alpha. \end{cases} \tag{1}$$

Proposition 3.1 Regularized equation to Schrödinger equation

$$\frac{1}{i}\partial_t u_\epsilon(t,x) = (\Delta - V(x))u_\epsilon(t,x) \tag{2}$$

has a unique solution in the space $\mathcal{G}^e_{L^\infty}([0,T)\times\mathbb{R}^n)$.

Proof. The integral form to equation (2)

$$u_\epsilon(t,x) = S_{n\epsilon}(t,x) * u_{0\epsilon}(x) + \int_0^t \int_{\mathbb{R}^n} S_{n\epsilon}(t-\tau, x-y) V_\epsilon(y) u_\epsilon(\tau, y) dy d\tau.$$

Denote by $S_{n\epsilon}(t,x) = S_n(t,x) * \phi_\epsilon(t)$, where $S_n = (4\pi t)^{\frac{-n}{2}} \exp(i|x|^2/4t)$. Then,

$$\sup_x |S_{n\epsilon}(t,x)| \leqslant \sup_x |\int_0^t S_n(t-\tau, y)\phi_\epsilon(\tau) d\tau|$$

$$\leqslant \sup_x \int_0^t |(4\pi(t-\tau))^{\frac{-n}{2}}|| \exp(i|x|^2/4(t-\tau))||\phi_\epsilon(\tau)| d\tau$$

$$\leqslant C \int_0^t |(t-\tau))^{\frac{-n}{2}}||\phi_\epsilon(\tau)| d\tau$$

This is the fractional derivative of δ-sequence and by (1) it follows,

$$\sup_x |S_{n\epsilon}(t,x)| \leqslant \begin{cases} C & n < 2, \\ C(\ln|\ln\epsilon|)^m & n \geqslant 2, m > \frac{n}{2} - 1. \end{cases} \tag{3}$$

In L^∞-norm we have

$$\|u_\epsilon(t,\cdot)\|_{L^\infty} \leqslant \|S_{n\epsilon}(t, x-\cdot)\|_{L^\infty} \|u_{0\epsilon}(\cdot)\|_{L^1} +$$

$$\int_0^t \|S_{n\epsilon}(t-\tau, x-\cdot)\|_{L^\infty} \|V_\epsilon(\cdot)\|_{L^1} \|u_\epsilon(\tau,\cdot)\|_{L^\infty} d\tau,$$

since $\|V_\epsilon(\cdot)\|_{L^\infty} \leqslant C|\ln\epsilon|^{n(c-1)}$ and by (3) we obtain

$$\|u_\epsilon(t,\cdot)\|_{L^\infty} \leqslant \begin{cases} C & n < 2, \\ C(\ln|\ln\epsilon|)^m & n \geqslant 2, m > \frac{n}{2} - 1 \end{cases} |\ln\epsilon|^{an-1}$$

$$+ \int_0^t \begin{cases} C & n < 2, \\ C(\ln|\ln\epsilon|)^m & n \geqslant 2, m > \frac{n}{2} - 1 \end{cases} |\ln\epsilon|^{n(c-1)} \|u_\epsilon(\tau,\cdot)\|_{L^\infty} d\tau.$$

By Gronwall inequality

$$\|u_\epsilon(t,\cdot)\|_{L^\infty} \leqslant \begin{cases} C & n<2, \\ C(\ln|\ln\epsilon|)^m & n\geqslant 2, m>\frac{n}{2}-1 \end{cases} |\ln\epsilon|^{an-1}$$

$$+\exp(CT \begin{cases} C & n<2, \\ C(\ln|\ln\epsilon|)^m & n\geqslant 2, m>\frac{n}{2}-1 \end{cases} |\ln\epsilon|^{n(c-1)}).$$

Thus,

$$\|u_\epsilon(t,.)\|_{L^\infty} \leqslant C\epsilon^{-N},\ \exists N>0,\ x\in\mathbb{R}^n,\ t\in[0,T],\ \epsilon<\epsilon_0.$$

For uniqueness suppose that $L_\epsilon(x,t)=u_{1\epsilon}(x,t)-u_{2\epsilon}(x,t)$ are two different solutions which make difference for equation (2)

$$\|L_\epsilon(t,\cdot)\|_{L^\infty} \leqslant \|S_{n\epsilon}(t,x-\cdot)\|_{L^\infty}\|N_{0\epsilon}(\cdot)\|_{L^1}+\int_0^t \|S_{n\epsilon}(t-\tau,x-\cdot)\|_{L^\infty}\|V_\epsilon(\cdot)\|_{L^1}\|L_\epsilon(\tau,\cdot)\|_{L^\infty}d\tau$$

$$+\int_0^t \|S_{n\epsilon}(t-\tau,x-\cdot)\|_{L^\infty}\|N_\epsilon(\tau,\cdot)\|_{L^1}d\tau,$$

then

$$\|L_\epsilon(t,\cdot)\|_{L^\infty} \leqslant \begin{cases} C & n<2, \\ C(\ln|\ln\epsilon|)^m & n\geqslant 2, m>\frac{n}{2}-1 \end{cases} \epsilon^s$$

$$+\int_0^t \begin{cases} C & n<2, \\ C(\ln|\ln\epsilon|)^m & n\geqslant 2, m>\frac{n}{2}-1 \end{cases} |\ln\epsilon|^{n(c-1)}\|L_\epsilon(\tau,\cdot)\|_{L^\infty}d\tau + \int_0^t C\epsilon^s d\tau.$$

By Gronwall inequality

$$\|L_\epsilon(t,\cdot)\|_{L^\infty} \leqslant \begin{cases} C & n<2, \\ C(\ln|\ln\epsilon|)^m & n\geqslant 2, m>\frac{n}{2}-1 \end{cases} \epsilon^s$$

$$+(\exp CT \begin{cases} C & n < 2, \\ C(\ln|\ln\epsilon|)^m & n \geqslant 2, m > \frac{n}{2}-1 \end{cases} |\ln\epsilon|^{n(c-1)}) \ +CT\epsilon^s.$$

Then, we obtain

$$\|L_\epsilon(t,.)\|_{L^\infty} \leqslant C\epsilon^s,\ \exists N>0,\ x\in\mathbb{R}^n,\ t\in[0,T],\ \epsilon<\epsilon_0.$$

Consider γth-derivative ,$\gamma \in \mathbb{N}_0^n$,

$$\partial_x^\gamma u_\epsilon(t,x) = \int_{\mathbb{R}^n} \partial_x^\gamma S_{n\epsilon}(t,x-y)u_{0\epsilon}(y)dy + \int_0^t \int_{\mathbb{R}^n} \partial_x^\gamma S_{n\epsilon}(t-\tau,x-y)V_\epsilon(y)u_\epsilon(\tau,y)dyd\tau.$$

Hence,

$$\|\partial_x^\gamma u_\epsilon(t,\cdot)\|_{L^\infty} \leqslant \begin{cases} C & n=1,\gamma=0, \\ C(\ln|\ln\epsilon|)^m & n\geqslant 2, m>\gamma+\frac{n}{2}-1 \end{cases} |\ln\epsilon|^{an-1}$$

$$+\begin{cases} C & n=1,\gamma=0, \\ C(\ln|\ln\epsilon|)^m & n\geqslant 2, m>\gamma+\frac{n}{2}-1 \end{cases} |\ln\epsilon|^{n(c-1)}\partial_x^\gamma\|u_\epsilon(t,\cdot)\|_{L^\infty}d\tau.$$

Employ Gronwall inequality to obtain

$$\|\partial_x^\gamma u_\epsilon(t,.)\|_{L^\infty} \leqslant \begin{cases} C & n=1,\gamma=0, \\ C(\ln|\ln\epsilon|)^m & n\geqslant 2, m>\gamma+\frac{n}{2}-1 \end{cases} |\ln\epsilon|^{an-1}$$

$$+\exp(CT \begin{cases} C & n=1,\gamma=0, \\ C(\ln|\ln\epsilon|)^m & n\geqslant 2, m>\gamma+\frac{n}{2}-1 \end{cases} |\ln\epsilon|^{n(c-1)}).$$

$$\|\partial_x^\gamma u_\epsilon(t,.)\|_{L^\infty} \leqslant C\epsilon^{-N},\ \exists N>0,\ x\in\mathbb{R}^n,\ t\in[0,T],\ \epsilon<\epsilon_0.$$

Consider the uniqueness

$$\partial_x^\gamma L_\epsilon(t,x) = \int_{\mathbb{R}^n} \partial_x^\gamma S_{n\epsilon}(t, x-y) u_{0\epsilon}(y)dy +$$

$$\int_0^t \int_{\mathbb{R}^n} \partial_x^\gamma S_{n\epsilon}(t-\tau, x-y) V_\epsilon(y) L_\epsilon(\tau, y) dy d\tau$$

$$+ \int_0^t \int \partial_x^\gamma S_{n\epsilon}(t-\tau, x-y) N_\epsilon(\tau, y) dy d\tau.$$

Then,

$$\|\partial_x^\gamma L_\epsilon(t,\cdot)\|_{L^\infty} \leqslant \begin{cases} C & n=1, \gamma=0, \\ C(\ln|\ln\epsilon|)^m & n \geqslant 2, m > \gamma + \frac{n}{2} - 1 \end{cases} C\epsilon^s$$

$$+ \begin{cases} C & n=1, \gamma=0, \\ C(\ln|\ln\epsilon|)^m & n \geqslant 2, m > \gamma + \frac{n}{2} - 1 \end{cases} |\ln\epsilon|^{n(c-1)} \partial_x^\gamma \|L_\epsilon(t,\cdot)\|_{L^\infty} + \int_0^t C\epsilon^s d\tau.$$

It results that,

$$\|\partial_x^\gamma L_\epsilon(t,\cdot)\|_{L^\infty} \leqslant \begin{cases} C & n=1, \gamma=0, \\ C(\ln|\ln\epsilon|)^m & n \geqslant 2, m > \gamma + \frac{n}{2} - 1 \end{cases} \epsilon^s$$

$$+ \exp(CT \begin{cases} C & n=1, \gamma=0, \\ C(\ln|\ln\epsilon|)^m & n \geqslant 2, m > \gamma + \frac{n}{2} - 1 \end{cases} |\ln\epsilon|^{n(c-1)}) + CT\epsilon^s.$$

By Gronwall inequality we obtain

$$\|\partial_x^\gamma L_\epsilon(t,\cdot)\|_{L^\infty} \leqslant C\epsilon^s,\ \exists N > 0,\ x \in \mathbb{R}^n,\ t \in [0,T],\ \epsilon < \epsilon_0.$$

Take the Marchaud fractional derivative for $0 < \gamma < 1$,

$$\tilde{D}^{\gamma}u_{\epsilon}(t,x)=\int_{\mathbb{R}^n}\tilde{D}_x^{\gamma}S_{n\epsilon}(t,x-y)u_{0\epsilon}(y)dy+\int_0^t\int_{\mathbb{R}^n}\tilde{D}_x^{\gamma}S_{n\epsilon}(t-\tau,x-y)V_{\epsilon}(y)u_{\epsilon}(\tau,y)dyd\tau.$$

$$\|\tilde{D}^{\gamma}u_{\epsilon}(t,\cdot)\|_{L^{\infty}}\leqslant\|\tilde{D}_x^{\gamma}S_{n\epsilon}(t,x-\cdot)\|_{L^{\infty}}\|u_{0\epsilon}(\cdot)\|_{L^1}$$

$$+\int_0^t\|\tilde{D}_x^{\gamma}S_{n\epsilon}(t-\tau,x-\cdot)\|_{L^{\infty}}\|V_{\epsilon}(\cdot)\|_{L^1}\|u_{\epsilon}(\tau,\cdot)\|_{L^{\infty}}d\tau.$$

$$\|\tilde{D}^{\gamma}u_{\epsilon}(t,\cdot)\|_{L^{\infty}}\leqslant\begin{cases}C & n<2,\\ C(\ln|\ln\epsilon|^m) & n\geqslant 2, m>\frac{n}{2}-1\end{cases}\qquad|\ln\epsilon|^{an-1}X^{1-\gamma}$$

$$+\int_0^t\begin{cases}C & n<2,\\ C(\ln|\ln\epsilon|)^m & n\geqslant 2, m>\frac{n}{2}-1\end{cases}\qquad|\ln\epsilon|^{n(c-1)}X^{1-\gamma}\|u_{\epsilon}(\tau,\cdot)\|_{L^{\infty}}d\tau.$$

The moderateness of $u_{\epsilon}(t,x)$

$$\|\tilde{D}^{\gamma}u_{\epsilon}(t,.)\|_{L^{\infty}}\leqslant\begin{cases}C & n<2,\\ C(\ln|\ln\epsilon|)^m & n\geqslant 2, m>\frac{n}{2}-1\end{cases}\qquad|\ln\epsilon|^{an-1}X^{1-\gamma}$$

$$+\exp(CT\begin{cases}C & n<2,\\ C(\ln|\ln\epsilon|)^m & n\geqslant 2, m>\frac{n}{2}-1\end{cases}\qquad|\ln\epsilon|^{n(c-1)}X^{1-\gamma}\epsilon^{-N}).$$

$$\|\tilde{D}^{\gamma}u_{\epsilon}(t,\cdot)\|_{L^{\infty}}\leqslant C\epsilon^{-N},\ \exists N>0,\ x\in\mathbb{R}^n,\ t\in[0,T],\ \epsilon<\epsilon_0.$$

For uniqueness suppose that $L_{\epsilon}(x,t)=u_{1\epsilon}(x,t)-u_{2\epsilon}(x,t)$ be two different solutions whose difference for equation (2)

$$\tilde{D}^{\gamma}L_{\epsilon}(t,x)=\int_{\mathbb{R}^n}\tilde{D}_x^{\gamma}S_{n\epsilon}(t,x-y)N_{0\epsilon}(y)dy+\int_0^t\int_{\mathbb{R}^n}\tilde{D}_x^{\gamma}S_{n\epsilon}(t-\tau,x-y)V_{\epsilon}(y)L_{\epsilon}(\tau,y)dyd\tau$$

$$+\int_0^t \int_{\mathbb{R}^n} \tilde{D}_x^\gamma S_{n\epsilon}(t-\tau, x-y) N_\epsilon(\tau, y) dy d\tau.$$

In L^∞-norm we obtain

$$\|\tilde{D}^\gamma L_\epsilon(t,\cdot)\|_{L^\infty} \leqslant \|\tilde{D}_x^\gamma S_{n\epsilon}(t, x-\cdot)\|_{L^\infty} \|N_{0\epsilon}(\cdot)\|_{L^1}$$

$$+\int_0^t \|\tilde{D}_x^\gamma S_{n\epsilon}(t-\tau, x-\cdot)\|_{L^\infty} \|V_\epsilon(\cdot)\|_{L^1} \|L_\epsilon(\tau,\cdot)\|_{L^\infty} d\tau$$

$$+\int_0^t \|\tilde{D}_x^\gamma S_{n\epsilon}(t-\tau, x-\cdot)\|_{L^\infty} \|N_\epsilon(\tau,\cdot)\|_{L^1} d\tau.$$

It leads to,

$$\|\tilde{D}^\gamma L_\epsilon(t,\cdot)\|_{L^\infty} \leqslant \begin{cases} C & n<2, \\ C(\ln|\ln\epsilon|)^m & n \geqslant 2, m > \frac{n}{2}-1 \end{cases} \quad C\epsilon^s$$

$$+\int_0^t \begin{cases} C & n<2, \\ C(\ln|\ln\epsilon|)^m & n \geqslant 2, m > \frac{n}{2}-1 \end{cases} \quad |\ln\epsilon|^{n(c-1)} \|L_\epsilon(\tau,\cdot)\|_{L^\infty} d\tau.$$

By using the moderateness of $L_\epsilon(t,x)$, it follows that

$$\|\tilde{D}^\gamma L_\epsilon(t,\cdot)\|_{L^\infty} \leqslant \begin{cases} C & n<2, \\ C(\ln|\ln\epsilon|)^m & n \geqslant 2, m > \frac{n}{2}-1 \end{cases} \quad C\epsilon^s$$

$$+\exp(CT \begin{cases} C & n<2, \\ C(\ln|\ln\epsilon|)^m & n \geqslant 2, m > \frac{n}{2}-1 \end{cases} |\ln\epsilon|^{n(c-1)} \epsilon^{-N}) + CT\epsilon^s.$$

Finally we conclude that,

$$\|\tilde{D}^\gamma L_\epsilon(t,\cdot)\|_{L^\infty} \leqslant C\epsilon^s,\ \exists N>0,\ x \in \mathbb{R}^n,\ t \in [0,T],\ \epsilon < \epsilon_0.$$

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