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On the construction of symmetric nonnegative matrix with prescribed Ritz values

A. M. Nazari[a*], E. Afshari[b]

[a] *Department of Mathematics, Arak University, P.O. Box 38156-8-8349, Iran.*
[b] *Department of Mathematics, Khomein Branch, Islamic Azad University, Khomein, Iran.*

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Abstract. In this paper for a given prescribed Ritz values that satisfy in the some special conditions, we find a symmetric nonnegative matrix, such that the given set be its Ritz values.

Keywords: Ritz values, Nonnegative matrix.

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1. Introduction

The Ritz values of a $n \times n$ matrix are the eigenvalues of its leading principal submatrices of order $m = 1, 2, \ldots, n$. In the symmetric case, the Ritz values are real and interlacing, i.e. the ordered values $\lambda_i^{(j)}$ at level j and $\lambda_i^{(j+1)}$ at the next level $j+1$ satisfy

$$\lambda_1^{(j+1)} \leqslant \lambda_1^{(j)} \leqslant \lambda_2^{(j+1)} \leqslant \lambda_2^{(j)} \leqslant \ldots \leqslant \lambda_{j+1}^{(j+1)}. \tag{1}$$

Kostant and Wallach [1,2], construct the unique unit Hessenberg matrix H with given R_B, where R_B is the list of Ritz values. Beresford Parlett and Gilbert Strang [3] solved the problem for symmetric arrow and tridiagonal matrices.

*Corresponding author.
E-mail address: A-nazari@araku.ac.ir (A. M. Nazari).

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In the section 2 of this paper similar Theorem 2.1 of [4], we prove a theorem which can be used to construction a nonnegative matrix for a given set of Ritz values in a systematic method. The algorithm of this method is presented in section 3.

2. Main Result

We will use some standard basic concepts and results about square nonnegative matrices such as reducible, irreducible, Frobenius normal form of a reducible matrix, irreducible component and Frobenius Theorem about the spectral structure of an irreducible matrix as are described in [1].

Theorem 2.1 Let B be an $n \times n$ $(n \geqslant 2)$ irreducible symmetric nonnegative matrix and let

$$R_B = \{\lambda_1^{(1)}; \lambda_1^{(2)}, \lambda_2^{(2)}; ...; \lambda_1^{(n)}, \lambda_2^{(n)}, ..., \lambda_n^{(n)}\}$$

be the Ritz values of B that satisfy in the interlacing condition (1) for $j = 1, 2, \ldots, n-1$. If $\lambda_n^{(n)}$ is the Perron eigenvalue of B and $A = \begin{pmatrix} \lambda & a \\ a & \lambda_n^{(n)} \end{pmatrix}$ is a 2×2 nonnegative symmetric matrix with Ritz values $\{\mu_1; \mu_2, \mu_3\}$, then there exist an $(n+1) \times (n+1)$ nonnegative symmetric matrix $C = \begin{pmatrix} B & sa \\ s^T a & \delta \end{pmatrix}$ with the following Ritz values

$$M = \{\lambda_1^{(1)}; \lambda_1^{(2)}, \lambda_2^{(2)}; ...; \lambda_1^{(n)}, \lambda_2^{(n)}, ..., \lambda_n^{(n)}; \lambda_1^{(n)}, \lambda_2^{(n)}, ..., \lambda_{n-1}^{(n)}, \mu_2, \mu_3\},$$

where s is the normalized eigenvector corresponding to the Perron eigenvalue $\lambda_n^{(n)}$ and $\delta = \mu_3 + \mu_2 - \lambda_n^{(n)}$.

Proof. Let s be the normalized eigenvector associated to the Perron eigenvalue of B. We find the $n \times (n-1)$ matrix V_1 such that $Y_1 = (s, V_1)$ be unitary and $BY_1 = (\lambda_n^{(n)} s, BV_1)$. Therefore

$$B_1 = Y_1^* B Y_1 = \begin{pmatrix} \lambda_n^{(n)} ss^* & s^* BV_1 \\ \lambda_n^{(n)} V_1^* s & V_1^* BV_1 \end{pmatrix} = \begin{pmatrix} \lambda_n^{(n)} & * \; * \cdots * \\ 0 & \\ \vdots & \widehat{B} \\ 0 & \end{pmatrix},$$

where $\widehat{B} = V_1^* BV_1$ is an $(n-1) \times (n-1)$ matrix and $\{\lambda_1^{(1)}, ..., \lambda_{n-1}^{(n)}\}$ is set of eigenvalues of $\widehat{B}$. On the other hand, by Schur decomposition theorem there exist the unitary matrix V_2 of order $(n-1)$, such that $V_2^* \widehat{B} V_2 = \widehat{T}_B$, where $\widehat{T}_B$ is upper triangular matrix with the eigenvalues of $\widehat{B}$ on its main diagonal. If

$$Y_2 = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & V_2 \\ 0 & \end{pmatrix},$$

whereas V_2 is a unitary matrix, then Y_2 is also unitary. Thus

$$Y_2^* B_1 Y_2 = Y_2^*(Y_1^* B Y_1) Y_2 = (Y_1 Y_2)^* B (Y_1 Y_2) = Y^* B Y, \qquad (\text{if} \quad Y = Y_1 Y_2)$$

$$Y = Y_1 Y_2 = (s \quad V_1 V_2) = (s \quad T) \Longrightarrow Y^* = \begin{pmatrix} s^* \\ T^* \end{pmatrix}. \qquad (\text{if} \quad T = V_1 V_2)$$

Since T is a unitary matrix, we can write

$$YY^* = ss^* + TT^* = I_n, \qquad Y^* Y = \begin{pmatrix} s^* s & s^* T \\ T^* s & T^* T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \end{pmatrix}, \tag{2}$$

and

$$Y^* B Y = \begin{pmatrix} s^* B s & s^* B T \\ T^* B s & T^* B T \end{pmatrix} = \begin{pmatrix} \lambda_n^{(n)} & * \\ 0 & \widehat{T}_B \end{pmatrix} = T_B. \tag{3}$$

Consequently, T_B is an upper triangular matrix with eigenvalues of B on its main diagonal. On the other hand by Schur decomposition theorem there exist a unitary matrix X, such that $X^* A X = T_A$, where T_A is an upper triangular matrix so that eigenvalues of A $(\{\mu_2, \mu_3\})$ lie on its main diagonal. Assume that the matrices X and X^* have the following partitions:

$$X = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \qquad X^* = \begin{pmatrix} K_1^* & K_2^* \end{pmatrix},$$

where K_1 and K_2 are 1×2 vectors. Since X is a unitary matrix we have

$$\begin{gathered} XX^* = \begin{pmatrix} K_1 K_1^* & K_1 K_2^* \\ K_2 K_1^* & K_2 K_2^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ XX^* = K_1^* K_2 + K_2^* K_2 = I_2. \end{gathered} \tag{4}$$

By (4) we have

$$\begin{aligned} &T_A = X^* A X = K_1^* \lambda K_1 + K_2^* a K_1 + K_1^* a K_2 + \lambda_n^{(n)} K_2^* K_2 = \\ &\lambda K_1^* K_1 + a(K_2^* K_1 + K_1^* K_2) + \lambda_n^{(n)} K_2^* K_2. \end{aligned} \tag{5}$$

Now we consider two matrices Z and Z^* and a nonnegative matrix C of order $(n+2-1) \times (n+2-1)$ in the following forms:

$$Z = \begin{pmatrix} sK_2 & T \\ K_1 & 0 \end{pmatrix}, \ Z^* = \begin{pmatrix} K_2^* s^* & K_1^* \\ T^* & 0 \end{pmatrix}, \ C = \begin{pmatrix} B & sa \\ as^* & \delta \end{pmatrix}.$$

Using relations (2) and (4), it is easy to show that Z is a unitary matrix. Now by relations $(2)-(5)$, we can calculate Z^*CZ as follows:

$$Z^*CZ = \begin{pmatrix} K_1^*\lambda K_1 + a(K_2^*K_1 + K_1^*K_2) + \lambda_n^n K_2^* K_1^* & K_1^* a s^* T + K_2^* s^* BT \\ T^* s a K_1 + TBsK_2 & T^*BT \end{pmatrix} = \begin{pmatrix} T_A & \star \\ 0 & \hat{T}_B \end{pmatrix} = T_C,$$

where T_C is an upper triangular matrix and the elements of its main diagonal are the elements of the last level of M. On the other hand by the above relation, C and T_C are similar, therefore C solves the problem which completes the proof. □

3. An Algorithm for construction of a symmetric nonnegative matrix

Let

$$R_A = \{\lambda_1^{(1)}; \lambda_1^{(2)}, \lambda_2^{(2)}; ...; \lambda_1^{(n)}, \lambda_2^{(n)}, ..., \lambda_n^{(n)}\},$$

where $\lambda_1^{(j)}, \lambda_2^{(j)}, ..., \lambda_j^{(j)}$ are the eigenvalues of leading principal submatrix A_j for $j = 1, 2, \cdots, n$ and assume that R_A satisfies the interlacing condition (1). Assume the eigenvalues of A_{j+1} are only different with eigenvalues of A_j in $\lambda_j^{(j+1)}, \lambda_{j+1}^{(j+1)}$ where $\lambda_{j+1}^{(j+1)}$ is the Perron eigenvalue of A_{j+1} and $\lambda_i^{(j)} = \lambda_i^{(j+1)}$, for $i = 1, 2, ..., j-1$ and $j = 1, 2, ..., n$. By following algorithm we construct a nonnegative symmetric $n \times n$ matrix A with Ritz values R_A and leading principal submatrices.

step 1 **:** Let $A_1 = (\lambda_1^{(1)})$.

Step 2 **:** Let

$$A_2 = \begin{pmatrix} \lambda_1^{(1)} & \sqrt{(\lambda_2^{(2)} - \lambda_1^{(1)})(\lambda_1^{(1)} - \lambda_1^{(2)})} \\ \sqrt{(\lambda_2^{(2)} - \lambda_1^{(1)})(\lambda_1^{(1)} - \lambda_1^{(2)})} & \lambda_2^{(2)} + \lambda_1^{(2)} - \lambda_1^{(1)} \end{pmatrix}.$$

By interlacing condition (1) the elements of A_2 are nonnegative.

Step j **:** We construct a nonnegative symmetric matrix with prescribed Ritz values

$$\{\lambda_1^{(1)}; \lambda_1^{(2)}, \lambda_2^{(2)}; ...; \lambda_1^{(j)}, \lambda_2^{(j)}, ..., \lambda_j^{(j)}\}$$

that satisfy in interlacing condition (1) and $\lambda_1^{(j-1)}, ..., \lambda_{j-2}^{(j-1)}$ lies in the set of eigenvalues of A_j. At first we construct the 2×2 symmetric nonnegative matrix A' with Ritz values $\{\mu_1; \mu_2, \mu_3\}$, where

$$\mu_3 = \lambda_j^{(j)}, \ \mu_2 = \lambda_{j-1}^{(j)}, \qquad \mu_1 = \mu_2 + \mu_3 - \lambda_{j-1}^{(j-1)}$$

and $A' = \begin{pmatrix} \mu_1 & a \\ a & \lambda_{j-1}^{(j-1)} \end{pmatrix}$ and $a = \sqrt{(\mu_3 - \mu_1)(\mu_1 - \mu_2)}$. The condition on μ_1 is the solvability condition for this problem. Now by Theorem 2.1 we construct A_j by combining two matrices A_{j-1} and A' in the form $A_j = \begin{pmatrix} A_{j-1} & as \\ as^T & \mu_1 \end{pmatrix}$, where s is the eigenvector corresponding to the Perron eigenvector of A'.

***Example* 3.1** We construct a matrix with the following prescribed Ritz values $\{1; -1, 3; -1, 2, 4; -1, 2, 3, 5; -1, 2, 2, 4, 6\}$.
$A_1 = (1)$,
$$A_2 = \begin{pmatrix} \lambda_1^{(1)} & a \\ a & \delta_1 \end{pmatrix},$$
where
$$a = \sqrt{(\lambda_1^{(1)} - \lambda_1^{(2)})(\lambda_2^{(2)} - \lambda_1^{(1)})} = 2\,,$$
$$\delta_1 = \lambda_2^{(2)} + \lambda_1^{(2)} - \lambda_1^{(1)} = 1\,.$$
The construction of the matrix A_3 needs the normalized eigenvector to associated eigenvalue 3. This vector is
$$s = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$
We must construct 2×2 matrix A' with Ritz values $\{\mu_1; \mu_2, \mu_3\} = \{\mu_1; 2, 4\}$, so that $\lambda_2^{(2)} = 3$ is lies on the main diagonal of A' .
$$A' = \begin{pmatrix} \mu_1 & a \\ a & \lambda_2^{(2)} \end{pmatrix},$$
where
$$\mu_1 = \mu_2 + \mu_3 - \lambda_2^{(2)} = 3\,,$$
$$a = \sqrt{(\mu_1 - \mu_2)(\mu_3 - \mu_1)} = 1\,.$$
Now by Theorem 2.1 , we combine A' and A_2 to construct A_3 as
$$A_3 = \begin{pmatrix} 1 & 2 & \sqrt{2}/2 \\ 2 & 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 3 \end{pmatrix}.$$

Similarly we can construct the matrices A_4 and A_5 .
$$A_4 = \begin{pmatrix} 1 & 2 & \sqrt{2}/2 & 1/2 \\ 2 & 1 & \sqrt{2}/2 & 1/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 3 & 1/\sqrt{2} \\ 1/2 & 1/2 & 1/\sqrt{2} & 4 \end{pmatrix},$$
$$A_5 = \begin{pmatrix} 1 & 2 & \sqrt{2}/2 & 1/2 & \sqrt{2}/4 \\ 2 & 1 & \sqrt{2}/2 & 1/2 & \sqrt{2}/4 \\ \sqrt{2}/2 & \sqrt{2}/2 & 3 & 1/\sqrt{2} & 1/2 \\ 1/2 & 1/2 & 1/\sqrt{2} & 4 & \sqrt{2}/2 \\ \sqrt{2}/4 & \sqrt{2}/4 & 1/2 & \sqrt{2}/2 & 5 \end{pmatrix}.$$

References

[1] Bertram Kostant, Nolan Wallach, GelfandZeitlin theory from the perspective of classical mechanics I, Prog. Math. 243 (2006) 319-364.

[2] Bertram Kostant, Nolan Wallach, GelfandZeitlin theory from the perspective of classical mechanics II, Prog. Math. 244 (2006) 387-420.

[3] Beresford Parlett, Gilbert Strang, Matrices with prescribed Ritz values, Linear Algebra and its Applications 428 (2008) 1725-1739.

[4] A. M. Nazari, F. Sherafat, On the inverse eigenvalue problem for nonnegative matrices of order two to five, Linear Algebra Appl. 436 (2012) 1771-1790.