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Numerical solution of Fredholm integral-differential equations on unbounded domain

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Abstract. In this study, a new and efficient approach is presented for numerical solution of Fredholm integro-differential equations (FIDEs) of the second kind on unbounded domain with degenerate kernel based on operational matrices with respect to generalized Laguerre polynomials(GLPs). Properties of these polynomials and operational matrices of integration, differentiation are introduced and are ultilized to reduce the (FIDEs) to the solution of a system of linear algebraic equations with unknown generalized Laguerre coefficients. In addition, two examples are given to demonstrate the validity, efficiency and applicability of the technique.

Keywords: Fredholm integro-differential equations, unbounded domain, generalized Laguerre polynomials, operational matrices.

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1. Introduction

The main object of this paper is to approximate the solution of Fredholm integro-differential equation of the second kind on unbounded domain of the following form:

$$\begin{cases} f'(x) - (\vartheta\ f)(x) = g(x), \\ f(0) = f_0, \end{cases} \tag{1}$$

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$$(\vartheta\ f)(x) := \rho \int_0^\infty w(t)k(x,t)f(t)dt, \quad x \in \mathbb{R}_+, \tag{2}$$

where $\rho \in \mathbb{R}$, $w(t) = t^\alpha e^{-t}(\alpha > -1)$ and $g(x)$ is continuous function and the kernel $k(x,t)$ might has singularity in the region $D = \{(x,t) : 0 \leqslant x, t < \infty\}$, and $f(x)$ is the unknown function which to be determined. The considered equation arise in a number of important problem of elasticity theory, neutron transport, particle scattering and the theory of mixed-type equations [1–4]. The analytical solutions of some FIDEs cannot be found, thus numerical methods are required. It's the reason of great interest for solving these equations. Moreover many researchers have developed the approximate method to solve infinite boundary integral equation using Galerkin and Collocation methods with Laguerre and Hermite polynomials as a bases function [5, 6]. However, method of solution for Eq. (1) is too rear in the literature. In the present work, we are going to use the operational matrices of generalized Laguerre polynomials to find the approximate solution of the FIDEs on unbounded domain. Our approach in the current paper is different. The organization of this paper is as follows: In section 2, we describe the basic formulation and give some relevant properties of the GPLs which is required for our subsequent development. Section 3 is devoted to the approximate of the function $g(x)$ and also the kernel function $k(x,t)$ by using GPLs basis. Also the upper bound of the approximation error is presented. In Section 4 we obtain the operational matrices of integration and differentiation by GPLs. In Section 5, the approximate solution of the Fredholm integral-differential equations on unbounded domain using generalized Laguerre polynomials basis is presented. In Section 6, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples. Finally, we conclude the article in Section 7.

2. The generalized Laguerre polynomials

In this part, for the reader's convenience, we present some necessary definitions which are used further in this paper.
Let $\mathbb{R}_+ := \Lambda = \{x : 0 \leqslant x < \infty\} = [0, \infty)$ and $w^{(\alpha)}(x) = x^\alpha e^{-x}$ be a weight function on Λ in the usual sense. We define the following:

$$L^2_{w^{(\alpha)}}(\Lambda) = \{v : v \ \ \textit{is measurable on} \ \ \Lambda \ \ \textit{and} \ \ \|v\|_{w^{(\alpha)}} < \infty\}, \tag{3}$$

equipped with the following inner product and norm:

$$(u,v)_{w^{(\alpha)}} = \int_\Lambda u(x)v(x)w^{(\alpha)}(x)dx, \quad \|v\|_{w^{(\alpha)}} = (v,v)^{\frac{1}{2}}_{w^{(\alpha)}}. \tag{4}$$

Next, suppose $L_n^{(\alpha)}(x)$ be the generalized Laguerre polynomials of degree n, defined by the following:

$$L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^x \partial_x^n \left(e^{-x} x^{n+\alpha}\right), \qquad n = 0, 1, ... \tag{5}$$

$L_n^{(\alpha)}(x)$(generalized Laguerre polynomials) are the nth eigenfunction of the Sturm-Liouville problem:

$$x^{-\alpha}e^{x}\left(x^{\alpha+1}e^{-x}\left(L_n^{(\alpha)}(x)\right)'\right)' + \lambda_n L_n^{(\alpha)}(x) = 0, \quad x \in \Lambda, \tag{6}$$

with the eigenvalues $\lambda_n = n$[7, 8].
Generalized Laguerre polynomials are orthogonal in $L^2_{w^{(\alpha)}}(\Lambda)$ Hilbert space with the weight function $w^{(\alpha)}(x) = x^{\alpha}e^{-x}$ satisfy in the following relation

$$\int_0^{\infty} x^{\alpha}e^{-x}L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)dx = \gamma_n^{\alpha}\delta_{n,m}, \qquad \forall n,m \geqslant 0, \tag{7}$$

where $\delta_{n,m}$ is the Kronecher delta function and $\gamma_n^{\alpha} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}$. The explicit form of these polynomials is in the form

$$L_n^{(\alpha)}(x) = \sum_{i=0}^{n} E_i^{\alpha}x^i, \tag{8}$$

where

$$E_i^{\alpha} = \frac{\binom{n+\alpha}{n-i}(-1)^i}{i!}. \tag{9}$$

These polynomials are satisfied in the following three terms recurrence formula

$$\begin{aligned} &L_0^{(\alpha)}(x) = 1, \quad L_1^{(\alpha)}(x) = 1+\alpha-x, \\ &L_{n+1}^{(\alpha)}(x) = \frac{1}{n+1}\left[(2n+\alpha+1-x)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x)\right], \; n = 1,2,.... \end{aligned} \tag{10}$$

The case $\alpha = 0$ leads to the classical Laguerre polynomials, which are used most frequently in practice and will simply be denoted by $L_n(x)$. An important property of the Laguerre polynomials is the following derivative relation [9]:

$$\left(L_n^{(\alpha)}(x)\right)' = \sum_{i=0}^{n-1} L_i^{(\alpha)}(x). \tag{11}$$

Further, $\left(L_i^{(\alpha)}(x)\right)^{(k)}$ are orthogonal with respect to the weight function $w^{(\alpha+k)}(x)$. i.e.

$$\int_0^{\infty} (L_i^{(\alpha)})^{(k)}(x)(L_j^{(\alpha)})^{(k)}(x)w^{(\alpha+k)}(x)dx = \gamma_{n-k}^{\alpha+k}\delta_{i,j}, \qquad \forall i,j \geqslant 0, \tag{12}$$

where $\gamma_{n-k}^{\alpha+k}$ is defined in (7).

3. Approximation of functions by using GLPs

A function $g(x) \in L^2_{w^{(\alpha)}}(\Lambda)$ may be expressed in terms of generalized Laguerre polynomials as:

$$g(x) = \sum_{i=0}^{\infty} g_i^{(\alpha)} L_i^{(\alpha)}(x), \tag{13}$$

where the generalized Laguerre coefficients $g_i^{(\alpha)}$ are given by

$$g_i^{(\alpha)} = \int_0^{\infty} \frac{L_i^{(\alpha)}(x)}{\binom{i+\alpha}{i}} \cdot \frac{x^{\alpha} e^{-x}}{\Gamma(\alpha+1)} \cdot g(x)dx, \qquad i = 0, 1, ... \tag{14}$$

The series converges in the associated Hilbert space $L^2_{w^{(\alpha)}}(\Lambda)$, iff

$$\|g\|^2_{L^2} := \int_0^{\infty} \frac{x^{\alpha} e^{-x}}{\Gamma(\alpha+1)} |g(x)|^2 dx = \sum_{i=0}^{\infty} \binom{i+\alpha}{i} |g_i^{(\alpha)}|^2 < \infty. \tag{15}$$

In practice, only the first $(n+1)$ terms of generalized Laguerre polynomials are considered. Then we have

$$g(x) \simeq \sum_{i=0}^{n} g_i^{(\alpha)} L_i^{(\alpha)}(x) = G^T L_x, \tag{16}$$

where the generalized Laguerre coefficient vector G and the generalized Laguerre vector L_x are given by as follows:

$$G = [g_0^{(\alpha)},\ g_1^{(\alpha)}, \ldots,\ g_n^{(\alpha)}]^T, \ \ and \ \ L_x = [L_0^{(\alpha)}(x),\ L_1^{(\alpha)}(x), \ldots,\ L_n^{(\alpha)}(x)]^T. \tag{17}$$

Now in the following lemma we present an upper bound to estimate the error.

Theorem 3.1 Suppose that the function $g : [0, \infty) \to \mathbb{R}$ is $n+1$ times continuously differentiable (i.e. $g \in C^{n+1}[0, \infty)$), and $Y = Span\{L_0^{(\alpha)}(x),\ L_1^{(\alpha)}(x), \ldots,\ L_n^{(\alpha)}(x)\}$. If $G^T L_x$ be the best approximation g out of Y then mean error bound is presented as follows:

$$\|g - G^T L_x\|_{L^2_{w^{(\alpha)}}(\Lambda)} \leqslant \frac{N\sqrt{(2n+\alpha+2)!}}{(n+1)!}, \tag{18}$$

where $N = \max_{x\in\Lambda} |g^{(n+1)}(x)|$.

Proof. We know that set $\{1, x, ..., x^n\}$ is a basis for polynomials space of degree n. Therefore we define $y_1(x) = g(0) + xg'(0) + \frac{x^2}{2!}g''(0) + \cdots + \frac{x^n}{n!}g^{(n)}(0)$. From Taylor expansion we have

$$|g(x) - y_1(x)| \leqslant |g^{(n+1)}(\eta_x) \frac{x^{n+1}}{(n+1)!}|, \tag{19}$$

where $\eta_x \in (0,\infty)$. Since $G^T L_x$ is the best approximation g out of Y, $y_1 \in Y$ and using (19) we have

$$\|g - G^T L_x\|^2_{L^2_{w^{(\alpha)}}(\Lambda)} \leqslant \|g - y_1\|^2_{L^2_{w^{(\alpha)}}(\Lambda)} = \int_0^\infty x^\alpha e^{-x}|g(x) - y_1(x)|^2 dx$$
$$\leqslant \frac{N^2(2n+\alpha+2)!}{(n+1)!^2}. \tag{20}$$

Then by taking square roots we have the above bound. The previous Lemma shows that the error vanishes as $n \to \infty$. ■

We can also approximate the function of two variables, $k(x,t) \in L^2_{w^{(\alpha)}}(\Lambda^2)$ as follows:

$$k(x,t) \simeq \sum_{i=0}^{n}\sum_{j=0}^{n} L_i^{(\alpha)}(x) k_{ij}^{(\alpha)} L_j^{(\alpha)}(t) = L_x^T K L_t. \tag{21}$$

Here the entries of matrix $K = [k_{ij}^{(\alpha)}]_{(n+1)\times(n+1)}$ will be obtained by

$$k_{ij}^{(\alpha)} = \frac{(L_i^{(\alpha)}(x), (k(x,t), L_j^{(\alpha)}(t)))}{(L_i^{(\alpha)}(x), L_i^{(\alpha)}(x))(L_j^{(\alpha)}(t), L_j^{(\alpha)}(t))}, \qquad for \quad i,j = 0,1,...,n. \tag{22}$$

where $(.,.)$ denotes the inner product.

4. The operational matrices

The main objective of this section is to obtain the operational matrices of the integration and differentiation by GPLs.

Theorem 4.1 Suppose L_x be the generalized Laguerre vector defined in (17) then

$$\int_0^x L_t dt \simeq P L_x, \tag{23}$$

where P is the $(n+1)\times(n+1)$ operational matrix for integration as follows:

$$P = \begin{bmatrix} \Omega(0,0,\alpha) & \Omega(0,1,\alpha) & \Omega(0,2,\alpha) & \cdots & \Omega(0,n,\alpha) \\ \Omega(1,0,\alpha) & \Omega(1,1,\alpha) & \Omega(1,2,\alpha) & \cdots & \Omega(1,n,\alpha) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega(i,0,\alpha) & \Omega(i,1,\alpha) & \Omega(i,2,\alpha) & \cdots & \Omega(i,n,\alpha) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega(n,0,\alpha) & \Omega(n,1,\alpha) & \Omega(n,2,\alpha) & \cdots & \Omega(n,n,\alpha) \end{bmatrix}, \tag{24}$$

where

$$\Omega(i,j,\alpha) = \sum_{k=0}^{i}\sum_{r=0}^{j} \frac{(-1)^{k+r} j!\Gamma(i+\alpha+1)\Gamma(k+\alpha+r+2)}{(i-k)!(j-r)!(k+1)!r!\Gamma(k+\alpha+1)\Gamma(r+\alpha+1)}. \tag{25}$$

Proof. The analytic form of the generalized Laguerre polynomials $L_i^{(\alpha)}(x)$ of degree i on Λ, is given as follows:

$$L_i^{(\alpha)}(x) = \sum_{k=0}^{i} (-1)^k \frac{\Gamma(i+\alpha+1)}{\Gamma(k+\alpha+1)(i-k)!k!} x^k, \quad i = 0, 1, ..., \tag{26}$$

where $L_0^{(\alpha)}(x) = 1$. Using Eq.(26), and since the integration is a linear operation, we get the following:

$$\begin{aligned}\int_0^x L_i^{(\alpha)}(t)dt &= \sum_{k=0}^{i} (-1)^k \frac{\Gamma(i+\alpha+1)}{(i-k)!k!\Gamma(k+\alpha+1)} \int_0^x t^k dt \\ &= \sum_{k=0}^{i} (-1)^k \frac{\Gamma(i+\alpha+1)}{(i-k)!(k+1)!\Gamma(k+\alpha+1)} x^{k+1}, \ i = 0, ..., n. \end{aligned} \tag{27}$$

Now, by approximating x^{k+1} by the $n+1$ terms of the generalized Laguerre series, we have the following:

$$x^{k+1} = \sum_{j=0}^{n} b_j L_j^{(\alpha)}(x), \tag{28}$$

where b_j is given from Eq. (14) with $g(x) = x^{k+1}$, that is,

$$b_j = \sum_{r=0}^{j} \frac{(-1)^r j!\Gamma(k+\alpha+r+2)}{(j-r)!r!\Gamma(r+\alpha+1)}, \quad j = 0, 1, ..., n. \tag{29}$$

In virtue of Eqs. (27) and (28), we get the following:

$$\int_0^x L_i^{(\alpha)}(t)dt = \sum_{j=0}^{n} \Omega(i, j, \alpha) L_j^{(\alpha)}(t), \quad i = 0, 1, ..., n, \tag{30}$$

where

$$\Omega(i, j, \alpha) = \sum_{k=0}^{i} \sum_{r=0}^{j} \frac{(-1)^{k+r} j!\Gamma(i+\alpha+1)\Gamma(k+\alpha+r+2)}{(i-k)!(j-r)!(k+1)!r!\Gamma(k+\alpha+1)\Gamma(r+\alpha+1)}. \tag{31}$$

Accordingly, Eq. (30) can be written in a vector form as follows:

$$\int_0^x L_i^{(\alpha)}(t)dt \simeq [\Omega(i, 0, \alpha), \Omega(i, 1, \alpha), ..., \Omega(i, n, \alpha)] L_x, \quad i = 0, 1, ..., n. \tag{32}$$

Eq. (32) leads to the desired result. ■

Theorem 4.2 Let L_x is the generalized Laguerre vector defined in (17) then

$$\frac{d}{dx} L_x \simeq D L_x, \tag{33}$$

where D is the $(n+1)\times(n+1)$ operational matrix for differentiation as follows:

$$D=\begin{bmatrix} D(0,0,\alpha) & D(0,1,\alpha) & D(0,2,\alpha) & \cdots & D(0,n,\alpha)\\ D(1,0,\alpha) & D(1,1,\alpha) & D(1,2,\alpha) & \cdots & D(1,n,\alpha)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ D(i,0,\alpha) & D(i,1,\alpha) & D(i,2,\alpha) & \cdots & D(i,n,\alpha)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ D(n,0,\alpha) & D(n,1,\alpha) & D(n,2,\alpha) & \cdots & D(n,n,\alpha) \end{bmatrix}, \tag{34}$$

where

$$D(i,j,\alpha)=\sum_{k=0}^{i}\sum_{r=0}^{j}\frac{(-1)^{k+r}j!\Gamma(i+\alpha+1)\Gamma(k+\alpha+r)}{(i-k)!(j-r)!(k-1)!r!\Gamma(k+\alpha+1)\Gamma(r+\alpha+1)}. \tag{35}$$

Proof. Applying the same procedure as in the previous theorem we arrive to (35). ■

5. Solution of the Fredholm integral-differential equations on unbounded domain

In this section, we consider (FIDE) of the second kind in (1) and approximate to solution by means of finite generalized Laguerre series defined in (16). The aim is to find generalized Laguerre coefficients, we approximate functions $g(x)$, $k(x,t)$ and $f'(x)$ with respect to (GPLs) by the way mentioned in before sections as follows:

$$g(x)\simeq G^TL_x,\quad f'(x)\simeq F'^TL_x,\quad f(0)\simeq F_0^TL_x,\quad k(x,t)\simeq L_x^TKL_t, \tag{36}$$

where L_x is defined in (17), the vectors G^T, F'^T, and matrix K are generalized Laguerre coefficients of $g(x)$, $f'(x)$, and $k(x,t)$, respectively. Then

$$\begin{aligned} f(x)&=\int_0^x f'(t)dt+f(0)\simeq\int_0^x F'^TL_tdt+F_0^TL_x\\ &\simeq F'^TPL_x+F_0^TL_x=(F'^TP+F_0^T)L_x, \end{aligned} \tag{37}$$

where P is a $(n+1)\times(n+1)$ matrix given in (23). With substituting the approximations (36) and (37) into equation (1), we have:

$$\begin{aligned} L_x^TF'&=L_x^TG+\rho\int_0^\infty t^\alpha e^{-t}L_x^TKL_tL_t^T(P^TF'+F_0)dt\\ &=L_x^TG+\rho L_x^TK\int_0^\infty t^\alpha e^{-t}L_tL_t^Tdt(P^TF'+F_0)\\ &=L_x^TG+\rho L_x^TKQ(P^TF'+F_0), \end{aligned} \tag{38}$$

then we have the following linear system:

$$(I-\rho KQP^T)F'=G+\rho KQF_0, \tag{39}$$

where

$$Q = \int_0^\infty t^\alpha e^{-t} L_t L_t^T \, dt = [q_{ij}^{(\alpha)}], \qquad i,j = 0,1,...,n, \tag{40}$$

and I is the unit matrix, and Q is a $(n+1)\times(n+1)$ matrix with elements

$$q_{ij}^{(\alpha)} = \int_0^\infty t^\alpha e^{-t} L_i^{(\alpha)}(t) L_j^{(\alpha)}(t) dt, \qquad i,j = 0,1,...,n. \tag{41}$$

E.q (39) is a linear system of algebraic equations that can be easily solved by direct or iterative methods. In equation (39), if $D(\rho,\alpha) = |I - \rho KQP^T| \neq 0$ we get

$$F' = (I - \rho KQP^T)^{-1}(G + \rho KQF_0), \qquad \rho \neq 0. \tag{42}$$

We can find the vector F', so

$$F^T = F'^T P + F_0^T \Longrightarrow f(x) \simeq F^T L_x \tag{43}$$

Remark 1 *$D(\rho,\alpha)$ is a polynomial in ρ of degree at most $n+1$, $D(\rho,\alpha)$ is not identically zero, since when ρ=0, $D(\rho,\alpha) = 1$.*

6. Numerical Examples

To demonstrate the effectiveness of the proposed method in the present paper, two test examples are carried out in this section. For each example we find the approximate solutions using different degree of generalized Laguerre polynomials. The results obtained by the present methods reveal that the present method is very effective and convenient for equation (1) on the half line. In all examples the package of Matlab (2013) has been used to solve the test problems considered in this paper.

***Example* 6.1** For the first example, consider the following of Fredholm integral-differential equation on unbounded domain (constructed):

$$f'(x) = -\frac{247131410303000045}{36028797018963968}x^2 - \frac{38903199231847830919}{144115188075855872} + \int_0^\infty t^{\frac{1}{2}} e^{-t}(x^2+t^2) f(t) dt, \quad f(0) = 1. \tag{44}$$

Exact solution of this problem is $f(x) = x^3 - 2x + 1$. If we apply the technique described in the section 5, with $\alpha = \frac{1}{2}$ and $n = 3$, then the approximate solution can be written as follows:

$$f(x) \simeq \sum_{i=0}^{3} f_i^{(\alpha)} \, L_i^{(\alpha)}(x) = F^T L_x, \tag{45}$$

where

$$F = [f_0^{(\alpha)},\ f_1^{(\alpha)},\ f_2^{(\alpha)},\ f_3^{(\alpha)}]^T. \tag{46}$$

Hence, from Eqs. (16), (21), (23), and (40), we find the matrices

$$G=\begin{bmatrix}-125363/424\\17011/496\\-6434/469\\0\end{bmatrix},\ K=\begin{bmatrix}15/2&-5&2&0\\-5&0&0&0\\2&0&0&0\\0&0&0&0\end{bmatrix},\ P=\begin{bmatrix}3/2&-1&0&0\\3/8&1&-1&0\\5/16&0&1&-1\\35/128&0&0&1\end{bmatrix}$$
$$,Q=\begin{bmatrix}148/167&0&0&0\\0&222/167&0&0\\0&0&555/334&0\\0&0&0&2053/1059\end{bmatrix}.$$

Next, we substitute these matrices into equation (42) and then simplify to obtain

$$\begin{bmatrix}f_0^{(\alpha)'}\\f_1^{(\alpha)'}\\f_2^{(\alpha)'}\\f_3^{(\alpha)'}\end{bmatrix}=\begin{bmatrix}-119/5475&-19/7262&-574/2251&-865/5409\\161/2349&347/1578&93/632&-247/3501\\-181/6602&552/1769&1487/1580&193/6839\\0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}-13873/48\\4211/141\\-6427/538\\0\end{bmatrix}\tag{47}$$

By solving the linear system (Equation (47)), we have the following:

$$f_0^{(\alpha)'}=\frac{37}{4},\ f_1^{(\alpha)'}=-15,\ f_2^{(\alpha)'}=6,\ f_3^{(\alpha)'}=0.\tag{48}$$

By substituting the obtained coefficients in (43) the solution of (44) becomes

$$f(x)\simeq\frac{89}{8}L_0^{(\alpha)}(x)-\frac{97}{4}L_1^{(\alpha)}(x)+21L_2^{(\alpha)}(x)-6L_3^{(\alpha)}(x),\tag{49}$$

or briefly

$$f(x)\simeq x^3-2x+1,\tag{50}$$

which is the exact solution. Also, if we choose $n\geqslant 4$, we get the same approximate solution as obtained in equation (50). Numerical results will not be presented since the exact solution is obtained.

Example **6.2** As the second example, consider the following of Fredholm integral-differential equation on a semi infinite interval (constructed):

$$f'(x)=e^{-x}-\frac{7}{4}\sqrt{x}+\int_0^\infty t^{\frac{1}{2}}e^{-t}\sqrt{xt}f(t)dt,\quad f(0)=1.\tag{51}$$

With the exact solution $f(x)=2-e^{-x}$. We apply the generalized Laguerre series approach and solve Eq. (51). Table 1 shows the absolute values of error $|e|=|f_{exact.}(x)-f_{app.}(x)|$ for $n=10$, and $n=12$ using the present method in equally divided interval $[0,1]$.

Absolute error for Example 6.2

i	x_i	$n = 10$	$n = 12$
0	0.0	$1.3000e-003$	$3.6116e-004$
1	0.1	$4.4620e-004$	$9.3426e-005$
2	0.2	$7.1845e-005$	$4.5500e-005$
3	0.3	$3.2705e-004$	$9.9666e-005$
4	0.4	$4.0407e-004$	$1.0182e-004$
5	0.5	$3.6812e-004$	$7.5666e-005$
6	0.6	$2.6834e-004$	$3.7765e-005$
7	0.7	$1.4076e-004$	$8.8208e-007$
8	0.8	$1.0787e-005$	$3.3527e-005$
9	0.9	$1.0468e-004$	$5.6602e-005$
10	1.0	$1.9540e-004$	$6.8843e-005$

7. Conclusion

Finding analytical-numerical solutions for Fredholm integral-differential equations on unbounded domain of the second kind are usually difficult, and therefore approximating these solutions is very important. In this article, we develop an efficient and powerful method for solving Fredholm integral-differential equations of the second kind along with initial condition on a semi-infinite domain by using of generalized Laguerre polynomials. By some useful properties of these polynomials such as, operational matrix, orthogonal basis, a (FIDE) can be transformed to a linear system of algebraic equations wherein the matrix of unknown coefficients is sparse and can be easily invertible. Therefore, the reduction of the volume of calculations and runtime of the method can be observed. The illustrations show that the proposed technique produces satisfactory results and yields the desired accuracy only in a few terms with high accuracy.

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