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The solutions to some operator equations in Hilbert C^*-module

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Abstract. In this paper, we state some results on product of operators with closed ranges and we solve the operator equation $TXS^* - SX^*T^* = A$ in the general setting of the adjointable operators between Hilbert C^*-modules, when $TS = 1$. Furthermore, by using some block operator matrix techniques, we find explicit solution of the operator equation $TXS^* - SX^*T^* = A$.

Keywords: Operator equation; Moore-Penrose inverse; Complemented submodule, Closed range, Hilbert C^*-module.

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1. Introduction and preliminaries

The equation $TXS^* - SX^*T^* = A$ was studied by Yuan [13] for finite matrices and Xu et al. [12] generalized the results to Hilbert C^*-modules, under the condition that ran(S) is contained in ran(T). When T equals an identity matrix or identity operator, this equation reduces to $XS^* - SX^* = A$, which was studied by Braden [2] for finite matrices, and Djordjevic [3] for the Hilbert space operators. In this paper, we state some results of product of operators with closed ranges, therefore we solve the operator equation $TXS^* - SX^*T^* = A$, when $TS = 1$. Furthermore, by using some block operator matrix techniques, we find explicit solution of the operator equation $TXS^* - SX^*T^* = A$ in the general setting of the adjointable operators between Hilbert C^*-modules.

Hilbert C^*-modules are objects like Hilbert spaces, except that the inner product take its values in a C^*-algebra, instead of being complex-valued. Throughout the paper $\mathcal{A}$ is a

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C*-algebra (not necessarily unital). A (right) pre-Hilbert module over a C^*-algebra $\mathcal{A}$ is a complex linear space $\mathcal{X}$, which is an algebraic right $\mathcal{A}$-module and $\lambda(xa) = (\lambda x)a = x(\lambda a)$ equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying,

(i) $\langle x, x \rangle \geqslant 0$, and $\langle x, x \rangle = 0$ iff $x = 0$,
(ii) $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle$,
(iii) $\langle x, ya \rangle = \langle x, y \rangle a$,
(iv) $\langle y, x \rangle = \langle x, y \rangle^*$.

for each $x, y, z \in \mathcal{X},\ \lambda \in \mathbb{C},\ a \in \mathcal{A}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called a Hilbert $\mathcal{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$. Left Hilbert $\mathcal{A}$-modules are defined in a similar way. For example every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with respect to inner product $\langle x, y \rangle = x^*y$, and every inner product space is a left Hilbert $\mathbb{C}$-module.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. Then, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the set of all maps $T : \mathcal{X} \to \mathcal{Y}$ for which there is a map $T^* : \mathcal{Y} \to \mathcal{X}$, the so-called adjoint of T such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for each $x \in \mathcal{X}$, $y \in \mathcal{Y}$. It is known that any element T of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa) = (Tx)a$ for $x \in \mathcal{X}$ and $a \in \mathcal{A}$ [7, Page 8]. We use the notations $\mathcal{L}(\mathcal{X})$ in place of $\mathcal{L}(\mathcal{X}, \mathcal{X})$, and $\ker(\cdot)$ and $\mathrm{ran}(\cdot)$ for the kernel and the range of operators, respectively. The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$ or 1 if there is no ambiguity.

Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $\mathcal{Y}$ is a closed submodule of $\mathcal{X}$. We say that $\mathcal{Y}$ is orthogonally complemented if $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}^{\perp}$, where $\mathcal{Y}^{\perp} := \{y \in \mathcal{X} :\ \langle x, y \rangle = 0 \ \text{ for } \ \text{all } x \in \mathcal{Y}\}$ denotes the orthogonal complement of $\mathcal{Y}$ in $\mathcal{X}$. The reader is referred to [5–7] and the references cited therein for more details.

Throughout this paper $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. Recall that a closed submodule in a Hilbert module is not necessarily orthogonally complemented, however Lance proved that certain submodules are orthogonally complemented as follows.

Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. A bounded adjointable operator $S \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ is called an inner inverse of T if $TST = T$. If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has an inner inverse S, then the bounded adjointable operator $T^\times = STS$ in $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ satisfies

$$TT^\times T = T, \quad T^\times TT^\times = T^\times. \tag{1}$$

The bounded adjointable operator $T^\times$ which satisfies (1) is called generalized inverse of T. It is known that a bounded adjointable operator T has a generalized inverse if and only if ran(T) is closed, see e.g. [1].

Theorem 1.1 [7, Theorem 3.2] Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then

- $\ker(T)$ is orthogonally complemented in $\mathcal{X}$, with complement ran(T*).
- ran(T) is orthogonally complemented in $\mathcal{Y}$, with complement $\ker(T^*)$.
- The map $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ has closed range.

Definition 1.2 Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The Moore-Penrose inverse $T^\dagger$ of T (if it exists) is an element in $L(\mathcal{Y}, \mathcal{X})$ which satisfies:

(i) $T\,T^\dagger T = T$,
(ii) $T^\dagger\,T\,T^\dagger = T^\dagger$,
(iii) $(T\,T^\dagger)^* = T\,T^\dagger$,
(iv) $(T^\dagger\,T)^* = T^\dagger T$.

Motivated by these conditions $T^\dagger$ is unique and $T^\dagger T$ and $T\,T^\dagger$ are orthogonal projections, in the sense that they are selfadjoint idempotent operators. Clearly, T is Moore-Penrose invertible if and only if T^* is Moore-Penrose invertible, and in this case $(T^*)^\dagger = (T^\dagger)^*$. The following theorem is known.

Theorem 1.3 [11, Theorem 2.2] Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then the Moore-Penrose inverse $T^\dagger$ of T exists if and only if T has closed range.

By Definition 1.2, we have

$$\begin{aligned} \mathrm{ran(T)} &= \mathrm{ran(T\,T^\dagger)}, & \mathrm{ran(T^\dagger)} &= \mathrm{ran(T^\dagger T)} = \mathrm{ran(T^*)}, \\ \ker(T) &= \ker(T^\dagger T), & \ker(T^\dagger) &= \ker(T\,T^\dagger) = \ker(T^*), \end{aligned}$$

and by Theorem 1.1, we have

$$\begin{aligned} \mathcal{X} &= \ker(T) \oplus \mathrm{ran(T^\dagger)} = \mathrm{ker(T^\dagger T)} \oplus \mathrm{ran(T^\dagger T)}, \\ \mathcal{Y} &= \ker(T^\dagger) \oplus \mathrm{ran(T)} = \mathrm{ker(T\,T^\dagger)} \oplus \mathrm{ran(T\,T^\dagger)}. \end{aligned}$$

A matrix form of a bounded adjointable operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ can be induced by some natural decompositions of Hilbert C^*-modules. Indeed, if $\mathcal{M}$ and $\mathcal{N}$ are closed orthogonally complemented submodules of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp, \quad \mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp$, then T can be written as the following 2×2 matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \tag{2}$$

where, $T_1 \in \mathcal{L}(\mathcal{M},\mathcal{N})$, $T_2 \in \mathcal{L}(\mathcal{M}^\perp,\mathcal{N})$, $T_3 \in \mathcal{L}(\mathcal{M},\mathcal{N}^\perp)$ and $T_4 \in \mathcal{L}(\mathcal{M}^\perp,\mathcal{N}^\perp)$. Note that $P_\mathcal{M}$ denotes the projection corresponding to $\mathcal{M}$.

In fact $T_1 = P_\mathcal{N} T P_\mathcal{M}, \quad T_2 = P_\mathcal{N} T(1-P_\mathcal{M}), T_3 = (1-P_\mathcal{N})TP_\mathcal{M}, \quad T_4 = (1-P_\mathcal{N})T(1-P_\mathcal{M})$.

Recall that if $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range, then $TT^\dagger = P_{\mathrm{ran(T)}}$ and $T^\dagger T = P_{\mathrm{ran(T^*)}}$.

The proof of the following Lemma can be found [9, Corollary 1.2.] or [4, Lemma 1.1.].

Lemma 1.4 Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then T has the following matrix decomposition with respect to the orthogonal decompositions of closed submodules $\mathcal{X} = \mathrm{ran(T^*)} \oplus \mathrm{ker(T)}$ and $\mathcal{Y} = \mathrm{ran(T)} \oplus \mathrm{ker(T^*)}$:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix}$$

where T_1 is invertible. Moreover

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix}$$

Lemma 1.5 (see [10, Lemma 1.2.]) Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Let $\mathcal{X}_1$, $\mathcal{X}_2$ be closed submodules of $\mathcal{X}$ and $\mathcal{Y}_1$, $\mathcal{Y}_2$ be closed submodules of $\mathcal{Y}$ such that $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$. Then the operator T has the following matrix representations with respect to the orthogonal sums of submodules $\mathcal{X} = \mathrm{ran(T^*)} \oplus \mathrm{ker(T)}$ and $\mathcal{Y} =$

ran(T) $\oplus$ ker(T*):

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix}$$

Then $D = T_1T_1^* + T_2T_2^* \in \mathcal{L}(\mathrm{ran(T)})$ is positive and invertible. Moreover,

$$T^\dagger = \begin{bmatrix} T_1^*D^{-1} & 0 \\ T_2^*D^{-1} & 0 \end{bmatrix}. \tag{3}$$

$$T = \begin{bmatrix} T_1 & 0 \\ T_3 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{bmatrix}, \tag{4}$$

where $F = T_1^*T_1 + T_3^*T_3 \in \mathcal{L}(\mathrm{ran(T^*)})$ is positive and invertible. Moreover,

$$T^\dagger = \begin{bmatrix} F^{-1}T_1^* & F^{-1}T_2^* \\ 0 & 0 \end{bmatrix}. \tag{5}$$

2. Main results

In this section we solve $TXS^* - SX^*T^* = A$ via the some results of product of operators with closed ranges.

Lemma 2.1 Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and let $Q \in \mathcal{L}(\mathcal{X})$ and $P \in \mathcal{L}(\mathcal{Y})$ be orthogonal projections and TQ and PT have closed ranges. Then

(i) $(TQ)^\dagger = Q(TQ)^\dagger$,
(ii) $(PT)^\dagger = (PT)^\dagger P$.

Proof. (i) Since ran(TQ) is closed, the operator $(TQ)^\dagger$ exists. Therefore, $\mathrm{ran((TQ)^\dagger)} = \mathrm{ran((TQ)^*)} = \mathrm{ran(QT^*)} \subseteq \mathrm{ranQ}$. Hence $Q((TQ)^\dagger) = (TQ)^\dagger$. The proof for (ii) is similar. ■

Lemma 2.2 Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $S \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that $TS = 1_\mathcal{Y}$. Then

(i) $(1_\mathcal{X} - ST)^\dagger = (1_\mathcal{X} - SS^\dagger)(1_\mathcal{X} - T^\dagger T)$,
(ii) $T^\dagger = P_{N(T)^\perp} S P_{R(T)}$.

Proof. (i) Since $TS = 1_\mathcal{Y}$, the operator S is generalized inverse of T and vise versa. Therefore T and S have closed ranges, hence $T^\dagger$ and $S^\dagger$ exist. Put $Q = 1_\mathcal{X} - ST$. From $TS = 1_\mathcal{Y}$, we have $QS = 0$ and $TQ = 0$. Put $M = (1_\mathcal{X} - SS^\dagger)(1_\mathcal{X} - T^\dagger T)$. Then

$$QM = (1_\mathcal{X} - ST)(1_\mathcal{X} - SS^\dagger)(1_\mathcal{X} - T^\dagger T) = (1_\mathcal{X} - ST)(1_\mathcal{X} - T^\dagger T) = 1_\mathcal{X} - T^\dagger T,$$
$$MQ = (1_\mathcal{X} - SS^\dagger)(1_\mathcal{X} - T^\dagger T)(1_\mathcal{X} - ST) = (1_\mathcal{X} - SS^\dagger)(1_\mathcal{X} - ST) = 1_\mathcal{X} - SS^\dagger.$$

Hence, $QMQ = (1_\mathcal{X} - T^\dagger T)(1_\mathcal{X} - ST) = (1_\mathcal{X} - ST) = Q$ and $MQM = (1_\mathcal{X} - SS^\dagger)(1_\mathcal{X} - SS^\dagger)(1_\mathcal{X} - ST) = M$. So $(1_\mathcal{X} - ST)^\dagger = (1_\mathcal{X} - SS^\dagger)(1_\mathcal{X} - T^\dagger T)$.

To prove (ii) By (i) we know that ran(T) is closed. Put $N = P_{N(T)^\perp} S P_{R(T)}$. Then

$$TN = TP_{N(T)^\perp} S P_{R(T)} = TSP_{R(T)} = P_{R(T)},$$

and

$$NT = P_{N(T)^{\perp}} S P_{R(T)} T = P_{N(T)^{\perp}}.$$

Obviously, we have $TNT = P_{R(T)}T = T$ and $NTN = N$. Hence, $T^{\dagger} = P_{N(T)^{\perp}} S P_{R(T)}$. ■

Corollary 2.3 Let $T, S \in \mathcal{L}(\mathcal{X})$ be such that $TS = 1_{\mathcal{X}}$. Then T^2 has closed range and $(T^2)^{\dagger} = P_{N(T^2)^{\perp}} S^2 P_{R(T^2)}$.

Proof. Since $TS = 1_{\mathcal{X}}$. Then $T^2S^2 = 1_{\mathcal{X}}$. Hence, the bounded adjointable operator S^2 is generalized inverse of T^2. Therefore T^2 has closed range. Lemma 2.2(ii) implies that $(T^2)^{\dagger} = P_{N(T^2)^{\perp}} S^2 P_{R(T^2)}$. ■

In the following theorems we obtain explicit solutions to the operator equation

$$TXS^* - SX^*T^* = A, \tag{6}$$

when $TS = 1$.

Theorem 2.4 Suppose $T, S \in \mathcal{L}(\mathcal{X})$ such that $TS = 1$ and $A \in \mathcal{L}(\mathcal{X})$. Then the following statements are equivalent:
(a) There exists a solution $X \in \mathcal{L}(\mathcal{X})$ to Eq. (6).
(b) $A = -A^*$ and $(1 - T^2(T^2)^{\dagger})TAT^*(1 - T^2(T^2)^{\dagger}) = 0$.
If (a) or (b) is satisfied, then any solution to Eq. (6) has the form

$$\begin{aligned} X = &\frac{1}{2} P_{N(T^2)^{\perp}} S^2 P_{R(T^2)} TAT^*T^2 P_{N(T^2)^{\perp}} S^2 P_{R(T^2)} + P_{N(T^2)^{\perp}} S^2 P_{R(T^2)} Z T^2 P_{N(T^2)^{\perp}} S^2 P_{R(T^2)} \\ &+ P_{N(T^2)^{\perp}} S^2 P_{R(T^2)} TAT^*(1 - T^2 P_{N(T^2)^{\perp}} S^2 P_{R(T^2)}) + (1 - P_{N(T^2)^{\perp}} S^2 P_{R(T^2)} T^2)Y, \end{aligned}$$

where $Z \in \mathcal{L}(\mathcal{X})$ satisfies $T^*(Z - Z^*)T = 0$, and $Y \in \mathcal{L}(\mathcal{X})$ is arbitrary.

Proof. By multiplication T of the left and T^* of the right of Eq. (6) get into

$$T^2X - X^*(T^*)^2 = TAT^*. \tag{7}$$

Corollary 2.3 implies that T^2 has closed range. Now, [8, Theorem 3] implies that (a) and (b) are equivalent. Again by [8, Theorem 3] implies that

$$\begin{aligned} X = &\frac{1}{2}(T^2)^{\dagger} TAT^*T^2(T^2)^{\dagger} + (T^2)^{\dagger} Z T^2 (T^2)^{\dagger} + (T^2)^{\dagger} TAT^*(1 - T^2(T^2)^{\dagger}) \\ &+ (1 - (T^2)^{\dagger} T^2)Y, \end{aligned}$$

where $Z \in \mathcal{L}(\mathcal{X})$ satisfies $T^*(Z - Z^*)T = 0$, and $Y \in \mathcal{L}(\mathcal{X})$ is arbitrary. Again by Corollary 2.3, equivalently

$$\begin{aligned} X = &\frac{1}{2} P_{N(T^2)^{\perp}} S^2 P_{R(T^2)} TAT^*T^2 P_{N(T^2)^{\perp}} S^2 P_{R(T^2)} + P_{N(T^2)^{\perp}} S^2 P_{R(T^2)} Z T^2 P_{N(T^2)^{\perp}} S^2 P_{R(T^2)} \\ &+ P_{N(T^2)^{\perp}} S^2 P_{R(T^2)} TAT^*(1 - T^2 P_{N(T^2)^{\perp}} S^2 P_{R(T^2)}) + (1 - P_{N(T^2)^{\perp}} S^2 P_{R(T^2)} T^2)Y \end{aligned}$$

■

The following remark is the same as in the matrix case.

Remark 1 *Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range and $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then to the operator equation*

$$TX = A \quad , \quad X \in \mathcal{L}(\mathcal{X}) \tag{8}$$

is solvable iff $TT^{\dagger}A = A$. Therefore

$$X = T^{\dagger}A. \tag{9}$$

is solution to Eq. (8).

Theorem 2.5 Suppose that $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are Hilbert $\mathcal{A}$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ and $(1 - P_{\mathrm{ran(S)}})T$ have closed ranges, $A \in \mathcal{L}(\mathcal{Y})$ and $\mathcal{X} = \mathrm{ran(S^*)} \oplus \mathrm{ker(S)}$ and $\mathcal{Y} = \mathrm{ran(S)} \oplus \mathrm{ker(S^*)}$ and $\mathcal{Z} = \mathrm{ran(T^*)} \oplus \mathrm{ker(T)}$. If the operator equation

$$TXS^* - SX^*T^* = A \quad , \quad X \in \mathcal{L}(\mathcal{X}, \mathcal{Z}) \tag{10}$$

is solvable, then

$$X = \begin{bmatrix} ((1 - P_{\mathrm{ran(S)}})T)^{\dagger}AS^{\dagger} & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(S^*)} \\ \ker(S) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix}$$

is solution to the operator equation (10) , such that X_2, X_3, X_4 are arbitrary operators.

Proof. Since S, T have closed ranges, we have $\mathcal{X} = \mathrm{ran(S^*)} \oplus \mathrm{ker(S)}$ and $\mathcal{Y} = \mathrm{ran(S)} \oplus \mathrm{ker(S^*)}$ and $\mathcal{Z} = \mathrm{ran(T^*)} \oplus \mathrm{ker(T)}$. Hence by (2) and orthogonal complemented submodules $\mathrm{ran(S^*)}$, $\mathrm{ran(T^*)}$ and $\mathrm{ran(S)}$ and using the matrix forms for X, A, that is,

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(S)} \\ \ker(S^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(S)} \\ \ker(S^*) \end{bmatrix},$$

and

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(S^*)} \\ \ker(S) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix},$$

and matrix forms for S, T as describe in Lemma 1.4 and Lemma 1.5, respectively. Now the operator equation $TXS^* - SX^*T^* = A$ can be written in an equivalent form

$$\begin{bmatrix} T_1 & 0 \\ T_3 & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} S_1^* & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1^* & X_3^* \\ X_2^* & X_4^* \end{bmatrix} \begin{bmatrix} T_1^* & T_3^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

That is,

$$\begin{bmatrix} T_1X_1S_1^* - S_1X_1^*T_1^* & -S_1X_1^*T_3^* \\ T_3X_1S_1^* & 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

Since Eq. (10) is solvable, then $A_4 = 0$. Therefore

$$T_1X_1S_1^* - S_1X_1^*T_1^* = A_1 \tag{11}$$
$$-S_1X_1^*T_3^* = A_2 \tag{12}$$
$$T_3X_1S_1^* = A_3 \tag{13}$$

This means that for every operators X_2, X_3, X_4, is a solution to Eq. (10). By Eq. (13) we have $T_3X_1S_1^* = A_3$. By Lemma 2, S_1^* is invertible. Hence we have

$$T_3X_1 = A_3(S_1^*)^{-1}. \tag{14}$$

By using the matrix form (2) implies that $T_3 = (1 - P_{\mathrm{ran(S)}})TP_{\mathrm{ran(T^*)}} = (1 - P_{\mathrm{ran(S)}})TT^\dagger T = (1 - P_{\mathrm{ran(S)}})T$ and $A_3 = (1 - P_{\mathrm{ran(S)}})AP_{\mathrm{ran(S)}}$, hence we have

$$\begin{aligned} ((1 - P_{\mathrm{ran(S)}})T)X_1 &= (1 - P_{\mathrm{ran(S)}})AP_{\mathrm{ran(S)}}(S_1^*)^{-1} \\ &= (1 - P_{\mathrm{ran(S)}})ASS^\dagger(S^*)^\dagger \\ &= (1 - P_{\mathrm{ran(S)}})AS^\dagger \end{aligned} \tag{15}$$

Since Eq. (10) is solvable then Eq. (15) is solvable. Since $T_3 = (1 - P_{\mathrm{ran(S)}})T$ has closed range, by Remark *1* Eq. (15) is solvable and

$$X_1 = ((1 - P_{\mathrm{ran(S)}})T)^\dagger(1 - P_{\mathrm{ran(S)}})AS^\dagger.$$

By Lemma 2.1, equivalently

$$X_1 = ((1 - P_{\mathrm{ran(S)}})T)^\dagger AS^\dagger.$$

■

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