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Solving the liner quadratic differential equations with constant coefficients using Taylor series with step size h

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Abstract. In this study we produced a new method for solving regular differential equations with step size h and Taylor series. This method analyzes a regular differential equation with initial values and step size h. this types of equations include quadratic and cubic homogenous equations with constant coefficients and cubic and second- level equations.

Keywords: Differential equation; initial value; step length; numerical methods; Taylor series.

1. Introduction

In the first and second sections of this paper, the numerical solution of a initial linear differential equations of cubic and quadratic homogeneous with constant coefficients is calculated by using Taylor series with length h. In this part the series $\sum_{n=0}^{\infty} a_n x_i^n$ will be replaced in Taylor expansion of $y(x_i), y'(x_i)$ and $y''(x_i)$, and then the obtained series replace in the given original differential equation, and we obtain $a_0, a_1, a_2, \cdots, a_n$. In Section 3 we will solve a quadratic homogenous differential equation

$$y'' + (A_0 x + B_0)y' + (A_1 x + B_1)y = 0,$$

using the mentioned method. Details are thoroughly discusses in the books [3] and [4].
An introduction to differential and their application of Zill et. al. [2], and numerical methods for partial differential equation of Ames[2].

2. Method of Solution

2.1 *Case 1.*

We consider the following initial value problem

$$y'' + Ay' + By = 0, \qquad y(x_0) = y_0, \quad y'(x_1) = y_1. \tag{1}$$

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We assume that the solution of Equation (1) has the following form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

According to Taylor series respect to x_0, we have

$$y(x_i) = y(x_0 + ih) = y(x_0) + ihy'(x_0) + \frac{(ih)^2}{2!} y''(x_0) + \frac{(ih)^3}{3!} y^{(3)}(x_0) + ... \tag{2}$$

and so

$$y'(x_i) = y'(x_0) + ihy''(x_0) + \frac{(ih)^2}{2!} y^{(3)}(x_0) + \frac{(ih)^3}{3!} y^{(4)}(x_0) + ... \tag{3}$$

therefore we have

$$y''(x_i) = y''(x_0) + ihy^{(3)}(x_0) + \frac{(ih)^2}{2!} y^{(4)}(x_0) + \frac{(ih)^3}{3!} y^{(5)}(x_0) + ... \tag{4}$$

If we set

$$y(x_0) = \sum_{n=0}^{\infty} a_n x_0^n,$$

then we have

$$y^{(k)}(x_0) = \sum_{n=k}^{\infty} n(n-1)...(n-k+1) a_n x_0^{n-k}, \qquad k = 1, 2, 3, ... \tag{5}$$

Now, by substituting the Equation (5) in Equation (2) we have

$$\begin{aligned} y(x_i) = & \sum_{n=0}^{\infty} a_n x_0^n + ih \sum_{n=1}^{\infty} n a_n x_0^{n-1} + \frac{(ih)^2}{2!} \sum_{n=2}^{\infty} n(n-1) a_n x_0^{n-2} \\ & + \frac{(ih)^3}{3!} \sum_{n=3}^{\infty} (n-1)(n-2) a_n x_0^{n-3} \\ & + \frac{(ih)^4}{4!} \sum_{n=4}^{\infty} (n-1)(n-2)(n-3) a_n x_0^{n-4} + ... \end{aligned} \tag{6}$$

or

$$\begin{aligned} y(x_i) = & \sum_{n=0}^{\infty} x_0^n \Big(a_n + ih(n+1) a_{n+1} + \frac{(ih)^2}{2!} (n+1)(n+2) a_{n+2} \\ & + \frac{(ih)^3}{3!} (n+1)(n+2)(n+3) a_{n+3} + ... \Big). \end{aligned} \tag{7}$$

If we substitute Equation (5) in Equation (3), we have

$$y'(x_i) = \sum_{n=0}^{\infty} x_0^n \Big((n+1)a_{n+1} + ih(n+1)(n+2)a_{n+2} + \frac{(ih)^2}{2!}(n+1)(n+2)(n+3)a_{n+3} + \frac{(ih)^3}{3!}(n+1)(n+2)(n+3)(n+4)a_{n+4} + ... \Big). \quad (8)$$

Finally, by substituting the Equation (5) in Equation (4) we have

$$y''(x_i) = \sum_{n=0}^{\infty} x_0^n \Big((n+1)(n+2)a_{n+2} + ih(n+1)(n+2)(n+3)a_{n+3} + \frac{(ih)^2}{2!}(n+1)(n+2)(n+3)(n+4)a_{n+4} + \frac{(ih)^3}{3!}(n+1)(n+2)(n+3)(n+4)(n+5)a_{n+5} + ... \Big). \quad (9)$$

By substituting Equations (7)-(9) in Equation (1), we have

$$\sum_{n=0}^{\infty}(x_0)^n \{((n+1)a_{n+1}(B(ih)+1) + \sum_{k=2}^{m} \left(a_{n+k} \left(B\frac{(ih)^k}{(k)!} + \frac{A(ih)^{k-1}}{(k-1)!} + \frac{(ih)^{k-2}}{(k-2)!} \Pi_{i=1}^{k}(n+i) \right) \right) \} = 0, \quad (10)$$

so

$$a_{n+k} = \frac{-(n+1)a_{n+1}(B(ih)+1) + \sum_{k=2}^{m-1}(a_{n+k}(B\frac{(ih)^k}{(k)!} + \frac{A(ih)^{k-1}}{(k-1)!} + \frac{(ih)^{k-2}}{(k-2)!}\alpha_{n,m})}{\left(B\frac{(ih)^m}{(m)!} + \frac{A(ih)^{m-1}}{(m-1)!} + \frac{(ih)^{m-2}}{(m-2)!} \right)\alpha_{n,m}}. \quad (11)$$

where $\alpha_{n,k} = \Pi_{i=1}^{k}(n+i))$.

Example 1. Consider the following initial value problem

$$y''(x) + 2y'(x) + y(x) = 0, \qquad y(0) = 0, y'(0) = 1,$$

according to the above algorithm for $y(0.1)$ and $h = 0.1$, if we set $m = 2$, then

$$a_0 = 0, a_1 = 1, a_{n+2} = \frac{-(a_{n+1}(n+1)(Bh + A))}{(B\frac{(ih)^2}{2!} + A(ih) + 1)\Pi_{i=1}^{2}(n+i)}$$

so $a_2 = -0.8714$. Also if we set $m = 3$, then $a_3 = 0.00067$ and $y(0.1) = 0.0913$. We know that the exact solution is $y(0.1) = 0.0905$ and absolute error is 8×10^{-4}.

2.2 *Case 2.*

In this case we consider the following problem

$$y''' + Ay'' + By' + Cy = 0, \quad y(x_0) = y_0, y'(x_1) = y_1, y''(x_2) = y_2, A, B, C \in R. \quad (12)$$

According the Case 1. we have

$$\sum_{n=0}^{\infty}(x_0)^n \left\{((n+1)a_{n+1}(B(ih)+1)\ + \right.$$
$$\left. \sum_{k=3}^{m}\left(a_{n+k}\left(B\frac{(ih)^{k-1}}{(k-1)!}+\frac{A(ih)^{k-2}}{(k-2)!}+\frac{(ih)^{k-3}}{(k-3)!}\right)\prod_{i=1}^{k}(n+i)\right)\right\}=0,$$

and then

$$a_{n+k}=\frac{-((Ca_n+B(n+1)a_{n+1}+\sum_{k=3}^{m-1}a_{n+k}(B\frac{(ih)^{k-1}}{(k-1)!}+\frac{A(ih)^{k-2}}{(k-2)!}+\frac{(ih)^{k-3}}{(k-3)!})\alpha_{n,k}}{\left(B\frac{(ih)^{m-1}}{(m-1)!}+\frac{A(ih)^{m-2}}{(m-2)!}+\frac{(ih)^{m-3}}{(m-3)!}\right)\alpha_{n,m}} \tag{13}$$

Example 2. Consider the following initial value problem

$$y'''(x)-6y''+11y'(x)-6y(x)=0, \qquad y(0)=0, y'(0)=1, y''(0)=0.$$

According to the above algorithm, we have

$$y(x)=x+x^2-29.01099x^3+918.9361x^4,$$

so $y(0.1)=0.1728$, and the absolute error is 0.0566.

3. Case 3.

Finally, we consider the following problem

$$y''(x)+(A_0x+B_0)y'(x)+(A_1x+B_1)y(x)=0, \qquad y(x_0)=y_0, \quad y'(x_1)=y_1 \tag{14}$$

According the above algorithm in Case 1., we have

$$a_{n+k}=\frac{-(a_nA_1+a_{n+1}(B_1+A_1(ih)(n+1)+A_0(ih)(n+1))+E}{\left(\left(\left(B_0\frac{(ih)^{m-2}}{(m-2)!}+B_1\frac{(ih)^{m-1}}{(m-1)!}\right)\prod_{i=2}^{k}(n+i\right)\right)+F} \tag{15}$$

where

$$E=\sum_{k=3}^{m}a_{n+k}\left(\left(B_0\frac{(ih)^{k-2}}{(k-2)!}+B_1\frac{(ih)^{k-1}}{(k-1)!}\right)\prod_{i=2}^{k}(n+i)\right)$$
$$+\left(A_0\frac{(ih)^{k-1}}{(k-1)!}+B_1\frac{(ih)^{k}}{(k)!}\prod_{i=1}^{k}(n+i)+\frac{(ih)^{k-3}}{(k-3)!}\right)\prod_{i=2}^{k}(n+i),$$

and

$$F=A_0\frac{(ih)^{m-1}}{(m-1)!}+A_1\frac{(ih)^{m}}{(m)!}\prod_{i=1}^{m}(n+i)+\frac{(ih)^{m-3}}{(m-3)!}\prod_{i=2}^{k}(n+i).$$

4. Conclusions

The results obtained from the method introduced in this paper, can be used to numerical solution of nth order differential equation with constant coefficient and with initial value and with step size h by series $\sum a_n x^n$, and thus for obtaining the answer of homogenous linear differential equation of nth order, $a_n(x)y^{(n)} + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y' + a_0 y = 0$ can get with initial values and with step length.

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