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Characterization of $G_2(q)$, where $2 < q \equiv 1(mod3)$ by order components

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Abstract. In this paper we will prove that the simple group $G_2(q)$ where $2 < q \equiv 1(mod3)$ is recognizable by the set of its order components, also other word we prove that if G is a finite group with $OC(G) = OC(G_2(q))$, then G is isomorphic to $G_2(q)$.

Keywords: Prime graph, order component, linear group.

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1. Introduction

Let G be a finite group. We denote by $\pi(n)$ the set of all prime divisors of n, where n is a natural number. The prime graph of G is a graph $\Gamma(G)$ with vertex set $\pi(G)$, the set of all prime divisors of $|G|$, and two distinct vertices p and q are adjacent by an edge if G has an element of order pq. Let $\pi_i = \pi_i(G)$, $1 \leqslant i \leqslant s(G)$, be the connected components of $\Gamma(G)$. For a group of even order we let $2 \in \pi_1(G)$. Then $|G|$ can be expressed as the product of $m_1, m_2, \ldots, m_{s(G)}$, where m_i's are positive integers with $\pi(m_i) = \pi_i$. These $m_i' s$ are called the order components of G. We write $OC(G) = \{m_1, m_2, \ldots, m_{s(G)}\}$ and call it the set of order components of G.

Definition 1.1 Given a finite group G, denote by $h(G)$ the number of non-isomorphism classes of finite groups S such that $OC(G) = OC(S)$ and this is called the h-function

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of G. A group G is called k-recognizable by its set of order components if $h(G) = k$. Moreover, if $h(G) = 1$ we say that G is characterizable by order components. In this case G is uniquely determined by the set of its order components.

Using [24] and [26] we list the order components for non-abelian finite simple groups P in the Table 1., Table 2. and Table 3. This information is used to prove our main theorem.

In a series of articles [3–5, 25] it is proved that the sporadic groups, and finite groups $PSL_2(q)$, ${}^3D_4(q)$, ${}^2D_n(3)$, where $9 \leqslant n = 2^m + 1$ not a prime, and ${}^2D_{p+1}(q)$, where $5 < p \neq 2^m - 1$, are characterized by order components. The recognizability of groups $L_{p+1}(2)$, ${}^2D_p(3)$, where $p \geqslant 5$ is a prime number not of the form $2^m + 1$, ${}^2D_n(2)$, where $n = 2^m + 1 \geqslant 5$, $D_{p+1}(2)$, $D_{p+1}(3)$ and $D_p(q)$, where $p \geqslant 5$ is a prime number and $q = 2, 3$ or 5, are proved by M.R. Darafsheh et. al. in [7–11]. Also characterizability of the groups $E_6(q)$, ${}^2E_6(q)$, ${}^2D_n(q)$, where $n = 2^m$, $PSL(p, q)$, $PSU(p, q)$, $PSL(p + 1, q)$, $PSU(p + 1, q)$, $PSL(3, q)$ where q is an odd prime power, $PSL(3, q)$ for $q = 2^n$ and $PSU(3, q)$ for $q > 5$ by their order components is proved in a series of articles by B. Khosravi et. al. [12–14, 16–20, 22, 23]. In addition, r-recognizability of $B_n(q)$ and $C_n(q)$, where $n = 2^m \geqslant 4$, are proved in [21].

The following open problem contains all remaining cases related to that all simple non-abelian groups, as P, with $s(P) = 2$ are characterizable by order components.

Open problem [15]. Are the groups $F_4(q)(q \text{ odd})$, $G_2(q)(2 < q \equiv \pm 1(mod3))$ and $C_p(2)$ characterizable by their order components?

In this paper we consider the simple group $G_2(q)$, where $2 < q \equiv 1(mod3)$, and prove that this group is characterizable by order components.

By [24] the prime graph of the group $G_2(q)$, where $2 < q \equiv 1(mod3)$, has two components $m_1 = q^6(q^3+1)(q^2-1)(q-1) = q^6(q+1)^2(q-1)^2(q^2+q+1)$ and $m_2 = q^2-q+1$.

Main Theorem. Let G be a finite group such that $OC(G) = OC(G_2(q))$,where $2 < q \equiv 1(mod3)$, then $G \cong G_2(q)$.

2. Preliminaries

Definition 2.1 A group G is called a 2-Frobenius group, if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ of G, such that K and G/H are Frobenius groups with kernels H and K/H respectively.

The following lemmas are taken from [1] and [2].

Lemma 2.2

(a) Let G be a Frobenius group of even order where H and K are Frobenius complement and Frobenius kernel of G, respectively. Then $s(G) = 2$ and the prime graph components of G are $\pi(H)$ and $\pi(K)$.

(b) Let G be a 2-Frobenius group of even order. Then $s(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $|K/H| = m_2$, $|H||G/K| = m_1$ and $|G/K| \mid (|K/H| - 1)$ and H is a nilpotent π_1-group.

Lemma 2.3 Let G be a finite group with $s(G) \geqslant 2$. If $H \trianglelefteq G$ is a π_i-group, then

$$(\prod_{j=1, j\neq i}^{s(G)} m_j) \mid (|H| - 1).$$

The structure of finite groups with non-connected prime graph is described in the following Lemma:

Lemma 2.4 Let G be a finite group with $s(G) \geqslant 2$. Then one of the following holds:

(a) G is a Frobenius or a 2-Frobenius group;
(b) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H is a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Moreover, any odd order components of G is also an odd order components of K/H.

The following Lemma of Zsigmondy is used to prove the main theorem.

Lemma 2.5 [27] Let n and a be integers greater than 1. There there exists a prime divisor p of $a^n - 1$ such that p dose not divide $a^i - 1$ for all i, $1 \leqslant i < n$, except in the following cases:

(a) $n = 2, a = 2^k - 1$, where $k \geqslant 2$,
(b) $n = 6, a = 2$.

The prime p in the above lemma is called a Zsigmondy prime for $a^n - 1$.

3. Proof of the main theorem

To prove the theorem we will use Lemma 2.4. But first we will prove the following Lemmas.

Lemma 3.1 Let $M = G_2(q)$ where $2 < q \equiv 1 (mod 3)$ and set $D(q) = q^2 - q + 1$,

(a) If $p \in \pi(M)$, then $|S_p| \leqslant q^6$, where $S_p \in Syl_p(M)$;
(b) If $p \in \pi_1(M)$, $p^\alpha \mid |M|$ and $p^\alpha - 1 \equiv 0 (mod D(q))$, then $p^\alpha = q^6$ or $p^\alpha = 27$, where $q = 4$.

Proof. We have

(a) $|G_2(q)| = q^6(q+1)^2(q-1)^2(q^2-q+1)(q^2+q+1) \qquad (1).$
An easy calculation shows that

$$\begin{array}{lll} (q+1, q-1) = (2, q-1) & (q-1, q^2+q+1) = (3, q-1) & \\ (q-1, q^2-q+1) = 1 & (q+1, q^2-q+1) = (3, q+1) & \\ (q+1, q^2+q+1) = 1 & (q^2+q+1, q^2-q+1) = 1 & (2) \end{array}$$

Where (.,.) denotes the greatest common divisor of two numbers. If $p^\alpha \mid |M|$, then regarding (1) and (2) we obtain $p^\alpha \mid q^6$, $2^2.3(q+1)^2$, $2^2.3(q-1)^2$, q^2+q+1 or $q^2 - q + 1$. Then (a) follows immediately.

(b) If $p^\alpha - 1 \equiv 0 (mod D(q))$, then we have $p^\alpha > D(q)$, since $q \geqslant 4$, we obtain $p^\alpha > 13$. Consider the following cases:
 (1) If $p^\alpha \mid 3^2(q^2+q+1)$, then $p^\alpha \mid 3^3$ or $p^\alpha \mid q^2+q+1$. If $p^\alpha \mid 3^3$, then we have $p^\alpha = 27$ and $q = 4$. If $p^\alpha \mid q^2+q+1$, then $p^\alpha = \frac{q^2+q+1}{t}$. Also, we have $p^\alpha - 1 = r.D(q)$, where $r \in N$, then $D(q) = \frac{q^2+q+1-t}{tr}$. But since

$\frac{q^2+q+1}{2} < D(q) = \frac{q^2+q+1-t}{t} < \frac{q^2+q+1}{tr}$, then $tr \leqslant 2$ and

$$(tr-1)q^2 - (tr+1)q + (tr+t-1) = 0.$$

From this equation we deduce $q \mid (tr+t-1)$, therefore $4 \leqslant q \leqslant tr+t-1 \leqslant 3$, which is impossible.

(2) If $p^\alpha \mid 2^2.3.(q\pm1)^2$, then $p^\alpha \mid 4(q\pm1)^2$ or $p^\alpha \mid 3.(q\pm1)^2$. Since the proof of these two cases are similar we only deal with one of them.
If $p^\alpha \mid 4(q+1)^2$, then $tp^\alpha = 4(q+1)^2$, i.e., $p^\alpha = 4(q+1)^2/t$, where t is a natural number. Also, we have $p^\alpha - 1 = r.D(q)$, where r is a natural number, then $D(q) = \frac{4(q+1)^2-t}{tr}$. But since $\frac{4(q+1)^2}{8} < D(q) = \frac{4(q+1)^2-t}{tr} < \frac{4(q+1)^2}{tr}$, then $tr \leqslant 8$ and $(tr-4)q^2 - (tr+8)q + (tr+t-4) = 0$. From this equation we deduce $q \mid (tr+t-4)$. Now using $tr = 1, 2, ..., 8$ we get contradictions.

(3) If $p^\alpha \mid q^6$, then we deduce

$$p^\alpha - 1 \leqslant (q^6-1) = (q^3+1)(q^3-1) \Rightarrow q^3+1 \leqslant p^\alpha - 1 = r(q^2-q+1)$$
$$\Rightarrow (q+1)(q^2-q+1) \leqslant r(q^2-q+1) \Rightarrow r \geqslant q+1.$$

From this we deduce that
$p^\alpha - 1 = r(q^2-q+1) \geqslant (q+1)(q^2-q+1) = q^3+1 \Rightarrow p^\alpha \geqslant q^3+2 > q^3$.
Therefore, $p^\alpha > q^3$, now we have $p^\alpha = q^3.p^m$, $m \geqslant 1$. Then

$$r.D(q) = p^\alpha - 1 = p^m.q^3 - 1 = p^m.q^3 + p^m - p^m - 1$$
$$= p^m(q+1)D(q) - (p^m+1),$$

which implies that $p^m + 1 \equiv 0(modD(q))$, then $p^m = q^3$ and $p^\alpha = q^6$.

■

Lemma 3.2 Let G be a finite group such that $OC(G) = OC(G_2(q))$, where $2 < q \equiv 1(mod3)$, then G is neither a Frobenius nor 2-Frobenius group.

Proof. If G is a Frobenius group, then $G = HK$ with Frobenius complement H and Frobenius kernel K. By Lemma 2.2 we have $OC(G) = \{|H|, |K|\}$. Since $|H| \mid (|K|-1)$, so $|H| < |K|$, therefore $|K| = m_1$ and $|H| = m_2$. There is a prime number p such that $p^\alpha \mid 4(q+1)^2$. If $S_p = S$ is a p-Sylow subgroup of K, then by nilpotency of K we have $S \trianglelefteq G$, and by Lemma 2.3, $m_2 \mid (|S|-1)$, hence $|S| \equiv 1(modD(q))$, then by Lemma 3.1, $p^\alpha = q^6$ or 27 and $q = 4$, which is impossible since $|S| \leqslant 4(q+1)^2 < q^6$ or if $|S| = 27$, then $27 \nmid 4.5^2$.

If G be a 2-Frobenius group, then there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, for G such that H is a nilpotent π_1-group, $|K/H| = m_2$ and $|G/K| \mid (|K/H|-1)$, hence $|G/K| \mid q(q-1)$. We have $|K/H| = q^2-q+1 < 4(q+1)^2$, thus there is a prime p such that $p \mid 4(q+1)^2$ and $p \mid |H|$. If $S = S_p \in Syl_p(H)$, then by nilpotency of H we have $S \trianglelefteq K$ and $|K| = (q^2-q+1)|H|$, so by Lemma 2.3, $m_2 \mid (|S|-1)$, hence $|S| \equiv 1(modD(q))$, then by Lemma 3.1, $|S| = q^6$ or 27 and $q = 4$, which is impossible since $|S| \leqslant 4(q+1)^2 < q^6$ or if $|S| = 27$, then $27 \nmid 4.5^2$. ■

Now we continue the proof of our main theorem. By Lemma 2.4, there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ for G such that K/H is a non-abelian simple group, H and G/K are π_1-group and H is a nilpotent group. Moreover $|G/K| \mid |Out(K/H)|$ and odd order components of G is one of the odd order components of K/H and $s(K/H) \geqslant 2$.

Since $P = K/H$ is a non-abelian simple group with disconnected prime graph, by the classification of finite simple groups we have one of the possibilities in Tables 1, 2 and 3 for P.

Case(1): $P \cong {}^2A_3(2)$, ${}^2F_4(2)'$, $A_2(2)$, $A_2(4)$, ${}^2A_5(2)$, $E_7(2)$, $E_7(3)$, ${}^2E_6(2)$ or one of the 26 Sporadic groups listed in Tables 1, 2 and 3.

The odd order component of G is equal to $m_2 = q^2 - q + 1$ and must be equal to one of the odd order components of the groups listed above. By inspecting Tables 1, 2 and 3, the largest odd order component of the above groups is 1093. Therefore $q^2 + q + 1 \leqslant 1093$, from which we obtain $q \leqslant 31$. Hence the possibilities for q are: $q = 4, 7, 13, 16, 19, 25, 31$ (note that $q \equiv 1(mod3)$). If $q = 7, 13, 16, 19, 25, 31$, then we have $m_2 = 43, 157, 241, 343, 601, 931$, respectively. But any group in Tables 1, 2 and 3 do not have these odd components. Therefore, we deduce $q = 4$. If $q = 4$, then $m_2 = 13$ corresponds to $P \cong Fi_{22}$ or Suz, but for both cases we have $7, 11 \mid |P|$ and $7, 11 \nmid |G|$, hence $|P| \nmid |G|$. Therefore the above possibilities are ruled out.

Case(2): $P \cong A_n$, where $n = p, p+1, p+2$, one of n or $n-2$ is prime, and $n = p, p-2$ are both prime where $p \geqslant 6$ is a prime number.

By Tables 1 and 2, the odd order components of A_n are p and $p-2$, hence $q^2-q+1 = p$ or $p-2$. If $q^2 - q + 1 = p$, then $p - 2 = q^2 - q - 1$, hence $q^2 - q - 1 \mid |G|$, which is impossible. If $q^2-q+1 = p-2$, hence $p = q^2-q+3$, therefore we deduce $q^2-q+3 \mid |G|$, which is impossible.

Case(3): $P \cong E_6(q')$. By Table 1, we have $\frac{q'^6+q'^3+1}{(3,q'-1)} = q^2 - q + 1$, now if $(3, q' - 1) = 3$, then

$$\frac{q'^6+q'^3+1}{3} = q^2 - q + 1 \Rightarrow q'^9 - 1 \equiv 0(modD(q)) \Rightarrow q'^9 \equiv 1(modD(q)).$$

Hence, by Lemma 3.1, we have $q'^9 = q^6$, therefore $q'^{36} = q^{24} > q^6$. Then P has a Sylow subgroup with order great than q^6, which is impossible by Lemma 3.1. But if $(3, q' - 1) = 1$, then $q'^6 + q'^3 + 1 = q^2 - q + 1$, therefore $q'^3(q'^3 + 1) = q(q - 1)$, hence $q = q'^3$, this implies $q(q - 1) = q'^3(q'^3 - 1) < q'^3(q'^3 + 1)$ a contradiction.

Case(4): $P \cong G_2(q')$, $2 < q' \equiv \pm1(mod3)$. By Table 1, if $\epsilon = -1$, then $q'^2 + q' + 1 = q^2 - q + 1$, hence $q'(q' + 1) = q(q - 1)$. Since $q \neq q'$, therefore $q' = q - 1 = 3k$, is a contradiction. If $\epsilon = 1$ we have $q'^2 - q' + 1 = q^2 - q + 1$ which implies $q = q'$. Therefore $P \cong G_2(q)$, since $|P| \mid |G|$ and $|P| = |G_2(q)| = |G|$, hence $P \cong G$. From this we deduce that $G \cong G_2(q)$.

Case(5): $P \cong B_p(3)$. By Table 1, we have $q^2 - q + 1 = \frac{3^p-1}{2}$, then $3^p \equiv 1(modD(q))$, therefore, by lemma 3.1, we have $3^p = q^6$, therefore $q \equiv 0(mod3)$, a contradiction, or $3^p = 27$ and $q = 4$, hence $p = 3$. Therefore we have $P \cong B_3(3)$, but we have $|B_3(3)| > |G_2(4)|$, a contradiction.

Case(6): $P \cong C_p(q'), q' = 2, 3$. If $q' = 3$ then by Table 1, we have $\frac{3^p-1}{2} = q^2 - q + 1$, so $3^p \equiv 1(modD(q))$. Therefore, by Lemma 3.1, we have $3^p = q^6$, therefore $q \equiv 0(mod3)$, a contradiction, or $3^p = 27$ and $q = 4$, hence $p = 3$. Therefore we have $P \cong C_3(3)$, but we have $|C_3(3)| > |G_2(4)|$, a contradiction..

If $q' = 2$, then by Table 1, we have $2^p - 1 = q^2 - q + 1$, so $2^p \equiv 1(modD(q))$. Therefore, by Lemma 3.1, we have $2^p = q^6$, therefore $q \equiv 0(mod2)$, i.e., $2 \mid q$. Also we have $2^p - 1 = q^2 - q + 1$, then $q(q - 1) = 2(2^{p-1} - 1)$, therefore we deduce $4 \nmid q$. Since

$2 \mid q$ and $4 \nmid q$, then $q = 2$ a contradiction. (we have $q \geqslant 4$)

Case(7): $P \cong D_p(q'), p \geqslant 5, q' = 2, 3, 5$. By Table 1, we have $q^2 - q + 1 = \frac{q'^p - 1}{q-1}$, therefore $q'^p \equiv 1(modD(q))$. Then, by Lemma 3.1, we have $q'^p = q^6$. Since $p \geqslant 5$ then $p(p-1) \geqslant 20$. From this we deduce that $q'^{p(p-1)} > q^6$, which is impossible, by Lemma 3.1.(Since $p > 3 \Rightarrow q'^p \neq 27$)

Case(8): $P \cong D_{p+1}(q'), q' = 2, 3$. By Table 1, if $q' = 2$, then we have $q^2 - q + 1 = 2^p - 1$, therefore, $2^p \equiv 1(modD(q))$. Then, by Lemma 3.1, we have $2^p = q^6$, hence $q \equiv 0(mod2)$. Also, $q^2 - q + 1 = 2^p - 1$, then $q(q-1) = 2(2^{p-1} - 1)$, therefore $4 \nmid q$, which imply $q = 2$, a contradiction. If $q' = 3$, then we have $q^2 - q + 1 = \frac{3^p - 1}{2}$, therefore $3^p \equiv 1(modD(q))$. Then, by Lemma 3.1, we have $3^p = q^6$, hence $q \equiv 0(mod3)$, a contradiction, or $3^p = 27$ and $q = 4$, hence $p = 3$. Therefore we have $P \cong D_4(3)$, but we have $|D_4(3)| > |G_2(4)|$, a contradiction..

Case(9): $P \cong F_4(q')$. By Tables 1 and 2, the odd order components of $F_4(q')$ are $q'^4 - q'^2 + 1$ and $q'^4 + 1$. If $q'^4 - q'^2 + 1 = q^2 - q + 1$, then $q'^2(q'^2 - 1) = q(q-1)$, hence $q = q'^2$, therefore we deduce $q'^{24} = q^{12} > q^6$, which is impossible, by Lemma 3.1.
If $q^2 - q + 1 = q'^4 + 1$, then $q(q-1) = q'^4$, that is impossible.

Case(10): $P \cong {}^2G_2(q'), q' = 3^{2m+1} > 3$. By Table 2, we have $q^2 - q + 1 = q' \pm \sqrt{3q'} + 1 = 3^{2m+1} \pm \sqrt{3^{2(m+1)}} + 1$, hence $q(q-1) = 3^{m+1}(3^m \pm 1)$, therefore $q = 3^{m+1}$ or $q = 3^m \pm 1$.
If $q = 3^{m+1}$, then $q \equiv 0(mod3)$, a contradiction.
If $q = 3^m \pm 1$, then from $q = 3^m + 1$ we deduce $q(q-1) = 3^m(3^m + 1)$, then $3^m(3^m + 1) = 3^{m+1}(3^m + 1)$, therefore $3^{m+1} = 3^m$, which is impossible and from $q = 3^m - 1$ we deduce $q(q-1) = (3^m - 1)(3^m - 2)$, then $(3^m - 2)(3^m - 1) = 3^{m+1}(3^m - 1)$, therefore $3^m - 2 = 3^{m+1}$, which is impossible.

Case(11): $P \cong E_8(q')$. By Table 3, the odd order components of $E_8(q')$ are $q'^8 - q'^4 + 1$, $\frac{q'^{10} \pm q'^5 + 1}{q'^2 \pm q' + 1}$ and $\frac{q'^{10}+1}{q'^2+1}$.
If $q^2 - q + 1 = q'^8 - q'^4 + 1$, then $q(q-1) = q'^4(q'^4 - 1)$. From this we deduce $q = q'^4$, then $q'^{120} = q^{30} > q^6$, which is impossible by Lemma 3.1.
If $q^2 - q + 1 = \frac{q'^{10} + q'^5 + 1}{q'^2 + q' + 1}$, then $q'^{15} \equiv 1(modD(q))$. Hence, by Lemma 3.1, $q'^{15} = q^6$, then $q'^{120} = q^{48} > q^6$, which is impossible by Lemma 3.1.
If $q^2 - q + 1 = \frac{q'^{10} - q'^5 + 1}{q'^2 - q' + 1}$ then $q'^{30} \equiv 1(modD(q))$. Hence, by Lemma 3.1, $q'^{30} = q^6$, then $q'^{120} = q^{24} > q^6$, which is impossible by Lemma 3.1.
If $q^2 - q + 1 = \frac{q'^{10}+1}{q'^2+1}$, then $q'^{20} \equiv 1(modD(q))$. Hence, by Lemma 3.1, $q'^{20} = q^6$, then $q'^{120} = q^{36} > q^6$, which is impossible by Lemma 3.1.

Case(12): $P \cong {}^2E_6(q')$, $q > 2$. By Table 1, we have $\frac{q'^6 - q'^3 + 1}{(3, q'+1)} = q^2 - q + 1$. Now if $(3, q'+1) = 1$, we have $q'^6 - q'^3 + 1 = q^2 - q + 1$, then $q'^3(q'^3 - 1) = q(q-1)$, therefore $q'^3 = q$. From this we deduce $q'^{36} = q^{12} > q^6$, which is impossible by Lemma 3.1.
If $(3, q'+1) = 3$, then $\frac{q'^6 - q'^3 + 1}{3} = q^2 - q + 1$. From this we deduce $q'^{18} \equiv 1(modD(q))$, then by Lemma 3.1, we have $q'^{18} = q^6$, this implies $q'^{36} = q^{12} > q^6$, which is impossible by Lemma 3.1.

Case(13): $P \cong {}^2D_n(2)$, $n = 2^m + 1 \geqslant 5$. By Table 1, $q^2 - q + 1 = 2^{n-1} + 1$, then $q(q-1) = 2^{n-1}$, a contradiction.

Case(14): $P \cong A_p(q')$, where $(q'-1) \mid (p+1)$. By Table 1, $q^2-q+1 = \frac{q'^p-1}{q'-1}$, then $q'^p \equiv 1(modD(q))$, therefore, by Lemma 3.1, we have $q'^p = q^6$, hence $q'^{p(p+1)/2} > q^6$, which is impossible by Lemma 3.1, or $q'^p = 27$ and $q = 4$, hence $p = 3$ and $q' = 3$. Therefore we have $P \cong A_3(3)$, but we have $3^6 \mid |A_3(3)|$ and $3^6 \nmid |G_2(4)|$, a contradiction.

Case(15): $P \cong {}^2D_p(3)$, where $5 \leqslant p$. By Tables 1 and 2, the odd order components of ${}^2D_p(3)$ are $\frac{3^p+1}{4}$ and $\frac{3^{p-1}+1}{2}$. If $q^2-q+1 = \frac{3^p+1}{4}$, then $3^{2p} \equiv 1(modD(q))$, so, by Lemma 3.1, $3^{2p} = q^6$, therefore $q \equiv 0(mod3)$, a contradiction or $3^{2p} = 27$ and $q = 4$, therefore $2p = 3$ is impossible. If $q^2-q+1 = \frac{3^{p-1}+1}{2}$, then $3^{2p-2} \equiv 1(modD(q))$, so, by Lemma 3.1, $3^{2p-2} = q^6$, therefore $q \equiv 0(mod3)$, a contradiction or $3^{2p-2} = 27$ and $q = 4$, therefore $2p-2 = 3$, hence $2p = 5$ is impossible.

Case(16): $P \cong {}^2D_n(3)$, where $5 \leqslant p \neq 2^m+1$. By Table 1, $q^2-q+1 = \frac{3^{n-1}+1}{2}$, we deduce $3^{2n-2} \equiv 1(modD(q))$. Then, by Lemma 3.1, we have $3^{2n-2} = q^6$, therefore $q \equiv 0(mod3)$, a contradiction or or $3^{2n-2} = 27$ and $q = 4$, therefore $2n-2 = 3$, hence $2n = 5$ is impossible.

Case(17): $P \cong {}^2B_2(q')$, where $q' = 2^{2m+1} > 2$. By Table 3, the odd order components of ${}^2B_2(q')$ are $q'-1$, $q'-\sqrt{2q'}+1$ and $q'+\sqrt{2q'}+1$. If $q^2-q+1 = q'-1$ then we have $q' \equiv 1(modD(q))$. Hence, by Lemma 3.1, $q' = q^6$, then we deduce $q'^2 = q^{12} > q^6$, which is impossible by Lemma 3.1

If $q^2-q+1 = q' \pm \sqrt{2q'}+1$, then, $q(q-1) = 2^{m+1}(2^m \pm 1)$. Therefore, $2^{m+1} \mid q$ or $2^{m+1} \mid (q-1)$. If $2^{m+1} \mid q$, then $q = 2^{m+1}$, hence $q(q-1) = 2^{m+1}(2^{m+1}-1) > 2^{m+1}(2^m-1)$, a contradiction. If $2^{m+1} \mid (q-1)$, then $q-1 = 2^{m+1} = 3k$, which is impossible.

Case(18): $P \cong {}^2F_4(q')$, where $q' = 2^{2m+1} > 2$. By Table 2, the odd order components of ${}^2F_4(q')$ are $q' \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'}+1$. Then $q^2-q+1 = q' \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'}+1$, hence $q(q-1) = 2^{m+1}(2^{3m+1} \pm 2^{2m+1} + 2^m \pm 1)$. From this equation we deduce $2^{m+1} \mid q$ or $2^{m+1} \mid (q-1)$. If $2^{m+1} \mid q$, then $q = 2^{m+1}$, which implies $2^{m+1}(2^{m+1}-1) = 2^{m+1}(2^{3m+1} \pm 2^{2m+1} + 2^m \pm 1)$, which is impossible. Similar to above case we can deduce $2^{m+1} \mid (q-1)$, which is impossible.

Case(19): $P \cong A_{p-1}(q'), (p,q') \neq (3,2),(3,4)$. By Table 1, $q^2-q+1 = \frac{q'^p-1}{(p,q'-1)(q'-1)}$. Then $q'^p \equiv 1(modD(q))$, therefore, by Lemma 3.1, we deduce $q'^p = q^6$. Since $p \geqslant 5$, then $q'^{\frac{p(p-1)}{2}} > q^6$, which is impossible by Lemma 3.1.

Case(20): $P \cong A_1(q')$, where q' is a power of 2. By Table 2, the odd order components of $A_1(q')$ are $q'+1$ and $q'-1$. If $q^2-q+1 = q'+1$, we deduce $q(q-1) = q'$, which is impossible. If $q^2-q+1 = q'-1$, then we deduce $q' \equiv 1(modD(q))$, therefore, by Lemma 3.1, we have $q' = q^6$. If $q' = q^6$, then $q^2-q+1 = q'-1 = q^6-1 = (q^3-1)(q+1)(q^2-q+1)$, from this we deduce $(q+1)(q^3-1) = 1$, which is impossible, since $q \geqslant 4$.

Case(21): $P \cong {}^2A_p(q')$, where $(q'+1) \mid (p+1)$ and $(p,q') \neq (3,3),(5,2)$. By Table 1, the odd order components of ${}^2A_p(q')$ is $\frac{q'^p+1}{q'+1}$. For both cases we have $q'^{2p} \equiv 1(modD(q))$, therefore by Lemma 3.1, we deduce $q'^{2p} = q^6$, hence $q'^p = q^3$. Since $(q'+1) \mid (p+1)$ and $q' \geqslant 4$, since if $q' = 3$, then we have $q \equiv 0(mod3)$. Hence $p > 5$, and $q'^{p(p+1)/2} > q^6$, which is impossible by Lemma 3.1, or $q'^{2p} = 27$ and $q = 4$, hence $2p = 3$ and $q' = 3$, is impossible, since $4 = (q' + 1) \mid (p + 1)$, therefore $p \geqslant 3$ and $2p \geqslant 6$.

Case(22): $P \cong {}^2D_n(q')$, $n = 2^m \geqslant 4$. By Table 1, the odd order component of ${}^2D_n(q')$ is $\frac{q'^n+1}{(2,q+1)}$. If $(2, q+1) = 1$, then we have $q^2-q+1 = q'^n+1$, hence $q(q-1) = q'^n$, which is impossible. If $(2, q + 1) = 2$, then we have $q'^{2n} \equiv 1(modD(q))$, therefore, by Lemma 3.1, $q'^{2n} = q^6$, hence $q'^n = q^3$. If $n > 4$, then we have $n - 1 > 3$, therefore $q^{n(n-1)} > q^6$, which is impossible by Lemma 3.1. Now if $n = 4$, then $P \cong {}^2D_4(q')$, hence we have $q'^4 = q^3$. By Table 1, we have
$|P| = q'^{12}(q'^2 - 1)(q'^4 - 1)(q'^6 - 1)(q'^4 + 1)/2 = q^6(q^3 - 1)(q^2 - q + 1)(q'^2 - 1)(q'^2 - 1)(q'^4 + q'^2 + 1) = q^6(q - 1)(q^2 + q + 1)(q^2 - q + 1)(q'^4 - 2q'^2 + 1)(q'^4 + q'^2 + 1) = q^6(q - 1)(q^2 + q + 1)(q^2 - q + 1)(q^3 - 2q'^2 + 1)(q^3 + q'^2 + 1) > q^6(q - 1)(q^2 + q + 1)(q^2 - q + 1)(q^3 - 2q'^2 + 1)(q^3 + 1) = q^6(q - 1)(q + 1)(q^2 + q + 1)(q^2 - q + 1)^2(q^3 - 2q'^2 + 1) > q^6(q - 1)^2(q + 1)(q^2 + q + 1)(q^2 - q + 1)(q^3 - 2q'^2 + 1) > |G|$
$((q^2 - q + 1) > (q - 1), (q^3 - 2q'^2 + 1) = (q^3 - 2q\sqrt{q} + 1) = (q(q^2 - 2\sqrt{q}) + 1) > (q + 1))$
a contradiction, or $q'^{2n} = 27$ and $q = 4$, hence $2n = 3$ is impossible.

Case(23): $P \cong C_n(q')$,$n = 2^m \geqslant 4$ or $P \cong B_n(q')$, $n = 2^m \geqslant 4$, q' odd.
In the above cases the odd order component is $\frac{q'^n+1}{2}$ and $q^2 - q + 1 = \frac{q'^n+1}{2}$, therefore, by Lemma 3.1, $q'^{2n} = q^6$, this implies $q'^n = q^3$, then we have $q'^{n^2} = q^{3n} \geqslant q^{12} > q^6$, which is impossible by Lemma 3.1 or $q'^{2n} = 27$ and $q = 4$, hence $2n = 3$ is impossible.

Case(24): $P \cong {}^2D_{p+1}(2)$, where $n \geqslant 2$ and $p = 2^n - 1$. By Table 2, the odd order components of ${}^2D_{p+1}(2)$ are 2^p+1 and $2^{p+1}+1$. If $q^2-q+1 = 2^p+1$, then $q(q-1) = 2^p$, which is impossible. If $q^2 - q + 1 = 2^{p+1} + 1$, then $q(q - 1) = 2^{p+1}$, which is impossible.

Case(25): $P \cong C_2(q')$, q' is odd. By Table 1, the odd order component of $C_2(q')$ is $\frac{q'^2+1}{2}$. If $q^2 - q + 1 = \frac{q'^2+1}{2}$, then $q'^2 = 2q^2 - 2q + 1$. From this we deduce $|C_2(q')| = q'^4(q'^2 - 1)^2(q'^2 + 1)/2 = 4q^2(q - 1)^2(q^2 - q + 1)(2q^2 - 2q + 1)^2$. Since $|P| \mid |G|$, hence $(2q^2 - 2q + 1) \mid q^6(q - 1)^2(q + 1)^2(q^2 + q + 1)$. Since

$$(2q^2 - 2q + 1, q + 1) = (5, q - 1)$$
$$(2q^2 - 2q + 1, q^2 + q + 1) = 1$$
$$(2q^2 - 2q + 1, q - 1) = 1 \qquad (3)$$

then we have $(2q^2 - 2q + 1) \mid 5^2$, this is not correct unless $q = 4$. If $q = 4$, then we have $|C_2(4)| = 2^7.3^2.5^2.17$ and $|G| = 2^{12}.3^3.5^2.7.13$. Since $|C_2(4)| \mid |G|$, but $17 \nmid |G|$, a contradiction.

Case(26): $P \cong {}^3D_4(q')$.By Table 1, we have $q^2 - q + 1 = q'^4 - q'^2 + 1$, then $q(q - 1) = q'^2(q'^2 - 1)$, therefore $q = q'^2$. From this we deduce that

$$|{}^3D_4(q')| = q'^{12}(q'^6-1)(q'^2-1)(q'^4+q'^2+1)(q'^4-q'^2+1) = q^6(q^3-1)(q-1)(q^2+q+1)(q^2 - q + 1) = q^6(q - 1)^2(q^2 + q + 1)^2(q^2 - q + 1)$$

Since $|{}^3D_4(q')| \mid |G|$, then we have $(q^2 + q + 1)^2 \mid |G|$. An easy calculation shows that

$$(q + 1, q^2 + q + 1) = 1$$
$$(q - 1, q^2 + q + 1) = (3, q - 1)$$
$$(q^2 + q + 1, q^2 - q + 1) = 1 \qquad (4)$$

Therefore $(q^2 + q + 1)^2 \mid |G| = q^6(q - 1)^2(q + 1)^2(q^2 + q + 1)(q^2 - q + 1)$ is impossible.

Case(27): $P \cong A_1(q')$, q' is not a power of 2. By Table 2, the odd order components of $A_1(q')$ are q' and $(q' + 1)/2$ or $(q' - 1)/2$. If $q^2 - q + 1 = q'$, then $|A_1(q')| = q'(q' +$

$1)(q'-1)/2 = (q^2-q+1)(q^2-q+2)q(q-1)/2$. Since $|P| \mid |G|$, we deduce $\frac{(q^2-q+2)}{2} \mid q^6(q-1)^2(q+1)^2(q^2-q+1)$. An easy calculation shows that;

$(q^2-q+2, q-1) = (2, q-1)$

$(q^2-q+2, q+1) = (4, q+1)$

$(q^2-q+2, q^2+q+1) = (7, q^2+5)$ (5)

Therefore $(q^2-q+2)/2 \mid 2^6.7$, this implies $(q^2-q+2)/2 = 2^4.7$, then for this equation we have $q = 11$, which is impossible$(q = 3k+1)$. Also we have $(q^2-q+2)/2 \mid 2^6$. From this we deduce $q^2-q+2 = 2^5$, then $q(q-1) = 6.5$, therefore we have $q = 6$, which is impossible because $q = 3k+1$.

If $4 \mid q'+1$, then $q^2-q+1 = q'-1/2$, hence $q' = 2q^2-2q+3$. From this we deduce $|A_1(q')| = 2(q^2-q+1)(q^2-q+2)(2q^2-2q+3)$. Since $|P| \mid |G|$, so we have $(q^2-q+2) \mid q^6(q-1)^2(q+1)^2(q^2+q+1)$. By (4) we have $(q^2-q+2) \mid 2^6.7$, this implies $(q^2+q+2) = 2^5.7$, then for this equation we have $q = 11$, which is impossible$(q = 3k+1)$. Also we have $(q^2-q+2) \mid 2^6$. From this we deduce $q^2-q+2 = 2^5$, then $q(q+1) = 6.5$, therefore we have $q = 6$, which is impossible because $q = 3k+1$.

If $4 \mid q'-1$, then $q^2-q+1 = q'+1/2$, hence $q' = 2q^2-2q+1$. From this we deduce $|A_1(q')| = 2q(q-1)(q^2-q+1)(2q^2-2q+1)$. Since $|P| \mid |G|$, so $(2q^2-2q+1) \mid q^6(q-1)^2(q+1)^2(q^2+q+1)$. Therefore, by (3) we have $2q^2-2q+1 \mid 5^2$, then $2q^2-2q+1 = 5$, this implies $2q(q-1) = 4$, then $q = 2$, a contradiction or $2q^2-2q+1 = 25$, this implies $2q(q-1) = 24$, then $q = 4$ and $q' = 25$. Therefore $P \cong A_1(25)$. By [6], we have $|Out(P)| = 4$ and by Lemma 2.4, we have $|G/K| \mid |Out(P)|$. Now we set $|G/K| = t$ and obtain $t = 1$, 2 or 4, and $t|H||P| = |G|$, then $t|H|(2^3.3.5^2.13) = 2^{12}.3^3.5^2.7.13$. Hence $|H| = 2^9.3^2.7/t$, where $t = 1$, 2 or 4. Now let $S \in Syl_7(H)$, then $|S| = 7$. Since H is nilpotent, therefore $S \trianglelefteq G$ and by Lemma 2.3 it follows that $m_2 \mid |S|-1$, i.e., $13 \mid 7-1$ which is impossible.

Case(28): $P \cong {}^2A_{p-1}(q')$. By Table 1, $q^2-q+1 = \frac{q'^p+1}{(q'+1)(p,q'+1)}$. Then $q'^{2p} \equiv 1(modD(q))$, therefore by Lemma 3.1, we deduce $q'^{2p} = q^6$, hence $q'^p = q^3$. Now if $p > 5$, we have $q^{p(p-1)/2} > q^6$, which is impossible by Lemma 3.1. If $p = 5$, by Table 1, $q^2-q+1 = \frac{q'^5+1}{(q'+1)(5,q'+1)}$ and $q'^5 = q^3$. Now if $(5, q'+1) = 1$, then we have $q^2-q+1 = (q^3+1)/(q+1) = (q'^5+1)/(q'+1) = (q^3+1)/(q'+1)$, then we deduce $q = q'$, which is impossible. Therefore, $(5, q'+1) = 5$, then we have $q^2-q+1 = (q^3+1)/(q+1) = \frac{q'^5+1}{5(q'+1)} = \frac{q^3+1}{5(q'+1)}$, then we have $(q+1) = 5(q'+1) = 5q'+5$, hence $q = 5q'+4$, which is impossible(q is power of a prime number). If $p = 3$, then, by Table 1, we have $q^2-q+1 = \frac{(q'^3+1)}{(q'+1)(3,q'+1)}$. Therefore, by Lemma 3.1, $q'^6 \equiv 1(modD(q))$, then $q'^6 = q^6$. From this we deduce that $q = q'$, then $q^2-q+1 = (q^3+1)/(q+1) = \frac{(q'^3+1)}{(q'+1)(3,q'+1)} = \frac{(q^3+1)}{(q+1)(3,q+1)}$, then $(q+1)(3, q+1) = (q+1)$. Therefore $(3, q+1) = 1$ and $|{}^2A_2(q)| = q^3(q+1)(q^2-1)(q^3+1)/(q+1) = q^3(q+1)^2(q-1)(q^2-q+1)$. By [6], we have $|Out(P)| = f$, such that $q^2 = r^f$, where r is a prime number. By Lemma 2.4, we have $|G/K| \mid |Out(P)|$. Now we set $|G/K| = t$ and obtain $t|H||P| = |G|$, then $t|H| = q^3(q-1)(q^2+q+1)$ and $t \mid f$. Since $q = 3k+1$ we have $q-1 = 3k$. If $t = 1$, then $|H| = q^3(q-1)(q^2+q+1)$. We have $(q-1, q^2+q+1) = 3$, therefore if we set $S \in Syl_3(H)$, then $|S| = 3(q-1)_3$. Since H is nilpotent, therefore $S \trianglelefteq G$ and by Lemma 2.3 it follows that $m_2 \mid |S|-1$, i.e., $q^2-q+1 \mid 3(q-1)_3-1$ which is impossible.

Table 1.: The order components of finite simple groups P with $s(P) = 2$

P	Restrictions on P	m_1	m_2
A_n	$6 < n = p, p+1, p+2$ one of n, $n-2$ is not a prime	$n!/2p$	p
$A_{p-1}(q)$	$(p,q) \neq (3,2),(3,4)$	$q^{\frac{p(p-1)}{2}} \prod_{i=1}^{p-1}(q^i - 1)$	$\frac{q^p-1}{(q-1)(p,q-1)}$
$A_p(q)$	$(q-1) \mid (p+1)$	$q^{\frac{p(p+1)}{2}}(q^{p+1}-1)\prod_{i=2}^{p-1}(q^i-1)$	$\frac{q^p-1}{q-1}$
$^2A_{p-1}(q)$		$q^{\frac{p(p-1)}{2}} \prod_{i=1}^{p-1}(q^i-(-1)^i)$	$\frac{q^p+1}{(q+1)(p,q+1)}$
$^2A_p(q)$	$(q+1) \mid (p+1)$ $(p,q) \neq (3,3),(5,2)$	$q^{\frac{p(p+1)}{2}}(q^{p+1}-1)\prod_{i=2}^{p-1}(q^i-1)$	$\frac{q^p+1}{q+1}$
$^2A_3(2)$		$2^6.3^4$	5
$B_n(q)$	$n = 2^m \geqslant 4$, q odd	$q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{2}$
$B_p(3)$		$3^{p^2}(3^p+1)\prod_{i=1}^{p-1}(3^{2i}-1)$	$\frac{3^p-1}{2}$
$C_n(q)$	$n = 2^m \geqslant 2$, q odd	$q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{(2,q-1)}$
$C_p(q)$	$q = 2,3$	$q^{p^2}(q^p+1)\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{(2,q-1)}$
$D_p(q)$	$p \geqslant 5, q = 2,3,5$	$q^{p(p-1)}\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{q-1}$
$D_{p+1}(q)$	$q = 2,3$	$\frac{1}{(2,q-1)}q^{p(p+1)}(q^p+1)$ $(q^{p+1}-1)\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{(2,q-1)}$
$^2D_n(q)$	$n = 2^m \geqslant 4$	$q^{n(n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{(2,q+1)}$
$^2D_n(2)$	$n = 2^m+1 \geqslant 5$	$2^{n(n-1)}(2^n+1)(2^{n-1}-1)$ $\prod_{i=1}^{n-2}(2^{2i}-1)$	$2^{n-1}+1$
$^2D_p(3)$	$5 \leqslant p \neq 2^m+1$	$3^{p(p-1)}\prod_{i=1}^{p-1}(3^{2i}-1)$	$\frac{3^p+1}{4}$
$^2D_n(3)$	$9 \leqslant 2^m+1 \neq p$	$\frac{1}{2}3^{n(n-1)}(3^n+1)(3^{n-1}-1)$ $\prod_{i=1}^{n-2}(3^{2i}-1)$	$\frac{3^{n-1}+1}{2}$
$G_2(q)$	$2 < q \equiv \epsilon(mod3), \epsilon = \pm 1$	$q^6(q^3-\epsilon)(q^2-1)(q+\epsilon)$	$q^2-\epsilon q+1$
$^3D_4(q)$		$q^{12}(q^6-1)(q^2-1)(q^4+q^2+1)$	q^4-q^2+1
$F_4(q)$	q odd	$q^{24}(q^8-1)(q^6-1)^2(q^4-1)$	q^4-q^2+1
$^2F_4(2)'$		$2^{11}.3^3.5^2$	13
$E_6(q)$		$q^{36}(q^{12}-1)(q^8-1)(q^6-1)$ $(q^5-1)(q^3-1)(q^2-1)$	$\frac{q^6+q^3+1}{(3,q-1)}$
$^2E_6(q)$	$q > 2$	$q^{36}(q^{12}-1)(q^8-1)(q^6-1)$ $(q^5+1)(q^3+1)(q^2-1)$	$\frac{q^6-q^3+1}{(3,q+1)}$
M_{12}		$2^6.3^3.5$	11
J_2		$2^7.3^3.5^2$	7
Ru		$2^{14}.3^3.5^3.7.13$	29
He		$2^{10}.3^3.5^2.7^3$	17
McL		$2^7.3^6.5^3.7$	11
Co_1		$2^{21}.3^9.5^4.7^2.11.13$	23
Co_3		$2^{10}.3^7.5^3.7.11$	23
Fi_{22}		$2^{17}.3^9.5^2.7.11$	13
HN		$2^{14}.3^6.5^6.7.11$	19

Table 2.: The order components of finite simple groups P with$s(P) = 3$

P	Restrictions on P	m_1	m_2	m_3
A_n	$n > 6, n = p$, $p-2$ are primes	$\frac{n!}{2n(n-2)}$	p	$p-2$
$A_1(q)$	$4 \mid (q+1)$	$q+1$	q	$\frac{q-1}{2}$
$A_1(q)$	$4 \mid (q-1)$	$q-1$	q	$\frac{q+1}{2}$
$A_1(q)$	$2 \mid q$	q	$q+1$	$q-1$
$A_2(2)$		8	3	7
${}^2A_5(2)$		$2^{15}.3^6.5$	7	11
${}^2D_p(3)$	$5 \leqslant p = 2^m + 1$	$2.3^{p(p-1)}(3^{p-1}-1) \prod_{i=1}^{p-2}(3^{2i}-1)$	$\frac{3^{p-1}+1}{2}$	$\frac{3^p+1}{4}$
${}^2D_{p+1}(2)$	$n \geqslant 2, p = 2^n - 1$	$2^{p(p+1)}(2^p-1) \prod_{i=1}^{p-1}(2^{2i}-1)$	2^p+1	$2^{p+1}+1$
$G_2(q)$	$q \equiv 0 (mod 3)$	$q^6(q^2-1)^3$	q^2-q+1	q^2+q+1
${}^2G_2(q)$	$q = 3^{2m+1} > 3$	$q^3(q^2-1)$	$q-\sqrt{3q}+1$	$q+\sqrt{3q}+1$
$F_4(q)$	q even	$q^{24}(q^6-1)^2(q^4-1)^2$	q^4+1	q^4-q^2+1
${}^2F_4(q)$	$q = 2^{2m+1} > 2$	$q^{12}(q^4-1)q^3+1)$	$q^2-\sqrt{2q^3}+ q-\sqrt{2q}+1$	$q^2+\sqrt{2q^3}+ q+\sqrt{2q}+1$
$E_7(2)$		$2^{36}.3^{11}.5^2.7^3.11.13$ $17.19.31.43$	73	127
$E_7(3)$		$2^{23}.3^{63}.5^2.7^3.11^2.13^2$ $19.37.41.61.73.547$	757	1093
M_{11}		$2^4.3^2$	5	11
M_{23}		$2^7.3^2.5.7$	11	23
M_{24}		$2^{10}.3^3.5.7$	11	23
J_3		$2^7.3^5.5$	17	19
HiS		$2^9.3^2.5^3$	7	11
Suz		$2^{13}.3^7.5^2.7$	11	13
Co_2		$2^{18}.3^6.5^3.7$	11	23
Fi_{23}		$2^{18}.3^{13}.5^2.7.11.13$	17	23
F_3		$2^{15}.3^{10}.5^3.7^2.13$	19	31
F_2		$2^{24}.3^{13}.5^6.7^2.$ $11.13.17.19.23$	31	47

Table 3.: The order components of finite simple groups P with $s(P) > 3$

P	Restrictions on P	m_1	m_2	m_3	m_4	m_5	m_6
$A_2(4)$		2^6	3	5	7		
${}^2B_2(q)$	$q = 2^{2m+1} > 2$	q^2	$q-1$	$q-\sqrt{2q}+1$	$q+\sqrt{2q}+1$		
${}^2E_6(2)$		$2^{36}.3^9.5^2.7^2.11$	13	17	19		
$E_8(q)$	$q \equiv 2,3$ $(mod5)$	$q^{120}(q^{20}-1)(q^{18}-1)$ $(q^{14}-1)(q^{12}-1)$ $(q^{10}-1)(q^8-1)$ $(q^4+1)(q^4+q^2+1)$	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4+1		
M_{22}		$2^7.3^2$	5	7	11		
J_1		$2^3.3.5$	7	11	19		
$O'N$		$2^9.3^4.5.7^3$	11	19	31		
LyS		$2^8.3^7.5^6.7.11$	31	37	67		
Fi'_{24}		$2^{21}.3^{16}.5^2.7^3.11.13$	17	23	29		
F_1		$2^{46}.3^{20}.5^9.7^6.11^2.13^3$ 17.19.23.29.31.47	41	59	71		
$E_8(q)$	$q \equiv 0,1,4$ $(mod5)$	$q^{120}(q^{18}-1)(q^{14}-1)$ $(q^{12}-1)^2(q^{10}-1)^2$ $(q^8-1)^2(q^4+q^2+1)$	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4+1	$\frac{q^{10}+1}{q^2+1}$	
J_4		$2^{21}.3^3.5.7.11^3$	23	29	31	37	43

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