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A note on the convergence of the Zakharov-Kuznetsov equation by homotopy analysis method

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Abstract. In this paper, the convergence of Zakharov-Kuznetsov (ZK) equation by homotopy analysis method (HAM) is investigated. A theorem is proved to guarantee the convergence of HAM and to find the series solution of this equation via a reliable algorithm.

Keywords: Homotopy analysis method, Zakharov-Kuznetsov equation, convergence, partial differential equation, recursive method.

2010 AMS Subject Classification: 65M12.

1. Introduction

The Zakharov-Kuznetsov equation ZK(m, n, k) governs the behavior of weakly nonlinear ion-acoustic waves in plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [12, 13]. The tanh method was applied by Wazwaz to solve the modified ZK equation [15]. Huang applied the polynomial expansion method to solve the coupled ZK equations [4]. Zhao et al. obtained numbers of solitary waves, periodic waves and kink waves using the theory of bifurcations of dynamical systems for the modified ZK equation [16]. Inc solved nonlinear dispersive ZK equations using the Adomian decomposition method [6]. Biazar et al. used the homotopy perturbation method to solve the Zakharov-Kuznetsov equations [3]. Hesam et al. applied the differential transform method to obtain the analytical solution of Zakharov-Kuznetsov equations [5] and Usman et al. obtained the series solution of Zakharov-Kuznetsov equations by

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homotopy analysis method [14].
In this work, we study on the convergence the HAM to use on the ZK equation and we prove a convergence theorem to illustrate if this method is convergent, it converges to the exact solution of the equation. We consider the following ZK(m, n, k) equation:

$$u_t + a(u^p)_x + b(u^l)_{xxx} + c(u^k)_{yyx} = 0, \qquad p, l, k \neq 0, \tag{1}$$

where a, b, c are arbitrary constants and m, n, k are positive integers. At first in section 2, we remind the main idea of HAM, then in sections 3, we prove the convergence theorem for ZK equation.

2. Preliminaries

In order to describe the HAM [1–3, 7–11], we consider the following differential equation:

$$N[u(x, y, t)] = 0, \tag{2}$$

where N is a nonlinear operator, x, y, t denote the independent variables and u is an unknown function. By means of the HAM, we construct the zeroth-order deformation equation

$$(1 - q)L[\Phi(x, y, t; q) - u_0(x, y, t)] = qhH(x, y, t)[\Phi(x, t; q)], \tag{3}$$

where $q \in [0, 1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator and $H(x, y, t)$ is an auxiliary function. $\Phi(x, y, t; q)$ is an unknown function and $u_0(x, y, t)$ is an initial guess of $u(x, y, t)$. It is obvious that when $q = 0$ and $q = 1$, we have:

$$\Phi(x, y, t; 0) = u_0(x, y, t), \qquad \Phi(x, y, t; 1) = u(x, y, t),$$

respectively. therefore, as q increase from 0 to 1, the solution $\Phi(x, y, t; q)$ varies from the $u_0(x, y, t)$ to the exact solution $u(x, y, t)$. By Taylor's theorem, we expand $\Phi(x, y, t; q)$ in a power series of the embedding parameter q as follows:

$$\Phi(x, y, t; q) = u_0(x, y, t) + \sum_{m=1}^{+\infty} u_m(x, y, t)q^m \tag{4}$$

where

$$u_m(x, y, t) = \frac{1}{m!}\frac{\partial^m \Phi(x, y, t; q)}{\partial q^m}\Big|_{q=0} \tag{5}$$

Let the initial guess $u_0(x, y, t)$, the auxiliary linear operator L, the nonzero auxiliary parameter h and the auxiliary function $H(x, y, t)$ be properly chosen so that the power series (4) converges at $q = 1$, then, we have:

$$u(x,y,t) = u_0(x,y,t) + \sum_{m=1}^{+\infty} u_m(x,y,t), \tag{6}$$

which must be one of the solution of the original nonlinear equation. Define the vectors

$$\overrightarrow{u}_n = \{u_0(x,y,t), u_1(x,y,t), \ldots, u_n(x,y,t)\}. \tag{7}$$

By differentiating the zeroth order deformation (3) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing theme by $m!$, we get the following $m-$th order deformation equation:

$$L[u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)] = hH(x,y,t)R_m(\overrightarrow{u}_{m-1}), \tag{8}$$

where

$$R_m(\overrightarrow{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\Phi(x,y,t;q)]}{\partial q^{m-1}}\Big|_{q=0}, \tag{9}$$

and

$$\chi_m = \begin{cases} 0, \, m \leqslant 1, \\ 1, \, m > 1. \end{cases} \tag{10}$$

It should be emphasized that $u_m(x,y,t)$ for $m \geqslant 1$ is governed by the linear equation (8) with initial conditions that come from the original problem [7].

3. Main Idea

In this section, at first a lemma is proved which is applied to complete the proof of the next theorem that proves the convergence of the HAM on Eq. (1).

Lemma 3.1 According to the concept of the HAM, for $r \in \mathbb{N}$,

$$\sum_{m=1}^{+\infty} \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}(\phi^r(x,y,t;q))\Big|_{q=0} = [\sum_{m=0}^{+\infty} u_m]^r.$$

Proof. The proof is by induction on r. At first, we suppose $r = 1$, therefore, according to the Eq.(5) we have, $\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}(\phi(x,y,t;q))\big|_{q=0} = u_{m-1}$. Therefore,

$$\sum_{m=1}^{+\infty} \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}(\phi(x,y,t;q))\Big|_{q=0} = \sum_{m=0}^{+\infty} u_m$$

If $r = 2$, we have,

$$\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}(\phi^2(x,y,t;q))\Big|_{q=0} =$$

$$\frac{1}{(m-1)!}\sum_{j=0}^{m-1}\frac{(m-1)!}{j!(m-1-j)!}\frac{\partial^j\phi(x,y,t;q)}{\partial q^j}\frac{\partial^{m-j-1}\phi(x,y,t;q)}{\partial q^{m-j-1}}\Big|_{q=0} = \sum_{j=0}^{m-1}u_j u_{m-j-1}.$$

Therefore, we have,

$$\sum_{m=1}^{+\infty}\sum_{j=0}^{m-1}u_j u_{m-j-1} = \sum_{j=0}^{+\infty}\sum_{m=j+1}^{+\infty}u_j u_{m-j-1} = \sum_{j=0}^{+\infty}u_j\sum_{i=0}^{+\infty}u_i = [\sum_{m=0}^{+\infty}u_m]^2. \qquad (11)$$

Now, we suppose , $\sum_{m=1}^{+\infty}\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}(\phi^r(x,y,t;q))\Big|_{q=0} = [\sum_{m=1}^{+\infty}u_m]^r$. It must be proved that for $r+1$. For this purpose,

$$\frac{1}{(m-1)!}\frac{\partial^{m-1}}{q^{m-1}}(\phi^{r+1}(x,y,t;q))\Big|_{q=0} = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}((\phi^r(x,y,t;q)\phi(x,y,t;q)))\Big|_{q=0}.$$

Therefore

$$\sum_{m=1}^{+\infty}\frac{1}{(m-1)!}\sum_{j=0}^{m-1}\frac{(m-1)!}{j!(m-1-j)!}\frac{\partial^j\phi^r(x,y,t;q)}{\partial q^j}\frac{\partial^{m-j-1}\phi(x,y,t;q)}{\partial q^{m-j-1}}\Big|_{q=0} =$$

$$\sum_{j=0}^{+\infty}\sum_{m=j+1}^{+\infty}\frac{1}{j!}\frac{\partial^j\phi^r(x,y,t;q)}{\partial q^j}u_{m-j-1} = \sum_{j=0}^{+\infty}\frac{1}{j!}\frac{\partial^j\phi^r(x,y,t;q)}{\partial q^j}\sum_{m=j+1}^{+\infty}u_{m-j-1} =$$

$$[\sum_{j=0}^{+\infty}u_j]^r\sum_{i=0}^{+\infty}u_i = [\sum_{m=0}^{+\infty}u_m]^{r+1}.$$

■

Theorem 3.2 If the series solution (6) of problem (1) obtained from the HAM and also the series $\sum_{m=0}^{+\infty}\frac{\partial u_m}{\partial t}$ $\sum_{m=0}^{+\infty}\frac{\partial u_m^p}{\partial x}$,$\sum_{m=0}^{+\infty}\frac{\partial^3 u_m^l}{\partial x^3}$ and $\sum_{m=0}^{+\infty}\frac{\partial^3 u_m^k}{\partial y^2\partial x}$ are convergent then (6) converges to the exact solution of the Eq. (1).

Proof. Let,

$$u(x,y,t) = \sum_{m=0}^{+\infty}u_m(x,y,t)$$

where

$$\lim_{m\to+\infty} u_m(x,y,t) = 0. \tag{12}$$

We can write,

$$\sum_{m=1}^{n}[u_m(x,y,t)-\chi_m u_{m-1}(x,y,t)] = u_1+(u_2-u_1)+(u_3-u_2)+\cdots+(u_n-u_{n-1}) = u_n(x,y,t),$$

using (12), we have,

$$\sum_{m=1}^{+\infty}[u_m(x,y,t)-\chi_m u_{m-1}(x,y,t)] = \lim_{n\to+\infty} u_n(x,y,t) = 0.$$

Since L is a linear operator, we can write

$$\sum_{m=1}^{+\infty} L[u_m(x,y,t)-\chi_m u_{m-1}(x,y,t)] = L\sum_{m=1}^{+\infty}[u_m(x,y,t)-\chi_m u_{m-1}(x,y,t)] = 0.$$

From above expression and equation (8), we obtain

$$\sum_{m=1}^{+\infty} L[u_m(x,y,t)-\chi_m u_{m-1}(x,y,t)] = hH(x,y,t)\sum_{m=1}^{+\infty}[R_m(\overrightarrow{u}_{m-1})].$$

Since $h \neq 0$ and $H(x,y,t) \neq 0$, we have

$$\sum_{m=1}^{+\infty}[R_m(\overrightarrow{u}_{m-1})] = 0. \tag{13}$$

From (9),it holds

$$\sum_{m=1}^{+\infty}[R_m(\overrightarrow{u}_{m-1})] =$$

$$\sum_{m=1}^{+\infty}\Big[\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}[\frac{\partial\phi(x,y,t;q)}{\partial t}+a\frac{\partial\phi^p(x,y,t;q)}{\partial x}+b\frac{\partial^3\phi^l(x,y,t;q)}{\partial x^3}+c\frac{\partial^3\phi^k(x,y,t;q)}{\partial y^2\partial x}]\Big|_{q=0}\Big]. \tag{14}$$

According to the hypotheses of the theorem and also lemma 3.1, we have,

$$\sum_{m=1}^{+\infty}[R_m(\overrightarrow{u}_{m-1})] =$$

$$\sum_{m=1}^{+\infty}\Big[\frac{1}{(m-1)!}[\frac{\partial\partial^{m-1}\phi(x,y,t;q)}{\partial t\partial q^{m-1}} + a\frac{\partial\partial^{m-1}\phi^p(x,y,t;q)}{\partial x\partial q^{m-1}} +$$

$$b\frac{\partial^3\partial^{m-1}\phi^l(x,y,t;q)}{\partial x^3\partial q^{m-1}} + c\frac{\partial^3\partial^{m-1}\phi^k(x,y,t;q)}{\partial y^2\partial x\partial q^{m-1}}]\Big|_{q=0}\Big] =$$

$$\frac{\partial}{\partial t}\sum_{m=0}^{+\infty}u_m + a\frac{\partial}{\partial x}[\sum_{m=0}^{+\infty}u_m]^p + b\frac{\partial^3}{\partial x^3}[\sum_{m=0}^{+\infty}u_m]^l + c\frac{\partial^3}{\partial y^2\partial x}[\sum_{m=0}^{+\infty}u_m]^k. \tag{15}$$

From (13) and (15), we have

$$u_t + a(u^p)_x + b(u^l)_{xxx} + c(u^k)_{yyx} = 0.$$

■

4. Conclusion

In this paper, we proved a theorem on the convergence of the homotopy analysis method to solve the Zakharov-Kuznetsov equation. Since the nonlinearity part of the equation is complicated, we applied an auxiliary relation which was proved in a lemma and used mathematical induction to complete the proof of the convergence theorem. Therefore, the HAM can be an efficient and reliable method to solve a nonlinear partial differential equation with strong nonlinearity like ZK equation.

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