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Upper and lower $\alpha(\mu_X, \mu_Y)$-continuous multifunctions

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Abstract. In this paper, a new class of multifunctions, called generalized $\alpha(\mu_X, \mu_Y)$-continuous multifunctions, has been defined and studied. Some characterizations and several properties concerning generalized $\alpha(\mu_X, \mu_Y)$-continuous multifunctions are obtained. The relationships between generalized $\alpha(\mu_X, \mu_Y)$-continuous multifunctions and some known concepts are also discussed.

Keywords: Generalized open sets, multifunction, generalized continuity.

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1. Introduction

There are various types of functions which play an important role in the classical theory of set topology. A great deal of works on such functions has been extended to the setting of multifunctions. A multifunction is a set-valued function. The theory of multifunctions was first codified by Berge [10]. In the last three decades, the theory of multifunctions has advanced in a variety of ways and applications of this theory, can be found for example, in economic theory, noncooparative games, artificial intelligence, medicine, information sciences and decision theory (See [14] and references therein). Continuity is a basic concept for the study of general topological spaces. This concept has been extended to the setting of multifunctions and has been generalized by weaker forms of open sets such as α-open sets [18], semiopen sets [17], preopen sets [9], $\beta-$

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open sets [16] and semi-preopen sets [13] . Multifunctions and of course continuous multifunctions stand among the most important and most researched points in the whole of the mathematical science. Many different forms of continuous multifunctions have been introduced over the years. Csaszar [1] introduced the notions of generalized topological spaces and neighborhood systems are contained in these classes, respectively. Specifically, he introduced the notions of continuous functions on generalized topological spaces and investigated the characterizations of generalized continuous functions. By using these consepts, Min [21] introduced the notions of $\alpha\left(g_X, g_Y\right)$-continuity, $\beta\left(g_X, g_Y\right)$ -continuity, $pre\left(g_X, g_Y\right)$-continuity and $semi\left(g_X, g_Y\right)$-continuity of functions on generalized topological spaces. Kanibir and Reilly [8] extended these concepts to multifunctions. Also, Boonpok [12] studied the $\beta\left(\mu_X, \mu_Y\right)$-continuous multifunctions. Then Akdağ and Erol [15], studied the $pre\left(\mu_X, \mu_Y\right)$ -continuous multifunctions. In this paper our purpose is to define $\alpha\left(\mu_X, \mu_Y\right)$-continuous multifunctions and to obtain some characterizations and several properties concerning such multifunctions. Moreover, the relationships between generalized $\alpha\left(\mu_X, \mu_Y\right)$-continuous multifunctions and some known concepts are also discussed.

2. Preliminaries

Let X be a nonempty set and μ be a collection of subsets of X. Then μ is called a generaized topology (briefly GT) on X iff $\emptyset \in \mu$ and an arbitrary union of elements of μ belongs to μ [1] . A set with a GT μ is said to be generalized topological spaces (briefly GTS) and denoted by (X, μ). The elements of μ are called $\mu - open$ sets and their complements are called $\mu - closed$ sets. The generalized interior of a subset A of X denoted by $i_\mu\left(A\right)$ is the union of $\mu - open$ sets contained in A, and the generalized closure of A denoted by $c_\mu\left(A\right)$ is the intersection of $\mu - closed$ sets containing A. It is easy to verify that $c_\mu\left(A\right) = X - i_\mu\left(X - A\right)$ and $i_\mu\left(A\right) = X - c_\mu\left(X - A\right)$. Let μ be a GT on a set $X \neq \varnothing$. Clear that $X \in \mu$ must not hold; if all the same $X \in \mu$, then we say that the GT μ is strong [2] . In general M_μ denote the union of all elements of μ, of course $M_\mu \in \mu$ and μ is strong GT if and only if $M_\mu = X$ [2] . Let us now consider those GTS μ satisfy the following condition: if $M, M' \in \mu$, then $M \cap M' \in \mu$. We call such a GT *quasitopology (briefly QT)* [3], the QTS clearly are very near to the topologies.

A subset A of a generalized topological spaces (X, μ) is said to be $\mu regular - open$ [3] (resp. $\mu regular - closed$) if $A = i_\mu(c_\mu(A))$ (resp. $A = c\mu(i_\mu(A))$). A subset A of a generalized topological spaces (X, μ) is said to be $\mu - semiopen$ [3] (resp. $\mu - preopen$, $\mu - \alpha - open$ and $\mu - \beta - open$) if $A \subseteq c\mu(i_\mu(A))$ (resp. $A \subseteq i_\mu(c_\mu(A))$, $A \subseteq i_\mu(c_\mu\left(i_\mu(A)\right))$, $A \subseteq c_\mu\left(i_\mu(c_\mu(A))\right)$). The family of all $\mu - semiopen$ (resp. $\mu - preopen$, $\mu - \alpha - open$ and $\mu - \beta - open$) sets of X is denoted by $\sigma\left(\mu\right)$ (resp. $\pi\left(\mu\right)$, $\alpha\left(\mu\right)$ and $\beta\left(\mu\right)$). It is shown in [3] , that $\alpha\left(\mu\right) = \pi\left(\mu\right) \cap \sigma\left(\mu\right)$ and it is obvious that $\sigma\left(\mu\right) \cup \pi\left(\mu\right) \subseteq \beta\left(\mu\right)$. The complement of a $\mu - semiopen$ (resp. $\mu - preopen$, $\mu - \alpha - open$ and $\mu - \beta - open$) set is said to be $\mu - semiclosed$ (resp. $\mu - preclosed$, $\mu - \alpha - closed$ and $\mu - \beta - closed$). The intersection of all $\mu - semiclosed$ (resp. $\mu - preclosed$, $\mu - \alpha - closed$ and $\mu - \beta - closed$) sets of X containing A is denoted by $c_\sigma\left(A\right)$, (resp. $c_\pi\left(A\right)$, $c_\alpha\left(A\right)$ and $c_\beta\left(A\right)$) are defined similarly. The union of all $\mu - \alpha - open$ ($\mu - \beta - open$) sets containing in A is denoted by $i_\alpha(A)$ ($i_\beta(A)$).

By a multifunction $F : X \longrightarrow Y$, we mean a point-to-set correspondence from X to Y, and we always assume that $F\left(x\right) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \longrightarrow Y$, following we shall denote the upper and lower inverse set of a set B of Y by $F^+\left(B\right)$ and $F^-\left(B\right)$ respectively, that is, $F^+\left(B\right) = \{x \in X : F\left(x\right) \subseteq B\}$ and $F^-\left(B\right) = \{x \in X :$

$F(x) \cap B \neq \emptyset\}$. In particular, $F^{-}(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. Also, $F(A) = \bigcup_{x \in X} F(x)$, for each $A \subseteq X$. Then F is said to be surjection if $F(X) = Y$, or equivalently, if for each $y \in Y$ there exists an $x \in X$ such that $y \in F(x)$. Throughout this paper (X, μ_X) and (Y, μ_Y) (or simply X and Y) always mean generalized topological spaces.

3. Upper and Lower $\alpha(\mu_X, \mu_Y)$-Continuous Multifunctions

Lemma 3.1 [21] Let A be a subset of a generalized topological space (X, μ_X). Then,
(i) $x \in c_{\alpha_X}(A)$ if and only if $A \cap U \neq \varnothing$ for each $U \in \alpha(\mu_X)$ containing x.
(ii) $c_{\alpha_X}(X - A) = X - i_{\alpha_X}(A)$.
(iii) $c_{\alpha_X}(A)$ is $\mu_X - \alpha-$ closed in X.

Definition 3.2 Let (X, μ_X) and (Y, μ_Y) be a generalized topological spaces. Then a multifunction $F : X \longrightarrow Y$ is said to be;
(i) upper $\alpha(\mu_X, \mu_Y)$ -continuous at a point $x \in X$ if for each μ_Y-open set V of Y containing $F(x)$, there exists $U \in \alpha(\mu_X)$ containing x such that $F(U) \subseteq V$.
(ii) lower $\alpha(\mu_X, \mu_Y)$ -continuous at a point $x \in X$ if for each μ_Y-open set V of Y such that $F(x) \cap V \neq \varnothing$, there exists $U \in \alpha(\mu_X)$ containing x such that $F(z) \cap V \neq \varnothing$ for every $z \in U$.
(iii) upper(lower) $\alpha(\mu_X, \mu_Y)-$ continuous if F has this property at each point of X.

Definition 3.3 [12] Let (X, μ_X) and (Y, μ_Y) be a generalized topological spaces. Then a multifunction $F : X \longrightarrow Y$ is said to be;
(i) upper $\beta(\mu_X, \mu_Y)$-continuous at a point $x \in X$ if for each μ_Y-open set V of Y containing $F(x)$, there exists $U \in \beta(\mu_X)$ containing x such that $F(U) \subseteq V$.
(ii) lower $\beta(\mu_X, \mu_Y)$-continuous at a point $x \in X$ if for each μ_Y-open set V of Y such that $F(x) \cap V \neq \varnothing$, there exists $U \in \beta(\mu_X)$ containing x such that $F(z) \cap V \neq \varnothing$ for every $z \in U$.
(iii) upper(lower) $\beta(\mu_X, \mu_Y)$- continuous if F has this property at each point of X.

Definition 3.4 [8] Let (X, μ_X) and (Y, μ_Y) be a generalized topological spaces. Then a multifunction $F : X \longrightarrow Y$ is said to be
(i) upper (μ_X, μ_Y)-continuous at a point $x \in X$ if for each μ_Y-open set V of Y containing $F(x)$, there exists $U \in (\mu_X)$ containing x such that $F(U) \subseteq V$.
(ii) lower (μ_X, μ_Y)-continuous at a point $x \in X$ if for each μ_Y-open set V of Y such that $F(x) \cap V \neq \varnothing$, there exists $U \in (\mu_X)$ containing x such that $F(z) \cap V \neq \varnothing$ for every $z \in U$.
(iii) upper(lower) (μ_X, μ_Y)- continuous if F has this property at each point of X.

Remark 1 *For a multifunction $F : X \longrightarrow Y$, following implications hold:*
upper(lower) (μ_X, μ_Y)-continuous $\Longrightarrow$ upper(lower) $\alpha(\mu_X, \mu_Y)$-continuous $\Longrightarrow$ upper(lower) $\beta(\mu_X, \mu_Y)$ -continuous.
The following examples shows that these implications are not reversible.

***Example* 3.5** Let $X = \{a, b, c, d\} = Y$ and $\mu_X = \{\emptyset, \{a\}, \{a, b, c\}\} = \mu_Y$. Consider a multifunction $F : (X, \mu_X) \longrightarrow (Y, \mu_Y)$ defined by $F(a) = \{a\} = F(b)$ and $F(c) = \{b\}$, $F(d) = \{d\}$. Then $F^{+}(\{a, b, c\}) = \{a, b, c\}$ and $F^{+}(\{a\}) = \{a, b\}$. Now $\{a, b\}$ is $\mu_X - \alpha-$open but not μ_X-open. Hence F is upper $\alpha(\mu_X, \mu_Y)$-continuous but not upper (μ_X, μ_Y)-continuous.

***Example* 3.6** Let $X = \{a, b, c\} = Y$ and $\mu_X = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} = \mu_Y$. Consider a multifunction $F:(X,\mu_X)\longrightarrow(Y,\mu_Y)$ defined by $F(a)=\{b\}=F(b)$ and $F(c)=\{a\}$. Then F is upper $\beta(\mu_X,\mu_Y)$-continuous but not upper $\alpha(\mu_X,\mu_Y)$-continuous.

Theorem 3.7 For a multifunction $F:(X,\mu_X)\longrightarrow(Y,\mu_Y)$ the following are equivalent.
(i) F is upper $\alpha(\mu_X,\mu_Y)$-continuous,
(ii) $F^+(V)$ is a $\mu_X-\alpha$-open set in X for each μ_Y-open set V of Y.
(iii) $F^-(K)$ is a $\mu_X-\alpha$-closed set in X for each μ_Y-closed set K of Y.
(iv) $c_{\alpha_X}(F^-(B))\subseteq F^-(c_Y(B))$ for each subset B of Y.
(v) $F^+(i_Y(B))\subseteq i_{\alpha_X}(F^+(B))$ for each subset B of Y.

Proof. $(i)\Longrightarrow(ii)$ Let V be any μ_Y-open set of Y and $x\in F^+(V)$. Then $F(x)\subseteq V$. There exists $U\in\alpha(\mu_X)$ containing x such that $F(U)\subseteq V$. Thus $x\in U\subseteq F^+(V)$. This implies that $x\in i_{\alpha_X}(F^+(V))$. This shows that $F^+(V)\subseteq i_{\alpha_X}(F^+(V))$. We have $i_{\alpha_X}(F^+(V))\subseteq F^+(V)$. Therefore, $i_{\alpha_X}(F^+(V))=F^+(V)$ and so $F^+(V)$ is $\mu_X-\alpha$-open set in X.

$(ii)\Longrightarrow(iii)$ Let K be any μ_Y-closed set of Y. If we take, $V=Y-K$, then V is a μ_Y-open set in Y. By (ii) $F^+(V)$ is a μ_X- open set in X. So $X-F^+(V)=X-F^+(Y-K)=F^-(K)$ is a μ_X-closed set in X.

$(iii)\Longrightarrow(iv)$ Let B be any subset of Y. Since $c_Y(B)$ is a μ_Y-closed set in Y, by (iii), $F^-(c_Y(B))$ is a $\mu_X-\alpha$-closed set in X. Thus $c_{\alpha_X}(F^-(c_Y(B)))\subseteq F^-(c_Y(B))$. So $c_{\alpha_X}(F^-(B))\subseteq F^-(c_Y(B))$.

$(iv)\iff(v)$ It follows from [Lemma 3.2, [18]].

$(v)\Longrightarrow(i)$ Let $x\in X$ and V be any μ_Y-open set V of Y containing $F(x)$. Then by (v), $x\in F^+(V)=F^+(i_Y(V))\subseteq i_{\alpha_X}(F^+(V))$. Therefore, there exists a $\mu_X-\alpha-$open set $U=F^+(V)$ of X containing x such that $F(U)\subseteq V$. This implies F is upper $\alpha(\mu_X,\mu_Y)$-continuous at x. ■

Theorem 3.8 For a multifunction $F:(X,\mu_X)\longrightarrow(Y,\mu_Y)$, the following are equivalent.
(i) F is lower $\alpha(\mu_X,\mu_Y)$-continuous.
(ii) $F^-(V)$ is a $\mu_X-\alpha$-open set in X for each μ_Y-open V of Y.
(iii) $F^+(K)$ is a $\mu_X-\alpha$-closed set in X for each μ_Y-closed set K of Y.
(iv) $c_{\alpha_X}(F^+(B))\subseteq F^+(c_Y(B))$ for each subset B of Y.
(v) $F^-(i_Y(B))\subseteq i_{\alpha_X}(F^-(B))$ for each subset B of Y.
(vi) $F(c_{\alpha_X}(A))\subseteq c_Y(F(A))$ for each subset A of X.

Proof. We prove only the implications $(iv)\Longrightarrow(vi)$ and $(vi)\Longrightarrow(v)$ with the proofs the other being similar to those of Theorem 3.7.

$(iv)\Longrightarrow(vi)$ Let A be any subset of X. By (iv), we have $c_{\alpha_X}(A)\subseteq c_{\alpha_X}F^+(F(A))\subseteq F^+(c_YF(A))$ and $F(c_{\alpha_X}(A))\subseteq c_Y(F(A))$.

$(vi)\Longrightarrow(v)$ Let B be any subset of Y. By (vi), we have $F(c_{\alpha_X}(F^+(Y-B)))\subseteq c_Y(F(F^+(Y-B)))\subseteq c_Y(Y-B)=Y-i_Y(B)$ and $F(c_{\alpha_X}(F^+(Y-B)))=F(c_{\alpha_X}(X-F^-(B)))=F(X-i_{\alpha_X}(F^-(B)))$.This implies $F^-(i_Y(B))\subseteq i_{\alpha_X}(F^-(B))$. ■

Theorem 3.9 Let (X,μ_X), (Y,μ_Y) be two generalized topological spaces and the multifunctions $F_1:X\longrightarrow Y$, $F_2:X\longrightarrow Y$ be upper $\alpha(\mu_X,\mu_Y)$-continuous. Then $F_1\cup F_2$ the combination of F_1 and F_2 is defined $(F_1\cup F_2)(x)=F_1(x)\cup F_2(x)$ is upper $\alpha(\mu_X,\mu_Y)$-continuous.

Proof. Let V be any μ_Y-open set of Y. Since F_1 and F_2 are upper $\alpha(\mu_X,\mu_Y)$-continuous then $F_1^+(V)$ and $F_2^+(V)$ are $\mu_X-\alpha$-open set in X. So $F_1^+(V)\cup F_2^+(V)=(F_1\cup F_2)^+(V)$ is $\mu_X-\alpha$-open set in X. Therefore $F_1\cup F_2$ is upper $\alpha(\mu_X,\mu_Y)$-continuous. ■

Theorem 3.10 Let (X, μ_X), (Y, μ_Y) be two generalized topological spaces and the multifunctions $F_1 : X \longrightarrow Y$, $F_2 : X \longrightarrow Y$ be lower $\alpha(\mu_X, \mu_Y)$-continuous. Then $F_1 \cup F_2$ is lower $\alpha(\mu_X, \mu_Y)$-continuous.

Proof. The proof is similar to Theorem 3.9. ■

Theorem 3.11 Let (X, μ_X), (Y, μ_Y) be generalized topological spaces and the multifunctions $F_1 : X \longrightarrow Y$, $F_2 : X \longrightarrow Y$ be upper $\alpha(\mu_X, \mu_Y)$ -continuous. Then $F_1 \times F_2$ the product of F_1 and F_2 is defined $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ is upper $\alpha(\mu_{X\times X}, \mu_{Y\times Y})$-continuous.

Proof. Let V be any μ_Y-open subset of Y. Since F_1 and F_2 are upper $\alpha(\mu_X, \mu_Y)$-continuous then $F_1^+(V)$ and $F_2^+(V)$ are $\mu_X - \alpha$-open set in X. So $F_1^+(V) \times F_2^+(V) = (F_1 \times F_2)^+(V)$ is $\mu_X - \alpha$-open set in $X \times X$. Therefore $F_1 \times F_2$ is upper $\alpha(\mu_{X\times X}, \mu_{Y\times Y})$-continuous. ■

Theorem 3.12 Let (X, μ_X) and (Y, μ_Y) be generalized topological spaces and the multifunctions $F_1 : X \longrightarrow Y$ and $F_2 : X \longrightarrow Y$ be lower $\alpha(\mu_X, \mu_Y)$-continuous. Then $F_1 \times F_2$ is lower $\alpha(\mu_{X\times X}, \mu_{Y\times Y})$-continuous.

Proof. The proof is similar to Theorem 3.11. ■

Theorem 3.13 Let (X, μ_X), (Y, μ_Y) and (Z, μ_Z) be three generalized topological spaces. The multifunctions $F_1 : X \longrightarrow Y$ be upper $\alpha(\mu_X, \mu_Y)$-continuous and $F_2 : Y \longrightarrow Z$ be upper (μ_Y, μ_Z)-continuous. Then the multifunction $F_1 \circ F_2$ is defined $(F_1 \circ F_2)(x) = F_1(F_2(x))$ is upper $\alpha(\mu_X, \mu_Z)$-continuous.

Proof. Let W be any μ_Z-open subset of Z. Since F_2 is upper (μ_Y, μ_Z)-continuous then $F_2^+(W)$ is μ_Y-open set in Y. Also, since F_1 is upper $\alpha(\mu_X, \mu_Y)$-continuous then $F_1^+\left(F_2^+(W)\right) = (F_1 \circ F_2)^+(W)$ is $\alpha - \mu_X$-open set in X. Therefore, $F_1 \circ F_2$ is upper $\alpha(\mu_X, \mu_Z)$-continuous. ■

Theorem 3.14 Let (X, μ_X), (Y, μ_Y) and (Z, μ_Z) be three generalized topological spaces. The multifunctions $F_1 : X \longrightarrow Y$ be lower $\alpha(\mu_X, \mu_Y)$-continuous and $F_2 : Y \longrightarrow Z$ lower (μ_X, μ_Z)-continuous. Then the multifunction $F_1 \circ F_2$ is lower $\alpha(\mu_X, \mu_Z)$ -continuous.

Proof. The proof is similar to Theorem 3.13. ■

Theorem 3.15 Let (X, μ_X), (Y, μ_Y) be two generalized topological spaces and $F(X)$ is endowed with subspace topology. If the multifunction $F : X \longrightarrow Y$ is upper $\alpha(\mu_X, \mu_Y)$-continuous then the multifunction $F : X \longrightarrow F(X)$ is upper $\alpha\left(\mu_X, \mu_{F(X)}\right)$-continuous.

Proof. Since F is upper $\alpha(\mu_X, \mu_Y)$ -continuous, for every μ_Y-open set V of Y, $F^+(V \cap F(X)) = F^+(V) \cap F^+(F(X)) = F^+(V) \cap X = F^+(V)$ is $\alpha - \mu_X$-open set in X. Therefore $F : X \longrightarrow F(X)$ is upper $\alpha\left(\mu_X, \mu_{F(X)}\right)$ -continuous. ■

Theorem 3.16 Let (X, μ_X), (Y, μ_Y) be two generalized topological spaces and $F(X)$ is endowed with subspace topology. If the multifunction $F : X \longrightarrow Y$ is lower $\alpha(\mu_X, \mu_Y)$-continuous then the multifunction $F : X \longrightarrow F(X)$ is lower $\alpha\left(\mu_X, \mu_{F(X)}\right)$-continuous.

Proof. The proof is similar to Theorem 3.15. ■

Let $K \neq \emptyset$ be an index set and $X_k \neq \varnothing$ for $k \in K$ and $X = \prod_{k\in K} X_k$ the cartesian product of the sets X_k. We denote by p_k the projection function $p_k : X \to X_k$ defined by $x_k = p_k(x)$, for each $x \in X$. Suppose that for $k \in K$, μ_k is a given generalized topological space on X_k. Let us consider all sets of the form $\prod_{k\in K} M_k$, where $M_k \in \mu_k$, with the exception of a finite number of indices k, $M_k = X_k = M_{\mu_k}$. We denote by $\wp$ the collection of all these sets. Clearly $\varnothing \in \wp$ so that we can define a GT $\mu = \mu(\wp)$ having $\wp$ for base. We call μ the product [11] of the GT's μ_k and denoted by P_{μ_k} $(k \in K)$.

Lemma 3.17 [20] Let $A = \prod_{k\in K} A_k \subseteq \prod_{k\in K} X_k$ and let K_0 be a finite subset of K. If $A_k \in (M_{\mu_k}, X_k)$ for each $k \in K - K_0$, then $iA = \prod_{k\in K} i_k A_k$.

Lemma 3.18 [11] Let $A = \prod_{k\in K} A_k \subseteq \prod_{k\in K} X_k$, then $cA = \prod_{k\in K} c_k A_k$.

Lemma 3.19 [11] If every μ_k is strong, then $\mu = \prod_{k\in K} \mu_k$ is strong and p_k is (μ, μ_k)-continuous for $k \in K$.

Lemma 3.20 [11] The projection p_k is (μ, μ_k)- open.

Lemma 3.21 Let $f : X \longrightarrow Y$ be (μ_X, μ_Y)-continuous and (μ_X, μ_Y)-open function. If A is a $\alpha - \mu_X$-open set in X, then $f(A)$ is $\alpha - \mu_Y$-open set in Y.

Proof. Let A be a $\alpha - \mu_X$-open set of X. Since f is (μ_X, μ_Y)-open and (μ_X, μ_Y)-continuous, then $f(i_{\mu_X}(A)) \subseteq i_{\mu_Y}(f(A))$ and $f(c_{\mu_X}(A)) \subseteq c_{\mu_Y}(f(A))$. Thus, $f(A) \subseteq f(i_{\mu_X}(c_{\mu_X}(i_{\mu_X}(A))))$ and $f(A) \subseteq f(i_{\mu_X}(c_{\mu_X}(i_{\mu_X}(A)))) \subseteq i_{\mu_Y}(c_{\mu_Y}(i_{\mu_Y}(f(A))))$. This show that, $f(A)$ is $\alpha - \mu_Y$-open in Y. ■

Theorem 3.22 Let X be a strong generalized topological space, $F : X \longrightarrow Y$ be a multifunction and $G_F : X \to X \times Y$ be the graph multifunction of F defined by $G_F(x) = (x, F(x))$ for each $x \in X$. If G_F is upper $\alpha(\mu_X, \mu_{X\times Y})$-continuous then F is upper $\alpha(\mu_X, \mu_Y)$-continuous.

Proof. Let V be any μ_Y-open set of Y. Then $X \times V$ is $\mu_{X\times Y}$-open set of $X \times Y$. Since G_F is upper $\alpha(\mu_X, \mu_{X\times Y})$-continuous, then $G_F^+(X \times V) = F^+(V) \cap X = F^+(V)$ is a $\alpha - \mu_X$-open set in X. Thus F is upper $\alpha(\mu_X, \mu_Y)$-continuous. ■

Theorem 3.23 Let X be a strong generalized topological space, $F : X \longrightarrow Y$ be a multifunction and $G_F : X \to X \times Y$ be the graph multifunction of F. If G_F is lower $\alpha(\mu_X, \mu_{X\times Y})$-continuous then F is lower $\alpha(\mu_X, \mu_Y)$-continuous.

Proof. The proof is similar to Theorem 3.22. ■

Let $\{X_k : k \in K\}$ and $\{Y_k : k \in K\}$ be any two families of generalized topological spaces with the same index set K. For each $k \in K$, let $F_k : X_k \longrightarrow Y_k$ be a multifunction. The product space $\prod_{k\in K} X_k$ is denoted by $\prod X_k$ and the product multifunction $\prod F_k : \prod X_k \longrightarrow \prod Y_k$, defined by $F(x) = \prod_{k\in K} F_k(x_k)$ for each $x = \{x_k\} \in \prod X_k$, is simply denoted by $F : \prod X_k \longrightarrow \prod Y_k$.

Theorem 3.24 Let $Y = \prod_{k\in K} Y_k$. If a multifunction $F : X \longrightarrow \prod_{k\in K} Y_k$ is upper $\alpha(\mu_X, \mu_Y)$-continuous and every $\mu_{Y k}$ is strong, then $p_k \circ F : X \to Y_k$ is upper $\alpha(\mu_X, \mu_{Y_k})$ -continuous, where p_k is the projection of $\prod_{k\in K} Y_k$ onto Y_k, for each $k \in K$.

Proof. Let $k \in K$ and V_k be any μ_{Y_k}-open set of Y_k. By Lemma 3.19, p_k is (μ_X, μ_{Y_k}) -continuous, so $p_k^{-1}(V_k)$ is a generalized open set in Y. Since F is upper $\alpha(\mu_X, \mu_{Y_k})$

-continuous, then $F^{-}\left(p_k^{-1}(V_k)\right) = (p_k \circ F)^{-}(V_k)$ is a $\alpha - \mu_X$-open set in X. Therefore, $p_k \circ F$ is upper $\alpha(\mu_X, \mu_{Y_k})$-continuous. ■

Theorem 3.25 Let $Y = \prod\limits_{k \in K} Y_k$. If a multifunction $F : X \longrightarrow \prod\limits_{k \in K} Y_k$ is lower $\alpha(\mu_X, \mu_Y)$-continuous and every μ_{Y_k} is strong, then $p_k \circ F : X \to Y_k$ is lower $\alpha(\mu_X, \mu_{Y_k})$ -continuous.

Proof. The proof is similar to Theorem 3.24. ■

Theorem 3.26 Let $k \in K$, μ_{Y_k} be strong generalized topological spaces and $F_k : X_k \to Y_k$ be multifunctions. If the product multifunction $F : X \to Y$ is upper $\alpha(\mu_X, \mu_Y)$-continuous where $X = \prod X_k$, $Y = \prod Y_k$, then $F_k : X_k \to Y_k$ is upper $\alpha(\mu_{X_k}, \mu_{Y_k})$-continuous for each $k \in K$.

Proof. Let k_0 be an arbitrary fixed index in K and V_{k_0} be any $\mu_{Y_{k_0}}$-open set of Y_{k_0}. Then $\prod\limits_{k \neq k_0} Y_k \times V_{k_0}$ is a μ_Y-open set in Y. Since F is upper $\alpha(\mu_X, \mu_Y)$-continuous, then $F^{-}\left(\prod\limits_{k \neq k_0} Y_k \times V_{k_0}\right) = \prod\limits_{k \neq k_0} X_k \times F_{k_0}^{-}(V_{k_0})$ is a $\alpha - \mu_X$-open set in X. By Lemma 3.21, $F_{k_0}^{-}(V_{k_0})$ is a $\mu_{Y_{k_0}}$-open set in X_{k_0}. This shows that F_{k_0} is upper $\alpha\left(\mu_{X_{k_0}}, \mu_{Y_{k_0}}\right)$-continuous. ■

Theorem 3.27 Let $k \in K$ and let X_k and μ_{Y_k} be strong generalized topological spaces and $F_k : X_k \to Y_k$ be multifunctions. If the product multifunction $F : X \to Y$ is lower $\alpha(\mu_X, \mu_Y)$-continuous where $X = \prod X_k$, $Y = \prod Y_k$, then $F_k : X_k \to Y_k$ is lower $\alpha(\mu_{X_k}, \mu_{Y_k})$-continuous for each $k \in K$.

Proof. The proof is similar to Theorem 3.26. ■

Definition 3.28 [7] A space X said to be μ-compact (resp. $\alpha - \mu$-compact) if every μ-open (resp. $\alpha - \mu-$open) cover of X has a finite subcover.

Theorem 3.29 Let a multifunction $F : X \longrightarrow Y$ be upper $\alpha(\mu_X, \mu_Y)$-continuous and X is $\alpha - \mu_X$-compact, then Y is μ_Y-compact.

Proof. Let χ be a cover of Y by μ_Y-open sets in Y. Since F is upper $\alpha(\mu_X, \mu_Y)$-continuous then $\{F^{+}(A) : A \in \chi\}$ is a $\alpha - \mu_X$- open cover of X. Also, since X is $\alpha - \mu_X$-compact, so the cover of X has a finite subcover $\{F^{+}(A) : A \in \chi'\}$, where χ' is a subfamily of χ. Then $Y \subset \bigcup\limits_{A \in \chi'} F(F^{+}(A)) = \bigcup\limits_{A \in \chi'} A$. Therefore Y is μ_Y-compact. ■

Theorem 3.30 Let (X, μ_X) be a generalized topological space and (Y, μ_Y) be a quasitopological space. If $F : X \longrightarrow Y$ be upper $\alpha(\mu_X, \mu_Y)$-continuous such that $F(x)$ is $\alpha - \mu_Y-$compact for each $x \in X$ and M is $\alpha - \mu_X-$compact set of X, then $F(M)$ is $\alpha - \mu_Y-$compact.

Proof. Let $\{V_i : i \in I\}$ be any cover of $F(M)$ by $\alpha - \mu_Y$-open sets. For each $x \in M$, $F(x)$ is $\alpha - \mu_Y$- compact and there exists a finite subset $I_0(x)$ of I such that $F(x) \subseteq \cup\{V_i : i \in I_0(x)\}$. Now set $V(x) = \cup\{V_i : i \in I_0(x)\}$. Then we have $F(x) \subseteq V(x)$ and $V(x)$ is $\alpha - \mu_Y$-open set of Y. Since F is upper $\alpha(\mu_X, \mu_Y)$ -continuous, there exists an $\alpha - \mu_X$-open set $U(x)$ containing x such that $F(U(x)) \subseteq V(x)$. The family $\{U(x) : x \in M\}$ is a cover of M by $\alpha - \mu_X$-open sets. Since M is $\alpha - \mu_X$-compact, there exists a finite number of points, say, $x_1, x_2, ..., x_n$ in M such that $M \subseteq \cup\{U(x_m) : x_m \in M,\ 1 \leqslant m \leqslant n\}$. Therefore, we obtain $F(M) \subseteq$

$\cup\{F(U(x_m)) : x_m \in M,\ 1 \leqslant m \leqslant n\} \subseteq \cup\{V_i : i \in i(x_m),\ x_m \in M,\ 1 \leqslant m \leqslant n\}$. This shows that $F(M)$ is $\alpha-\mu_Y$-compact. ■

Corollary 3.31 Let (X, μ_X) be a generalized topological space and (Y, μ_Y) be a quasitopological space. If $F : X \longrightarrow Y$ be upper $\alpha(\mu_X, \mu_Y)$ -continuous such that $F(x)$ is $\alpha-\mu_Y$-compact for each $x \in X$ and (X, μ_X) is $\alpha-\mu_X$-compact, then (Y, μ_Y) is $\alpha-\mu_Y$-compact.

Definition 3.32 [10] A space X is said to be μ_X-connected if there are no nonempty disjoint sets $U, V \subset \mu_X$ such that $U \cup V = X$.

Definition 3.33 A space X is said to be $\alpha-\mu_X$-connected if there are no nonempty disjoint α-open sets $U, V \subset \mu_X$ such that $U \cup V = X$.

Theorem 3.34 Let (X, μ_X), (Y, μ_Y) be generalized topological spaces and the multifunctions $F : X \longrightarrow Y$ be upper $\alpha(\mu_X, \mu_Y)$-continuous. If (X, μ_X) is $\alpha-\mu_X$-connected, then (Y, μ_Y) is μ_Y-connected.

Proof. Suppose there are two nonempty disjoint μ_Y-open subsets U, V of Y, such that $U \cup V = Y$. Since F is upper $\alpha(\mu_X, \mu_Y)$-continuous, so $F^+(U)$,$F^+(V)$ are $\alpha-\mu_X$-open subsets of X. Also $F^+(U) \cap F^+(V) = F^+(U \cap V) = F^+(\emptyset) = \emptyset$ and $F^+(U) \cup F^+(V) = F^+(U \cup V) = F^+(Y) = X$. So (X, μ_X) is α-disconnected. Therefore (Y, μ_Y) is μ_Y-connected. ■

Lemma 3.35 Let A and B be subsets of a GTS (X, μ).
(a) If $A \in \sigma(\mu) \cup \pi(\mu)$ and $B \in \alpha(\mu)$, then $A \cap B \in \alpha(\mu)$.
(b) If $A \subseteq B \subseteq X$, $A \in \alpha(B\mu)$ and $B \in \alpha(\mu)$, then $A \in \alpha(\mu)$.

Proof. Obvious. ■

Theorem 3.36 Let (X, μ_X), (Y, μ_Y) be generalized topological spaces and the multifunctions $F : X \longrightarrow Y$ be upper (resp. lower) $\alpha(\mu_X, \mu_Y)$-continuous. If $A \in \sigma(\mu) \cup \pi(\mu)$,then defined as $(F|_A)(x) = F(x)$ the restriction $F|_A : A \to Y$ is upper (resp. lower) $\alpha(\mu_A, \mu_Y)$-continuous.

Proof. We prove only the assertion for F upper $\alpha(\mu_A, \mu_Y)$-continuous, the proof for F lower $\alpha(\mu_A, \mu_Y)$-continuous being analogous. Let $x \in A$ and V be μ_Y-open set of Y such that $(F|_A)(x) \subseteq V$. Since F is upper $\alpha(\mu_X, \mu_Y)$-continuous and $(F|_A)(x) = F(x)$, there exists $U \in \alpha(\mu_X)$ containing x such that $F(U) \subseteq V$. Set $U_0 = U \cap A$, then by Lemma 5, we have $x \in U_0 \in \alpha(\mu_A)$ and $(F|_A)(U_0) \subseteq V$.This shows that $(F|_A) : A \to Y$ is upper $\alpha(\mu_A, \mu_Y)$-continuous. ■

Theorem 3.37 Let (X, μ_X), (Y, μ_Y) be generalized topological spaces and the multifunctions $F : X \longrightarrow Y$ is upper (resp. lower) $\alpha(\mu_X, \mu_Y)$-continuous. If for each $x \in X$, there exists $A \in \alpha(\mu_X)$ containing x, the restriction $(F|_A) : A \to Y$ is upper (resp. lower) $\alpha(\mu_A, \mu_Y)$-continuous.

Proof. The proof is similar to Theorem 3.36. ■

Corollary 3.38 Let (X, μ_X), (Y, μ_Y) be generalized topological spaces and $\{A_i : i \in I\}$ be an $\alpha-\mu_X$-open cover of X . A multifunctions $F : X \longrightarrow Y$ is upper (resp. lower) $\alpha(\mu_X, \mu_Y)$-continuous if and only if $(F|_A) : A_i \to Y$ is upper (resp. lower) $\alpha(\mu_X, \mu_Y)$-continuous for each $i \in I$.

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