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Some results for cyclic weak contractions in modular metric space

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Abstract. In this paper, we present some fixed point results for cyclic weak ϕ-contractions in ω-complete modular metric spaces and ω-compact modular metric spaces, respectively. Some results for contractions that have the zero cyclic property are also provided.

Keywords: Modular metric space, fixed point, cyclic ϕ-contraction, convex modular.

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1. Introduction and preliminaries

The concept of modular spaces was introduced by Nakano [17] and was later reconsidered in detail by Musielak and Orlicz [15, 16]. In 2010, Chistyakov [3] introduced a new metric structure, which has a physical interpretation, and generalized modular spaces and complete metric spaces by introducing modular metric spaces [1, 6–8, 11]. For more features of concepts of modular metric spaces, see [2, 4]. On the other hand, fixed point theory involves many branches of applied science and fields of mathematics such as functional analysis, mathematical analysis, general topology and operator theory [5, 9]. In 2003, Kirk et al. [14] introduced cyclic contraction in metric spaces and investigated the existence of proximity points and fixed points for cyclic contraction mapping [21]. Later, Karapinar and Erhan [13] proved the existence of fixed points for various types of cyclic contractions in a metric space. In 2011, Karapinar [12] proved a fixed point theorem for an operator T on a complete metric space X when X has a cyclic representation with respect to T.

In this paper, we improve and generalize the fixed point results for mappings satisfying cyclical contractive conditions established by Karapinar [12] in modular metric spaces.

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Karapinar's cyclic weak contractions [12] have been generalized by Harjani et al. [10]. We present some fixed point theorems for cyclic weak contractions in compact metric spaces. Some auxiliary results for contractions that have the zero cyclic property are also provided.

We first review some definitions from [3, 13, 19].

Definition 1.1 [13] Let X be an arbitrary set. The function $\omega : (0,\infty) \times X \times X \longrightarrow [0,\infty]$, that will be written as $\omega_\lambda(x,y) = \omega(\lambda, x, y)$ for all $x, y \in X$ and for all $\lambda > 0$, is said to be a modular metric on X (or simply a modular if no ambiguity arises) if it satisfies the following three conditions:

(i) $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$ and $x, y \in X$ if and only if $x = y$;
(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$ and $x, y \in X$;
(iii) $\omega_{\lambda+\mu}(x,y) \leqslant \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If instead of (i), we have only the following condition

(iv) $\omega_\lambda(x,x) = 0$ for all $\lambda > 0$ and $x \in X$,

then ω is said to be a (metric) pseudomodular on X and if ω satisfies (iv) and the following condition

(v) given $x, y \in X$, if there exists $\lambda > 0$ possibly depending on x and y such that $\omega_\lambda(x,y) = 0$ implies that $x = y$,

then ω is called a strict modular on X.

If instead of (iii) we replace the following condition for all $\lambda, \mu > 0$ and $x, y, z \in X$,

$$\omega_{\lambda+\mu}(x,y) \leqslant \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y), \tag{1}$$

then ω is called a convex modular on X.

Definition 1.2 [3] Given a modular ω on X, the sets

$$X_\omega \equiv X_\omega(x_\circ) = \{x \in X : \omega_\lambda(x, x_\circ) \to 0 \ as \ \lambda \to \infty\},$$
$$X^*_\omega \equiv X^*_\omega(x_\circ) = \{x \in X : \omega_\lambda(x, x_\circ) < \infty \ \ for\ some \ \ \lambda > 0\}$$

are said to be modular spaces (around $x_\circ$). The modular spaces X_ω and X^*_ω can be equipped with metrics d_ω and d^*_ω generated by ω and given by setting

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leqslant \lambda\}, \quad x, y \in X_\omega,$$
$$d^*_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leqslant 1\}, \quad x, y \in X^*_\omega.$$

If ω is a convex modular on X, then according to [3, Theorem 3.6] the two modular spaces coincide. Indeed, we have $X_\omega = X^*_\omega$.

Definition 1.3 Let X_ω be a modular metric space. A sequence $\{x_n\}_{n=1}^\infty$ in X_ω is said to be modular convergent (ω-convergent) to an element $x \in X_\omega$ if there exists $\lambda > 0$ possibly depending on $\{x_n\}$ and x such that $\lim\limits_{n\to\infty} \omega_\lambda(x_n, x) = 0$. We write $x_n \overset{\omega}{\to} x$ as $n \to \infty$.

Definition 1.4 A sequence $\{x_n\} \subset X_\omega$ is said to be ω-Cauchy if there exists $\lambda > 0$ such that $\lim_{m,n\to\infty} \omega_\lambda(x_n, x_m) = 0$, i.e.,

$$\forall \varepsilon > 0 \;\; \exists n_\circ(\varepsilon) \in \mathbb{N} \;\; \forall n, m \in \mathbb{N} \; (n \geqslant m \geqslant n_\circ(\varepsilon) \Longrightarrow \omega_\lambda(x_n, x_m) \leqslant \varepsilon).$$

A modular metric space is said to be ω-complete if each ω-Cauchy sequence in X_ω is modular convergent to an $x \in X_\omega$.

Remark 1 *A modular $\omega = \omega_\lambda$ on a set X is non-increasing on λ. Indeed, if $0 < \lambda < \mu$, then we have*

$$\omega_\mu(x, y) \leqslant \omega_{\mu-\lambda}(x, x) + \omega_\lambda(x, y) = \omega_\lambda(x, y)$$

for all $x, y \in X$.

2. A fixed point theorem for contractions in ω-complete modular metric space

In this section, we prove some fixed point results for cyclic weak ϕ-contractions in ω-complete modular metric spaces and we obtain some results for contractions that have the zero cyclic property.

Definition 2.1 Let X_ω be a modular metric space and $\{A_i\}_{i=1}^m$ be nonempty subsets of X_ω and $T : X_\omega \to X_\omega$. We say that the set $\cup_{i=1}^m A_i$ is a cyclic representation of X_ω with respect to T if the following two conditions hold

(i) $X_\omega = \cup_{i=1}^m A_i$;
(ii) $T(A_i) \subseteq A_{i+1}$ for $1 \leqslant i \leqslant m-1$, and $T(A_m) \subseteq A_1$.

Definition 2.2 Let X_ω be a modular metric space and $\{A_i\}_{i=1}^m$ be ω-closed nonempty subsets of X_ω and $Y = \cup_{i=1}^m A_i$. The operator $T : Y \to Y$ is called a cyclic weak ϕ-contraction if the following two conditions hold

(i) $\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;
(ii) there exists a non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) > 0$, $\phi(t) = 0$ has a unique solution at $t = 0$ and

$$\omega_\lambda(Tx, Ty) \leqslant \omega_\lambda(x, y) - \phi(\omega_\lambda(x, y)) \tag{2}$$

for all $\lambda > 0$ and all $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, ..., m$ where $A_{m+1} = A_1$.

We say that the operator $T : Y \to Y$ has the zero cyclic property if there exists a sequence $\{y_n\} \subset Y$ with $\lim_{n\to\infty} \omega_\lambda(y_n, Ty_n) = 0$. Also, we define

$$ZC(T) = \{\{y_n\} \subset Y : \lim_{n\to\infty} \omega_\lambda(y_n, Ty_n) = 0\}.$$

Indeed, the set $ZC(T)$ is the set of all sequences in Y that the operator T has the zero cyclic property at them.

Remember that a point x is called an isolated point of a subset S in a modular metric space X_ω if x is an element of S and there exists a neighborhood of x that does not contain any other points of S.

Example 2.3 Let $X_\omega = [0, 1]$. Consider the mapping $\omega : (0, \infty) \times [0, 1] \times [0, 1] \to [0, \infty]$

by $\omega_\lambda(x,y) = \dfrac{|x-y|}{\lambda}$ for all $x, y \in X = X_\omega$ and $\lambda > 0$. Also, consider the ω-closed nonempty subsets of X_ω as follows:

$$A_1 = A_6 = [0,1],\ A_2 = [0, \frac{2}{3}],\ A_3 = [0, \frac{1}{2}],\ A_4 = [0, \frac{5}{12}],\ A_5 = [0, \frac{3}{8}].$$

Let $Y = \cup_{i=1}^{6} A_i$ and $T : Y \to Y$ be the mapping defined by $Tx = \dfrac{3x+1}{6}$ and $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = \frac{t}{2}$. Hence, $\phi(0) = 0$, ϕ is strictly increasing, $T(A_i) \subseteq A_{i+1}$, $i = 1, 2, ..., 5$ and

$$\omega_\lambda(Tx, Ty) = \frac{|\frac{3x+1}{6} - \frac{3y+1}{6}|}{\lambda} = \frac{1}{\lambda}\frac{|x-y|}{2} \leqslant \omega_\lambda(x,y) - \frac{1}{2}\omega_\lambda(x,y).$$

This indicates that T is a cyclic weak ϕ-contraction.

Remark 2 *It follows from the inequality* (1.1) *that*

$$(\lambda+\mu)\omega_{\lambda+\mu}(x,y) \leqslant \lambda\omega_\lambda(x,z) + \mu\omega_\mu(y,z)$$

This shows the function ω is a convex modular on X if and only if the function $\widehat{\omega}_\lambda$ defined by $\widehat{\omega}_\lambda(x,y) = \lambda\omega_\lambda(x,y)$ is a modular on X for all $\lambda > 0$ and $x, y \in X$ and the function $\lambda \mapsto \widehat{\omega}_\lambda(x,y) = \lambda\omega_\lambda(x,y)$ is non-increasing on $(0,\infty)$. So, for $0 < \lambda \leqslant \mu$, one can deduce

$$\omega_\mu(x,y) \leqslant \frac{\lambda}{\mu}\omega_\lambda(x,y) \leqslant \omega_\lambda(x,y).$$

On the other hand, we can find $k \in \mathbb{R}_+$ such that $\mu = k\lambda$ for any $\lambda \leqslant \mu$. This implies

$$\omega_{k\lambda}(x,y) \leqslant \frac{1}{k}\omega_\lambda(x,y). \tag{3}$$

Theorem 2.4 Let ω be a convex modular on X and X_ω an ω-complete modular metric space. Let $\{A_i\}_{i=1}^m$ be ω-closed subsets of X_ω and $Y = \cup_{i=1}^m A_i$. If $T : Y \to Y$ is a cyclic weak ϕ-contraction, then T has a unique fixed point at $z \in \cap_{i=1}^m A_i$.

Proof. Let $x_\circ$ be an arbitrary point in $Y = \cup_{i=1}^m A_i$ and consider a Picard sequence as $x_{n+1} = T(x_n)$ for $n = 0, 1, 2, ...$ We realize two cases:

(1) If there exists $n_\circ$ such that $x_{n_\circ+1} = x_{n_\circ}$, then $Tx_{n_\circ} = x_{n_\circ+1} = x_{n_\circ}$ and the existence of the fixed point is proved.

(2) Suppose $x_{n+1} \neq x_n$ for all n. Then by Definition 2.1 for any $n \geqslant 1$ there exists $i_n \in \{1, 2, ..., m\}$ such that $x_n \in A_{i_n}$ and $x_n \in A_{i_n+1}$. Using (2) for each $\lambda > 0$, we get

$$\begin{aligned} \omega_\lambda(x_{n+1}, x_{n+2}) &= \omega_\lambda(Tx_n, T_{n+1}) \\ &\leqslant \omega_\lambda(x_n, x_{n+1}) - \phi(\omega_\lambda(x_n, x_{n+1})) \\ &\leqslant \omega_\lambda(x_n, x_{n+1}). \end{aligned} \tag{4}$$

Therefore, $\{\omega_\lambda(x_n, x_{n+1})\}$ is a decreasing sequence of non-negative real numbers. So, this sequence is convergent. Assume that $\omega_\lambda(x_n, x_{n+1}) \to \eta$ as $n \to \infty$. Now, taking $n \to \infty$

in (4), we obtain

$$\eta \leqslant \eta - \lim_{n\to\infty} \phi(\omega_\lambda(x_n, x_{n+1})) \leqslant \eta$$

and hence,

$$\lim_{n\to\infty} \phi(\omega_\lambda(x_n, x_{n+1})) = 0. \tag{5}$$

Define $t_n = \omega_\lambda(x_n, x_{n+1})$. By (4), we find that

$$t_{n+1} \leqslant t_n - \phi(t_n) \leqslant t_n. \tag{6}$$

Suppose that $\eta > 0$. Since $\eta = \inf\{\omega_\lambda(x_n, x_{n+1}) : n \in \mathbb{N}\}$, one can deduce $0 < \eta \leqslant t_n$ for $n = 0, 1, 2, ...$ and so $0 < \phi(\eta) \leqslant \phi(t_n)$. According to (6), we have $t_{n+1} \leqslant t_n - \phi(t_n) \leqslant t_n - \phi(\eta)$. This ensures

$$t_{n+2} \leqslant t_{n+1} - \phi(t_{n+1}) \leqslant t_n - \phi(t_n) - \phi(t_{n+1}) \leqslant t_n - 2\phi(t_n) \leqslant t_n - 2\phi(\eta).$$

Inductively, we obtain $t_{n+p} \leqslant t_n - p\phi(\eta)$ which is a contradiction for large enough $p \in \mathbb{N}$ and hence $\eta = 0$. Now, we prove that $\{x_n\}$ is an ω-Cauchy sequence. First, we claim that there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, we can find $p > q \geqslant n$ with $p - q \equiv 1 \pmod{m}$ satisfying $\omega_\lambda(x_p, x_q) \geqslant \epsilon$. Using (1)-(3), we get

$$\begin{aligned}\epsilon \leqslant \omega_\lambda(x_p, x_q) &\leqslant \frac{1}{2}\omega_{\frac{\lambda}{2}}(x_p, x_{p+1}) + \frac{1}{2}\omega_{\frac{\lambda}{2}}(x_{p+1}, x_q) \\ &\leqslant \omega_\lambda(x_p, x_{p+1}) + \omega_\lambda(x_{p+1}, x_q) \\ &\leqslant \omega_\lambda(x_p, x_{p+1}) + \frac{1}{2}\omega_{\frac{\lambda}{2}}(x_{p+1}, x_{q+1}) + \frac{1}{2}\omega_{\frac{\lambda}{2}}(x_{q+1}, x_q) \\ &\leqslant \omega_\lambda(x_p, x_{p+1}) + \omega_\lambda(x_{p+1}, x_{q+1}) + \omega_\lambda(x_{q+1}, x_q) \\ &= \omega_\lambda(x_p, x_{p+1}) + \omega_\lambda(Tx_p, Tx_q) + \omega_\lambda(x_{q+1}, x_q). \end{aligned} \tag{7}$$

Since x_p and x_q lie in different adjacently labeled sets A_i and A_{i+1} for certain $i \in \{1, 2, ..., m\}$, from (7) and using (2), we obtain

$$\epsilon \leqslant \omega_\lambda(x_p, x_q) \leqslant \omega_\lambda(x_p, x_{p+1}) + \omega_\lambda(x_p, x_q) - \phi(\omega_\lambda(x_p, x_q)) + \omega_\lambda(x_{q+1}, x_q)$$

and so,

$$\phi(\omega_\lambda(x_p, x_q)) \leqslant \omega_\lambda(x_p, x_{p+1}) + \omega_\lambda(x_{q+1}, x_q). \tag{8}$$

Since $\epsilon \leqslant \omega_\lambda(x_p, x_q)$ and ϕ is non-decreasing, it follows that

$$0 < \phi(\epsilon) \leqslant \phi(\omega_\lambda(x_p, x_q). \tag{9}$$

By (8) and (9), we get

$$0 < \phi(\epsilon) \leqslant \phi(\omega_\lambda(x_p, x_q)) \leqslant \omega_\lambda(x_p, x_{p+1}) + \omega_\lambda(x_{q+1}, x_q)$$

Passing to the limit as $p, q \to \infty$ in the above inequality with $p - q \equiv 1 \pmod{m}$ and using (5), we reach

$$0 < \phi(\epsilon) \leqslant \lim_{\substack{p,q\to\infty \\ p-q\equiv 1 \pmod{m}}} \phi(\omega_\lambda(x_p, x_q)) \leqslant 0$$

which is a contradiction. This means that for every $\epsilon > 0$ there exists $n_\circ \in \mathbb{N}$ such that if $p, q \geqslant n_\circ$ with $p - q \equiv 1 \pmod{m}$, then $\omega_\lambda(x_p, x_q) < \epsilon$.

In order to prove that $\{x_n\}$ is ω-Cauchy, we fix $\epsilon > 0$. By the above claim there exists $n_\circ \in \mathbb{N}$ such that if $p, q \geqslant n_\circ$ with $p - q \equiv 1 \pmod{m}$, then $\omega_\lambda(x_p, x_q) \leqslant \frac{\epsilon}{m}$. On the other hand, since $\omega_\lambda(x_n, x_{n+1}) \to 0$ as $n \to \infty$, one can find $n'_\circ \in \mathbb{N}$ such that if $n \geqslant n'_\circ$, then $\omega_\lambda(x_n, x_{n+1}) \leqslant \frac{\epsilon}{m}$. We take $r, t \geqslant \max\{n_\circ, n'_\circ\}$ with $t > r$. Then there exists $k \in \{1, 2, ..., m\}$ such that $t - r \equiv k \pmod{m}$. Putting $j = m - k + 1$, so $t - r + j \equiv 1 \pmod{m}$ and by using (1) and (3), we get

$$\omega_\lambda(x_r, x_t) \leqslant \omega_\lambda(x_r, x_{t+j}) + \omega_\lambda(x_{t+j}, x_{t+j-1}) + ... + \omega_\lambda(x_{t+1}, x_t).$$

So, $\omega_\lambda(x_r, x_t) \leqslant \frac{\epsilon}{m} + (j-1)\frac{\epsilon}{m} = j\frac{\epsilon}{m} \leqslant \epsilon$ and $\{x_n\}$ is ω-Cauchy. Since the sets A_i are ω-closed, $Y = \cup_{i=1}^m A_i$ is ω-closed and so Y is ω-complete. This implies that there exists $x \in Y$ such that $x_n \overset{\omega}{\to} x$ as $n \to \infty$. We prove that x is a fixed point of T. Since $\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T, it is clear that infinitely many terms of $\{x_n\}$ lie in each A_i for $i = 1, 2, ..., m$. From the fact that each A_i is ω-closed for $i = 1, 2, ..., m$, we have $x \in \cap_{i=1}^m A_i$. Suppose $x \in A_i$ for certain $i \in \{1, 2, ..., m\}$ and $Tx \in A_{i+1}$. We take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_i$ (where $A_{m+1} = A_1$) and ω-convergent to x. Using (1) and (2) for all $\lambda > 0$, we observe that

$$\begin{aligned} \omega_\lambda(x, Tx) &\leqslant \omega_\lambda(x, x_{n_k+1}) + \omega_\lambda(x_{n_k+1}, Tx) \\ &= \omega_\lambda(x, x_{n_k+1}) + \omega_\lambda(Tx_{n_k}, Tx) \\ &\leqslant \omega_\lambda(x, x_{n_k+1}) + \omega_\lambda(x_{n_k}, x) - \phi(\omega_\lambda(x_{n_k}, x)) \\ &\leqslant \omega_\lambda(x, x_{n_k+1}) + \omega_\lambda(x_{n_k}, x). \end{aligned}$$

Passing to the limit as $k \to \infty$ in the above inequality, we obtain $\omega_\lambda(x, Tx) \leqslant 0$ for all $\lambda > 0$ and so, $\omega_\lambda(x, Tx) = 0$ for all $\lambda > 0$. This implies that x is a fixed point of T.

For the uniqueness, suppose that y, z are two fixed points of T. Since $\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T, we see that $z, y \in \cup_{i=1}^m A_i$. Since $\phi(\omega_\lambda(y, z)) > 0$ for $\omega_\lambda(y, z) > 0$, using the contractive condition (2) for all $\lambda > 0$, we have

$$\omega_\lambda(y, z) = \omega_\lambda(Ty, Tz) \leqslant \omega_\lambda(y, z) - \phi(\omega_\lambda(y, z)) \leqslant \omega_\lambda(y, z).$$

This implies that $\phi(\omega_\lambda(y, z)) = 0$ for all $\lambda > 0$ and hence $\omega_\lambda(y, z) = 0$ for all $\lambda > 0$. Therefore, $y = z$. ■

Corollary 2.5 Let ω be a convex modular on X and X_ω an ω-complete modular metric space. If $T : X_\omega \to X_\omega$ is a cyclic weak ϕ-contraction, then T has a unique fixed point at $z \in X_\omega$.

Proof. Define $A_i = X_\omega$ and set $Y = \cup_{i=1}^m A_i$. So, by assumption $T : Y \to Y$ is a cyclic weak ϕ-contraction. Apply Theorem 2.4 to conclude that T has a unique fixed point at $z \in A_1 = X_\omega$. ■

Corollary 2.6 If $T : [0, 1] \to [0, 1]$ is an operator defined by $Tx = \dfrac{3x+1}{6}$, then T has a unique fixed point at $z \in [0, \frac{3}{8}]$.

Proof. Let $X_\omega = [0, 1]$. Consider the mapping ω and the ω-closed nonempty subsets A_i of X_ω for $i = 1, ..., 6$ as in Example 2.3. Define $Y = \cup_{i=1}^{6} A_i$. According to Example 2.3 the operator $T : Y \to Y$ is a cyclic weak ϕ-contraction. Apply now Theorem 2.4 to conclude that T has a unique fixed point at $z \in \cap_{i=1}^{6} A_i = [0, \frac{3}{8}]$. ■

Note that Corollary 2.6 works properly, since we know that in this corollary $T(\frac{1}{3}) = \frac{1}{3} \in [0, \frac{3}{8}]$.

Theorem 2.7 Under the hypotheses of Theorem 2.4, if the operator $T : Y \to Y$ has the zero cyclic property, then

(i) T has a unique non-isolated fixed point;
(ii) there exists a sequence $\{y_n\} \in ZC(T)$ such that $\lim_{n\to\infty} \omega_\lambda(y_n, T^n x) = 0$ for every $x \in Y$.

Proof. Since T has the zero cyclic property, there exists a sequence $\{y_n\} \in ZC(T)$ with

$$\lim_{n\to\infty} \omega_\lambda(y_n, Ty_n) = 0.$$

(i) According to Theorem 2.4 for any initial value $x_\circ \in Y$, one can deduce $z \in \cap_{i=1}^{m} A_i$ is the unique fixed point of T and so, $\omega_\lambda(y_n, z)$ is well defined. By (1) and (2) we have

$$\omega_\lambda(y_n, z) \leqslant \omega_\lambda(y_n, Ty_n) + \omega_\lambda(Ty_n, Tz) \leqslant \omega_\lambda(y_n, Ty_n) + \omega_\lambda(y_n, z) - \phi(\omega_\lambda(y_n, z))$$

for all $\lambda > 0$ and so, $\phi(\omega_\lambda(y_n, z)) \leqslant \omega_\lambda(y_n, Ty_n)$ for all $\lambda > 0$. Passing to the limit as $n \to \infty$, we get $\lim_{n\to\infty} \phi(\omega_\lambda(y_n, z)) = 0$ for all $\lambda > 0$. Therefore, $\lim_{n\to\infty} \omega_\lambda(y_n, z) = 0$ for all $\lambda > 0$. This indicates that z is not an isolated point.
(ii) By part (i), there exists a sequence $\{y_n\} \in ZC(T)$ such that $\lim_{n\to\infty} \omega_\lambda(y_n, z) = 0$ for all $\lambda > 0$. For initial value $x \in Y$ and by the proof of Theorem 2.4, it is clear that the sequence $\{T^n x\}$ is ω-convergent to z. This implies $\omega_\lambda(y_n, T^n x) \leqslant \omega_\lambda(y_n, z) + \omega_\lambda(z, T^n x)$. Now, by passing to the limit as $n \to \infty$ for all $\lambda > 0$ in the above inequality, we get $\lim_{n\to\infty} \omega_\lambda(y_n, T^n x) = 0$. ■

3. A fixed point theorem for contractions in ω-compact modular metric space

Recently, Ozkan et al. [18] have defined compact modular metric spaces and have proved some new fixed points theorems for contractive mappings in compact modular metric spaces. In 2013, Harjani et al. [10] generalized Karapinar's cyclic weak contractions [12] and gave some fixed point theorems for such contractions in compact metric spaces.

In this section, we verify some fixed point results for cyclic weak ϕ-contractions in the context of compact modular metric spaces and we deduce some results for contractions that have the zero cyclic property.

Definition 3.1 [20] Let X_ω be a modular metric space. We say that $T : X_\omega \to X_\omega$ is modular continuous (ω-continuous) if $x_n \xrightarrow{\omega} x$ implies that $Tx_n \xrightarrow{\omega} Tx$ as $n \to \infty$ for each $\{x_n\} \subset X_\omega$.

Definition 3.2 [18] Let X_ω be a modular metric space. We say that $B \subseteq X_\omega$ is modular compact (ω-compact) if and only if every sequence in B has an ω-convergent subsequence in B.

In the next theorem, we consider the operator $T : X_\omega \to X_\omega$ is ω-continuous and then under the assumption that X_ω is ω-compact we can drop the ω-closedness of the sets A_i $(i = 1, 2, ..., m)$ in Theorem 2.4.

Theorem 3.3 Let ω be a convex modular on X and X_ω an ω-compact modular metric space. Let $\{A_i\}_{i=1}^m$ be nonempty subsets of X_ω and $X_\omega = \cup_{i=1}^m A_i$. If $T : X_\omega \to X_\omega$ is an ω-continuous cyclic weak ϕ-contraction, then T has a unique fixed point at $z \in \cap_{i=1}^m A_i$.

Proof. Let $x_\circ$ be an arbitrary point in $X_\omega = \cup_{i=1}^m A_i$ and consider a Picard sequence $x_{n+1} = T(x_n)$ for $n = 0, 1, 2, ...$ We realize two cases:

(1) If there exists $n_\circ$ such that $x_{n_\circ+1} = x_{n_\circ}$, then $Tx_{n_\circ} = x_{n_\circ+1} = x_{n_\circ}$ and the existence of the fixed point is proved.

(2) Suppose $x_{n+1} \neq x_n$ for all n. Then by Definition 2.1 for any $n \geqslant 1$ there exists $i_n \in \{1, 2, ..., m\}$ such that $x_n \in A_{i_n}$ and $x_n \in A_{i_n+1}$. Now similar to the proof of Theorem 2.4 we have $\lim_{n\to\infty} \omega_\lambda(x_n, x_{n+1}) = 0$. Since $x_{n+1} = Tx_n$, this fact gives us that

$$\inf\{\omega_\lambda(x, Tx) : x \in X_\omega\} = 0 \text{ for all } \lambda > 0. \tag{10}$$

We define the function $F : X_\omega \to \mathbb{R}^+$ by $Fx = \omega_\lambda(x, Tx)$ for all $\lambda > 0$. Since the mapping T is ω-continuous, so F is continuous. Since X_ω is ω-compact, one can find $z \in X_\omega$ such that $\omega_\lambda(z, Tz) = \inf\{\omega_\lambda(x, Tx) : x \in X_\omega\}$ for all $\lambda > 0$. By (10), we find that $\omega_\lambda(z, Tz) = 0$ for all $\lambda > 0$ and so $z = Tz$.

For the uniqueness, suppose that y, z are two fixed points of T. Since $\cup_{i=1}^m A_i$ is a cyclic representation of X_ω with respect to T, we see that $z, y \in \cup_{i=1}^m A_i$. Since $\phi(\omega_\lambda(y, z)) > 0$ for $\omega_\lambda(y, z) > 0$, using the contractive condition (2) for all $\lambda > 0$, we have

$$\omega_\lambda(y, z) = \omega_\lambda(Ty, Tz) \leqslant \omega_\lambda(y, z) - \phi(\omega_\lambda(y, z)) \leqslant \omega_\lambda(y, z).$$

This implies that $\phi(\omega_\lambda(y, z)) = 0$ for all $\lambda > 0$ and hence $\omega_\lambda(y, z) = 0$ for all $\lambda > 0$. Therefore, $y = z$. ■

***Example* 3.4** Let $M = M_1 \cup M_2$ and

$$M_1 = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{ij} \in [0, 1] \right\} \quad \text{and} \quad M_2 = \left\{ \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} : b_{ij} \in [-1, 0] \right\}.$$

Let $\omega : (0, \infty) \times M \times M \to [0, \infty]$ be a metric modular on M with

$$\omega_\lambda(A, B) = \frac{1}{\lambda} \max_{1\leqslant i,j\leqslant 2} |a_{ij} - b_{ij}|$$

for all $A, B \in M = X_\omega$ and $\lambda > 0$. It is clear that $M = X_\omega$ is ω-compact and ω is convex modular. In fact,

$$\begin{aligned}(\lambda+\mu)\omega_{\lambda+\mu}(A,B) &= \max_{1\leqslant i,j\leqslant 2}|a_{ij}-b_{ij}| \\ &\leqslant \max_{1\leqslant i,j\leqslant 2}|a_{ij}-b_{ij}| + \max_{1\leqslant i,j\leqslant 2}|a_{ij}-b_{ij}| \\ &\leqslant \lambda\omega_\lambda(A,B)+\mu\omega_\mu(A,B).\end{aligned}$$

Define the mapping $T: M \to M$ by $T(A) = -\frac{A}{2}$ and the mapping $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = \frac{t}{2}$. Hence, T is ω-continuous and we have $\phi(0) = 0$, ϕ is strictly increasing, $T(M_1) \subseteq M_2$, $T(M_2) \subseteq M_1$ and

$$\begin{aligned}\omega_\lambda(T(A),T(B)) &= \omega_\lambda(-\frac{A}{2},-\frac{B}{2}) \\ &= \frac{1}{2\lambda}\max_{1\leqslant i,j\leqslant 2}|a_{ij}-b_{ij}| \\ &\leqslant \omega_\lambda(A,B) - \frac{1}{2}\omega_\lambda(A,B) \\ &= \omega_\lambda(A,B) - \phi(\omega_\lambda(A,B)).\end{aligned}$$

This indicates that T is a cyclic weak ϕ-contraction. Apply Theorem 3.3 to conclude that T has a unique fixed point at $X_\omega = M_1 \cup M_2$ (which is indeed $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$).

Corollary 3.5 Under the hypotheses of Theorem 3.3, if the operator $T : X_\omega \to X_\omega$ has the zero cyclic property, then

(i) T has a unique non-isolated fixed point;
(ii) there exists a sequence $\{y_n\} \in ZC(T)$ such that $\lim_{n\to\infty} \omega_\lambda(y_n, T^n x) = 0$ for every $x \in X_\omega$.

Proof. The proof is similar to that of Theorem 2.7. ■

4. Conclusions

We provided some fixed point results for cyclic weak ϕ-contractions in ω-complete modular metric spaces. Some other some fixed point results are proved in ω-compact modular metric spaces. Also, some results for contractions that have the zero cyclic property are proved. We now propose two questions for interested readers as follows:

(1) Let ω be a convex modular on X. Can we prove the results for ω-closed modular metric spaces?

(2) All results are proved for ω-complete modular metric spaces or ω-compact modular metric spaces, where ω was a convex modular on X. Can we prove the results for a non-convex modular on X?

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