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Minimal continuous multifunctions

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Abstract. In this paper, we introduce a new strong form of the continuity of multifunctions with the help of minimal open sets. We give some characterizations for this new continuity and investigate fundamental properties of it. Additionally, we use this type of multifunctions to characterize Alexandroff spaces.

Keywords: Minimal open set, Continuity, multifunction, Alexandroff space.

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1. introduction

The importance of multifunctions or set-valued functions, which are applied in many different fields from social sciences to physical and biological sciences, was understood at the beginning of the 20th century and various results had been obtained on this subject by many leading mathematicians such as Hausdorff [13], Vietoris [35, 36], Hahn [11] and Kuratowski [18, 19]. However, systematic studies emerged in the 1960s due to the need for applied fields such as control theory and mathematical economics. In this period, the concepts of continuity, differentiability, measurability, integrability, homotopicity and fixed point for multifunctions have been studied by different researchers (see [3, 5, 9, 10, 15, 16, 20, 33, 34]).

Today, multifunctions and multi-valued analysis appear as important topics in mathematics on their own. Additionally, multifunctions are an important tool that can be used in solving problems encountered in studies in many fields of science (see [14]). For example, problems that arise in nonlinear analysis, nonlinear programming, mathematical

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economics and business, optimal control theory, biology, artificial intelligence and many other research fields can be solved by set-valued transformations and their theories [8].

The concept of continuity for multifunctions, which will be discussed in this study, first mentioned by Wallace [37] in 1941. In 1946, Eilenberg and Montgomery [10] proved a fixed point theorem for multifunctions. Later, different definitions of continuity for multifunctions was given by different researchers (e.g. [7, 29]). In 1955, Stroter [33] and in 1979, Lechicki [20] examined the relations between the definitions of continuity given before them and investigated their equivalence. The definition of continuity of multifunction used today was proposed by Smithson [31] using Berge's [4] notations.

After this definition of continuity for multifunctions, many kinds of continuity existing in the literature for single-valued functions has been extended to the setting of multifunctions. The relationships between multifunctions which have these new continuities and continuous multifunctions were examined and the topological properties preserved by these new classes of multifunctions were emphasized (e.g [1, 17, 25–28, 40]). Today, studies on multifunctions and their applications are still continuing intensively (e.g [8, 12, 24, 30, 39]).

The concepts of minimal open sets and maximal closed sets in topological spaces were introduced and considered by Nakaoka and Oda in [21, 22]. More precisely, in 2001, Nakaoka and Oda [22] characterized the notions of minimal open sets and proved that any subset of a minimal open set is preopen. Also, as an application of a theory of minimal open sets, they obtained a sufficient condition for a locally finite space to be a pre-Hausdorff space. Recently, Carpintero et al [6] introduced the notion of minimal open set in a generalized topological space (X, μ) and investigated some of their fundamental properties and proved that any subset of a minimal open set on a GTS (X, μ) is a μ-preopen set.

In this study, we introduce a new strong form of continuity of multifunctions, called minimal continuity by using minimal open sets. We give some characterizations for this new kind of continuity and examine the properties of this new class of functions. In addition, Alexandroff spaces have been tried to be characterized by using minimal continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. A nonempty open set A of X is said to be a minimal open set if any open set which is contained in A is $\emptyset$ or A. By a multifunction $F : X \rightrightarrows Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction, $F : X \rightrightarrows Y$, the upper and lower inverse of any subset B of Y, denoted by $F^+(B)$ and $F^-(B)$ respectively, are the subsets $F^+(B) = \{x \in X : F(x) \subseteq B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each $y \in Y$, and the image of an $A \subseteq X$ under F is $F(A) = \cup\{F(x) : x \in A\}$. Note that $X - F^+(B) = F^-(Y - B)$ for each $B \subseteq Y$. A multifunction $F : (X, \tau) \rightrightarrows (Y, \sigma)$ is said to be upper semi continuous [4], abbreviated as u.s.c., (resp. lower semi continuous [4] or l.s.c.) at $x \in X$ if for each open $V \subseteq Y$ with $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \emptyset$), there is an open neighbourhood U of x such that $F(U) \subseteq V$ (resp. $F(z) \cap V \neq \emptyset$ for all $z \in U$). F is u.s.c. (resp. l.s.c.) iff it is u.s.c. (resp. l.s.c.) at each point of X. Then F is called semi continuous iff it is both u.s.c.and l.s.c. For each $x \in X$, if $F(x)$ is a compact (closed) set, then F is called point compact (closed) multifunction (see [4, 31]).

3. Minimal continuous multifunctions

Definition 3.1 A multifunction $F : (X, \tau) \rightrightarrows (Y, \sigma)$ is said to be:

(i) upper minimal continuous (briefly, u.m.c.) at a point $x \in X$ if for each open set V of Y containing $F(x)$, there exists a minimal open neigborhood U of x such that $F(U) \subseteq V$.

(ii) lower minimal continuous (briefly, l.m.c.) at a point $x \in X$ if for each open set V of Y that satisfies $F(x) \cap V \neq \emptyset$, there exists a minimal open neigborhood U of x such that $F(z) \cap V \neq \emptyset$ for all $z \in U$.

(iii) upper (lower) minimal continuous if F has this property at each point of X.

***Example* 3.2** Let $Y = \mathbb{R}$ with the usual topology $\tau_{\mathcal{U}}$ and $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Define a multifunction $F : (X, \tau) \rightrightarrows (\mathbb{R}, \tau_{\mathcal{U}})$ by $F(a) = (0, 1)$, $F(b) = F(c) = (0, +\infty)$. Then F is u.m.c. and l.m.c.

Remark 1 *It is clear from the definitions that upper (resp. lower) minimal continuous multifunctions are upper (resp. lower) semi continuous. The following examples show that the inverses of these requirements are not generally true.*

***Example* 3.3** Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$. Define a topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ on X and a topology $\sigma = \{\emptyset, Y, \{1, 2\}\}$ on Y. Let $F : X \rightrightarrows Y$ be a multifunction defined by $F(a) = \{1, 2\}$, $F(b) = \{2\}$, $F(c) = Y$. Then F is u(l).s.c., but it is not u(l).m.c.

Proposition 3.4 Let $F : (X, \tau) \rightrightarrows (Y, \sigma)$ be a multifunction.

(1) If $F^+(V)$ is a minimal open in X for every $V \in \sigma$, then F is u.m.c.

(2) If $F^-(V)$ is a minimal open in X for every $V \in \sigma$, then F is l.m.c.

The inverses of these requirements are generally not true, as will be seen in the following example.

***Example* 3.5** Let's consider Example 3.2. We know that F is u(l).m.c., but for the open set $V = (0, +\infty)$, neither $F^+(V) = X$ nor $F^-(V) = X$ are minimal open sets.

Definition 3.6 [2] Let X be a topological space. Then X is an Alexandroff space if arbitrary intersections of open sets are open.

It is noted in [32] that the original definition given for an Alexandroff space is easy to state, however it is not too useful for proving theorems about Alexandroff spaces. In addition, they used a different but equivalent definition to fix this. So, in an Alexandroff space, every point must have a minimal open set containing it.

Theorem 3.7 [32] X is an Alexandroff space iff each point in X has a minimal open neighborhood (briefly nbd).

Theorem 3.8 Let X be an Alexandroff space, $x \in X$ and $F : X \rightrightarrows Y$ be a multifunction. If F is u(l).s.c. at x, then it is u(l).m.c. at x.

Proof. We will only prove for u.s.c. The other can be done in a similar way. Let $F : X \rightrightarrows Y$ be an u.s.c multifunction and $x \in X$. Let us take an arbitrary open set $V \subseteq Y$ satisfying the condition $F(x) \subseteq V$. Since F is u.s.c, there is an open set U with $x \in U \subseteq F^+(V)$. On the other hand, since X is an Alexandroff space, the point x has a minimal open neighborhood W and then $W \subseteq U$. Therefore, we find the minimal open set W with $x \in W \subseteq F^+(V)$, which shows that F is u.m.c. at $x \in X$. ■

Corollary 3.9 Let X be an Alexandroff space. Then a multifunction $F : X \rightrightarrows Y$ is u(l).m.c. iff F is u(l).s.c.

Corollary 3.10 Let X be any topological space. Then the multifunction $F : X \rightrightarrows X$, $F(x) = \{x\}$ is u(l).m.c. iff X is Alexandroff space.

Proof. ($\Rightarrow$) It is obvious.
($\Leftarrow$) Since $F : X \rightrightarrows X$, $F(x) = \{x\}$ is u(l).s.c., it is a corollary of Theorem 3.8. ■

Theorem 3.11 A multifunction $F : X \rightrightarrows Y$ is u.m.c. at $x \in X$ iff for every closed set $K \subseteq Y$ satisfying the condition $F(x) \cap K = \emptyset$, there exists a minimal open nbd U of x such that $U \cap F^-(K) = \emptyset$.

Proof. ($\Rightarrow$) Let K be a closed set with $F(x) \cap K = \emptyset$. Then $Y - K$ is an open set and $F(x) \subseteq Y - K$. Since F is u.m.c., there exists a minimal open nbd U of x such that $U \subseteq F^+(Y - K)$. Therefore, we have $U \subseteq X - F^-(K)$ and so $U \cap F^-(K) = \emptyset$. This completes the proof.
($\Leftarrow$) Let V be an open set with $F(x) \subseteq V$. Then $Y - V$ is a closed set and $F(x) \cap (Y - V) = \emptyset$. From the hypothesis, we have a minimal open nbd U of x such that $U \cap F^-(Y - V) = \emptyset$. Hence, $U \cap (X - F^+(V)) = \emptyset$ and so $U \subseteq F^+(V)$. This shows that F is u.m.c. at x. ■

Theorem 3.12 A multifunction $F : X \rightrightarrows Y$ is l.m.c. at $x \in X$ iff for every closed set $K \subseteq Y$ satisfying the condition $x \notin F^+(K)$, there exists a minimal open nbd U of x such that $U \cap F^+(K) = \emptyset$.

Proof. It is similar to that of upper theorem. ■

4. Some properties of minimal continuous multifunctions

Let $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ be two multifunctions. In this case, for each $V \subseteq Z$, $(G \circ F)^+(V) = F^+(G^+(V))$ and $(G \circ F)^-(V) = F^-(G^-(V))$ where $G \circ F$ is the composition of F and G.

Theorem 4.1 If the multifunctions $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ are u(l).m.c., then the multifunction $G \circ F : X \rightrightarrows Z$ is u(l).m.c.

Proof. We prove only for the case of upper minimal continuity, the other is similar. Let $x \in X$, W be an open set in Z with $(G \circ F)(x) \subseteq W$. Since G is u.m.c., there exists a minimal open nbd V_y of $y \in F(x)$ such that $V_y \subseteq G^+(W)$ for each $y \in F(x)$. Then $\bigcup_{y \in F(x)} V_y$ is an open set in Y and $F(x) \subseteq \bigcup_{y \in F(x)} V_y$. By u.m.c. of F, we obtain a minimal open nbd U of x such that $U \subseteq F^+\left(\bigcup_{y \in F(x)} V_y\right)$. Therefore, we have $U \subseteq F^+\left(\bigcup_{y \in F(x)} V_y\right) \subseteq F^+(G^+(W)) = (G \circ F)^+(W)$. This shows that $G \circ F$ is u.m.c. at $x \in X$. ■

Theorem 4.2 If the multifunction $F : X \rightrightarrows Y$ is u(l).m.c. and $G : Y \rightrightarrows Z$ is u(l).s.c., then the multifunction $G \circ F : X \rightrightarrows Z$ is u(l).m.c.

Proof. Let $x \in X$, W be an open set in Z with $(G \circ F)(x) \cap W \neq \emptyset$. Since G is l.s.c., $G^-(W)$ is an open set in Y and $F(x) \cap G^-(W) \neq \emptyset$. Since F is l.m.c, there exist a minimal open nbd U of x such that $U \subseteq F^-(G^-(W)) = (G \circ F)^-(W)$. This shows that $G \circ F$ is l.m.c. at $x \in X$. The proof of the other case can be done in a similar way. ■

Proposition 4.3 [32] If U is minimal open in the space X and $A \subseteq X$, then $U \cap A$ is minimal open in the subspace A.

Proposition 4.4 Let X be a topological space and $A \subseteq X$. If A is an open set in X and W is a minimal open set in the subspace A, then W is minimal open in X.

Proof. If we assume that W is not minimal open in the space X, then there exists a nonempty open set U such that $U \subsetneq W$. But, in this case $\emptyset \neq U \cap A \subsetneq W \cap A = W$ and $U \cap A$ is an open in A. This contradicts with the minimal openness of W in the subspace A. ■

Theorem 4.5 If $F : X \rightrightarrows Y$ is u(l).m.c. and $A \subseteq X$, then $F|_A : A \rightrightarrows Y$ is u(l).m.c.

The proof is obvious from the above Proposition 4.3 and we omit it.

Theorem 4.6 Let $\{A_\alpha : \alpha \in I\}$ be open cover of X. Then a multifunction $F : X \rightrightarrows Y$ is u(l).m.c. iff the restrictions $F|_{A_\alpha} : A_\alpha \rightrightarrows Y$ are u(l).m.c. for every $\alpha \in I$.

Proof. ($\Rightarrow$) Theorem 4.5.

($\Leftarrow$) We prove for the lower minimal continuity. The other is analogous. Let $x \in X$ and V be any open set with $F(x) \cap V \neq \emptyset$. Since $\{A_\alpha : \alpha \in \Lambda\}$ is a cover of X, there exists an $\alpha_0 \in \Lambda$ such that $x \in A_{\alpha_0}$. By hypothesis, there exists a minimal open nbd U of x in A_{α_0} such that $U \subseteq \left(F|_{A_{\alpha_0}}\right)^-(V)$. Then U is a minimal open nbd of x in X from Proposition 4.4. Moreover, since $U \subseteq \left(F|_{A_{\alpha_0}}\right)^-(V) \subseteq F^-(V)$, we have that F is l.m.c. at x. ■

Theorem 4.7 If the multifunction $F : X \rightrightarrows Y$ is u.m.c. and Y is a normal space, then the multifunction $\overline{F} : X \rightrightarrows Y$, $\overline{F}(x) = \overline{F(x)}$ is u.m.c.

Proof. Let $x \in X$ and V be any open set with $\overline{F(x)} \subseteq V$. Since Y is normal, there exits an open set G such that $\overline{F(x)} \subseteq G \subseteq \overline{G} \subseteq V$. Then we have $F(x) \subseteq G$. Since F is u.m.c., there exists a minimal open nbd U of x such that $F(U) \subseteq G$. Therefore, we have $\overline{F(U)} \subseteq \overline{G} \subseteq V$ and this shows that $\overline{F}$ is u.m.c. ■

The following example shows that the converse of upper theorem is not necessarily true.

Example **4.8** Let $Y = \mathbb{R}$ with the cofinite topology τ_c and let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Define a multifunction $F : (X, \tau) \rightrightarrows (\mathbb{R}, \tau_c)$ by $F(a) = \mathbb{R} - \{1\}$, $F(b) = \mathbb{R} - \{2\}$, $F(c) = \mathbb{R} - \{3\}$. Then $\overline{F}$ is u.m.c., but F is not u.m.c.

Theorem 4.9 A multifunction $F : X \rightrightarrows Y$ is l.m.c. iff $\overline{F} : X \rightrightarrows Y$, $\overline{F}(x) = \overline{F(x)}$ is l.m.c.

Proof. Let $x \in X$ and V be any open set with $\overline{F(x)} \cap V \neq \emptyset$. Then $F(x) \cap V \neq \emptyset$ since V is open. By the hypothesis, there exists a minimal open nbd U of x such that $z \in U$ implies $F(z) \cap V \neq \emptyset$. and so $\overline{F(z)} \cap V \neq \emptyset$. This shows that $\overline{F}$ is l.m.c.

Conversely, If $x \in X$ and V is an open set which satisfies $F(x) \cap V \neq \emptyset$, then $\overline{F(x)} \cap V \neq \emptyset$. By the hypothesis, there exists a minimal open nbd U of x such that $z \in U$ implies $\overline{F(z)} \cap V \neq \emptyset$ and so $F(z) \cap V \neq \emptyset$ because V is open. This shows that F is l.m.c. ■

Theorem 4.10 If the multifunctions $F_1 : X \rightrightarrows Y$ and $F_2 : X \rightrightarrows Y$ are u(l).m.c., then the multifunction $F_1 \cup F_2 : X \rightrightarrows Y$, $(F_1 \cup F_2)(x) = F_1(x) \cup F_2(x)$ is u(l).m.c.

Proof. We first prove for upper minimal continuity. If $x \in X$ and V is an open set with $(F_1 \cup F_2)(x) = F_1(x) \cup F_2(x) \subseteq V$, then $F_1(x) \subseteq V$ and $F_2(x) \subseteq V$. By upper minimal continuities of F_1 and F_2, there exist minimal open nbd's U_1 and U_2 of x such that $F_1(U_1) \subseteq V$ and $F_2(U_2) \subseteq V$. Then we have a minimal open nbd $U := U_1 = U_2$ of x such that $(F_1 \cup F_2)(U) = F_1(U) \cup F_2(U) \subseteq V$. This shows that $F_1 \cup F_2$ is u.m.c.

Now let's we prove for lower minimal continuity. Let $x \in X$ be an arbitrary point and V be any open set which satisfies $(F_1 \cup F_2)(x) \cap V \neq \emptyset$. In this case, there are three situations.

If $F_1(x) \cap V \neq \emptyset$, then there exists a minimal open nbd U of x such that $z \in U$ implies $F_1(z) \cap V \neq \emptyset$ and so $(F_1 \cup F_2)(z) \cap V \neq \emptyset$. This completes the proof.

If $F_2(x) \cap V \neq \emptyset$, then the proof is made similar to the previous case.

If $F_1(x) \cap V \neq \emptyset$ and $F_2(x) \cap V \neq \emptyset$, then a similar proof is made. ■

Theorem 4.11 If the multifunctions $F_1 : X \rightrightarrows Y$ and $F_2 : X \rightrightarrows Y$ are l.m.c., then the multifunction $F_1 \cap F_2 : X \rightrightarrows Y$, $(F_1 \cap F_2)(x) = F_1(x) \cap F_2(x)$ is l.m.c.

Proof. The proof can be done in a similar way. ■

Theorem 4.12 If Y is a normal space and the multifunctions $F_1 : X \rightrightarrows Y$, $F_2 : X \rightrightarrows Y$ are u.m.c. and point closed, then the multifunction $F_1 \cap F_2 : X \rightrightarrows Y$, $(F_1 \cap F_2)(x) = F_1(x) \cap F_2(x)$ is u.m.c.

Proof. Let $x \in X$ and V be any open set with $(F_1 \cap F_2)(x) \subseteq V$. Then we have $F_1(x) \cap F_2(x) \cap (Y \backslash V) = \emptyset$. Since $F_1(x)$ and $F_2(x) \cap (Y \backslash V)$ are disjoint closed sets and Y is normal, there exist open sets V_1 and V_2 such that $F_1(x) \subseteq V_1$ and $F_2(x) \cap (Y \backslash V) \subseteq V_2$. If $V_3 := V_2 \cup V$, then we get $F_2(x) \subseteq V_3$. Since F_1 and F_2 are u.m.c., there exists a minimal open nbd U of x such that $F_1(U) \subseteq V$ and $F_2(U) \subseteq V$. Hence we have

$$(F_1 \cap F_2)(U) = F_1(U) \cap F_2(U) \subseteq V_1 \cap V_3 = V_1 \cap (V_2 \cup V) \subseteq V$$

This completes the proof. ■

For a multifunction $F : X \rightrightarrows Y$, the graph multifunction is defined as $G_F : X \rightrightarrows X \times Y$, $G_F(x) = \{x\} \times F(x)$. Moreover, the following hold:

(1) $G_F^+(A \times B) = A \cap F^+(B)$,

(2) $G_F^-(A \times B) = A \cap F^-(B)$ for any subsets $A \subseteq X$ and $B \subseteq Y$ [23].

Theorem 4.13 A multifunction $F : X \rightrightarrows Y$ is l.m.c. iff the graph multifunction G_F is l.m.c.

Proof. ($\Rightarrow$) Let $x \in X$ and W be any open set of $X \times Y$ such that $x \in G_F^-(W)$. Since $W \cap (\{x\} \times F(x)) \neq \emptyset$, there exists $y \in F(x)$ such that $(x, y) \in W$ and hence $(x, y) \in U \times V \subseteq W$ for some open sets U and V of X and Y, respectively. Since $F(x) \cap V \neq \emptyset$ and F is l.m.c., there exists a minimal open nbd G of x such that $G \subseteq F^-(V)$. Then we have $G \subseteq U \cap G \subseteq U \cap F^-(V) = G_F^-(U \times V) \subseteq G_F^-(W)$. This shows that G_F is l.m.c.

($\Leftarrow$) Let $x \in X$ and V be any open set of Y such that $x \in F^-(V)$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is l.m.c., there exists a minimal open nbd U of x such that $U \subseteq G_F^-(X \times V) = X \cap F^-(V)$. Hence we have $U \subseteq F^-(V)$. This shows that F is l.m.c. ■

Theorem 4.14 A multifunction $F : X \rightrightarrows Y$ is u.m.c. iff the graph multifunction G_F is u.m.c.

Proof. ($\Rightarrow$) Suppose that $G_F : X \rightrightarrows X \times Y$ is u.m.c. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subseteq X \times V$, there exists a minimal open nbd U of x such that $G_F(U) \subseteq X \times V$. Then we have $U \subseteq G_F^+(X \times V) = F^+(V)$ and $F(U) \subseteq V$. This shows that F is u.m.c.

($\Leftarrow$) Suppose that F is u.m.c. and suppose that $G_F(x) \subseteq U \times V$ where U is open in X and V is open in Y. Since F is u.m.c., there exists a minimal open nbd G of x such that $F(G) \subseteq V$. Hence we have $G \subseteq U \cap G \subseteq U \cap F^+(V) = G_F^+(U \times V)$. This completes the proof. ■

It is known that the graph $G(F)$ of the multifunction $F : X \rightrightarrows Y$ is said to be closed if for each $(x, y) \notin G(F)$, there exist an open set U containing x and an open set V containing y such that $(U \times V) \cap G(F) = \emptyset$.

Definition 4.15 [38] A subset A of a topological space X is called α-paracompact if every open cover of A in X has a locally finite open refinement in X which covers A.

Theorem 4.16 If $F : X \rightrightarrows Y$ is u.m.c. and point α-paracompact multifunction into a Hausdorff space Y, then the graph $G(F)$ is closed.

Proof. Let $(x_0, y_0) \notin G(F)$. Then $y_0 \notin F(x_0)$. Therefore, for every $y \in F(x_0)$, there exists an open set V_y and an open set W_y in Y containing y and y_0, respectively such that $V_y \cap W_y = \emptyset$. Then $\{V_y : y \in F(x_0)\}$ is a open cover of $F(x_0)$, thus there is a locally finite open cover $\Psi = \{U_\beta | \beta \in \Delta\}$ of $F(x_0)$ which refines $\{V_y : y \in F(x_0)\}$. So there exists an open neighborhood W_0 of y_0 such that W_0 intersect only finitely many members $U_{\beta_1}, U_{\beta_2}, ..., U_{\beta_n}$ of Ψ. Chose finitely many points $y_1, y_2, ..., y_n$ of $F(x_0)$ such that $U_{\beta_k} \subset V_{y_k}$ of each $1 \leqslant k \leqslant n$ and set $W = W_0 \cap [\bigcap_{k=1}^{n} W_{y_k}]$. Then W is an open neighborhood of y_0 such that $W \cap (\cup\Psi) = \emptyset$. Since F is u.m.c., then there exists a minimal open nbd U of x_0 such that $U \subseteq F^+(\cup\Psi)$. Therefore, we have that $(U \times W) \cap G(F) = \emptyset$. Thus, $G(F)$ is closed set $X \times Y$. ■

In the upper Theorem, for upper minimal continuous multifunction F, if F is taken as a point closed multifunction and Y is taken as a regular space, then we get also same result.

Theorem 4.17 Let X and X_α be topological spaces for $\alpha \in \Lambda$. If a multifunction $F : X \rightrightarrows \prod_{\alpha\in\Lambda} X_\alpha$ is u(l).m.c., then $p_\alpha \circ F$ is u(l).m.c. where $p_\alpha : \prod_{\alpha\in\Lambda} X_\alpha \rightrightarrows X_\alpha$ is the projection for each $\alpha \in \Lambda$.

Proof. We prove for upper minimal continuity. The other can be done similarly. Let $\alpha \in \Lambda$ be an arbitrary index and $x \in X$ be an arbitrary point. Suppose that $V_\alpha \subseteq Y_\alpha$ is an open set with $(p_\alpha \circ F)(x) \subseteq V_\alpha$. Then we get $x \in (p_\alpha \circ F)^+(V_\alpha) = F^+(p_\alpha^+(V_\alpha)) = F^+(V_\alpha \times \prod_{\beta\neq\alpha} X_\beta)$. Since $V_\alpha \times \prod_{\beta\neq\alpha} X_\beta$ is open in $\prod_{\alpha\in\Lambda} X_\alpha$ and F is u.m.c., there exists a minimal open nbd U of x such that $U \subseteq F^+(V_\alpha \times \prod_{\beta\neq\alpha} X_\beta) = (p_\alpha \circ F)^+(V_\alpha)$. This shows that $p_\alpha \circ F$ is u.m.c. ■

We know that for the multifunctions $F_1 : X \rightrightarrows Y$ and $F_2 : X \rightrightarrows Z$, and for any subsets $B \subseteq Y$ and $C \subseteq Z$, the following equations are true:

(1) $(F_1 \times F_2)^+ (B \times C) = F_1^+ (B) \cap F_2^+ (C)$,

(2) $(F_1 \times F_2)^- (B \times C) = F_1^- (B) \cap F_2^- (C)$.

Theorem 4.18 Let X, Y, and Z be topological spaces and $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$ be multifunctions. If the multifunction $H : X \rightrightarrows Y \times Z$, $H(x) = F(x) \times G(x)$ is u(l).m.c., then F and G are u(l).m.c.

Proof. Let $x \in X$ be a arbitrary point and V and W be open sets such that $F(x) \subseteq V$ and $G(x) \subseteq W$. Then we get $H(x) = F(x) \times G(x) \subseteq V \times W$. Since H is u.m.c., there exists a minimal open nbd U of x such that $U \subseteq H^+ (V \times W)$. Therefore, we have

$U \subseteq F^+(V) \cap G^+(W)$ and so $U \subseteq F^+(V)$ and $U \subseteq G^+(W)$. This completes the proof. ■

The proof in the case of lower minimal continuity can be done similarly to the above.

Theorem 4.19 Let Y be a normal topological space and $F : X \rightrightarrows Y$ be a multifunction which satisfies $F(x) \cap F(y) = \emptyset$ for each distinct points $x, y \in X$. If F is point closed and u.m.c., then X is a Hausdorff space.

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \emptyset$. Since Y is a normal space, then there exists disjoint open sets V and W such that $F(x) \subseteq V$ and $F(y) \subseteq W$. By upper minimal continuity of F, there exist minimal open nbd's U and U' of x and y, respectively such that $U \subseteq F^+(V)$ and $U' \subseteq F^+(W)$. Hence U and U' are disjoint open sets, and so we have that X is Hausdorff space. ■

Theorem 4.20 Let Y be a normal topological space. If $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Y$ are point closed and u.m.c. multifunctions, then the set $A = \{x : F(x) \cap G(x) \neq \emptyset\}$ is closed in X.

Proof. Let $x \in X - A$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are point closed multifunctions and Y is a normal space, then there exist disjoint open sets V and W containing $F(x)$ and $G(x)$, respectively. Since F and G are u.m.c., there exists a minimal open nbd U of x such that $U \subseteq F^+(V)$ and $U \subseteq G^+(W)$. Therefore, we have $U \cap A = \emptyset$ and so $U \subseteq X - A$. This shows that A is closed in X. ■

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