Journal of Linear and Topological Algebra
Vol. 11*, No.* 03*,* 2022*,* 189- 203
DOR: 20.1001.1.22520201.2022.11.03.4.9
DOI: 10.30495/JLTA.2022.695232

A generalization of weighted versions of the determinant, permanent and the generalized inverse of rectangular matrices

M. Bayat[a]

[a]*Department of Mathematics, Zanjan Branch, Islamic Azad University, P.O. Box 100190, Zanjan, Iran.*

Received 3 July 2022; Revised 23 September 2022; Accepted 26 September 2022.
Communicated by Hamidreza Rahimi

Abstract. In this paper, we first generalized the weighted versions of determinants, permanents and the generalized inverses of rectangular matrices. We also investigate some of their algebraic properties. As a by product of the above investigation, we then present a determinantal representation for the general and Moore-Penrose inverses which satisfy on certain conditions. Finally, we give a general algorithm for determining the inverse of some certain class of the rectangular matrices defined based on weighted determinants.

Keywords: The generalized weighted determinant, the generalized weighted permanent, the generalized Cauchy-Binet formula, the generalized Laplace expansion formula, the generalized determinantal inverse, Moore-Penrose weighted inverse.

2010 AMS Subject Classification: 15A15, 51M25.

1. Introduction and preliminaries

The generalized inverses of matrices play an essential role in both theoretical and practical applications. In particular, the Moore-Penrose inverse of a matrix and its weighted versions have many interesting applications in various fields of science and engineering including optimization problems, machine learning regularization problems, singularity of matrices in data science and statistical problems. Here, we will consider a more generalized version of this problem on the class of rectangular matrices. Next, we will quickly review some important research works in this respect.

Next, we introduce some notations that we need throughout this paper. Let $\mathbb{C}^n$ be the vector space over the complex field $\mathbb{C}$. We also let $\mathbb{C}^{m\times n}$ be the set of all m by n matrices with complex entries and $\mathbb{C}_r^{m\times n}$ is the subclass of these matrices with the rank

E-mail address: baayyaatt@gmail.com (M. Bayat).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.ctb.iau.ir

exactly equal to r. We reserve the notations $\overline{A}, A^T$ and A^* for the conjugate, transpose and conjugate transpose of the matrix A, respectively. The determinant of a square matrix A is denoted by $\det(A)$ or $|A|$. The submatrix of $A \in \mathbb{C}^{m\times n}$ containing rows set $I = \{\alpha_1, \ldots, \alpha_t\}$ and columns $J = \{\beta_1, \ldots, \beta_t\}$ is denoted by $A\left[{I \atop J}\right]$. Moreover, its corresponding minor will be denoted by $A\binom{I}{J}$, while its algebraic complement corresponding to the element $a_{i,j}$ is defined by

$$A_{i,j}\begin{pmatrix} \alpha_1 & \cdots & \alpha_{p-1} & i & \alpha_{p+1} & \cdots & \alpha_t \\ \beta_1 & \cdots & \beta_{q-1} & j & \beta_{q+1} & \cdots & \beta_t \end{pmatrix} = (-1)^{p+q} A\begin{pmatrix} \alpha_1 & \cdots & \alpha_{p-1} & \alpha_{p+1} & \cdots & \alpha_t \\ \beta_1 & \cdots & \beta_{q-1} & \beta_{q+1} & \cdots & \beta_t \end{pmatrix}.$$

In [10], Penrose showed the existence and uniqueness of a solution $X \in \mathbb{C}^{n\times m}$ of the following system of equations

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

For simplicity of presentation and arguments, we will use the notations introduced in [16]. For a given subset $\mathcal{S}$ of $\{1, 2, 3, 4\}$, the collection of matrices X satisfying the conditions represented in $\mathcal{S}$ will be denoted by $A(\mathcal{S})$. For example, if $\mathcal{S} = \{1, 2\}$, then

$$A\{1, 2\} = \big\{X \in \mathbb{C}^{n\times m} : AXA = A, XAX = X\big\}.$$

A matrix $X \in A(\mathcal{S})$ is called an $\mathcal{S}$-inverse of A and is denoted by $A^{(\mathcal{S})}$. In particular, for any $A \in \mathbb{C}^{m\times n}$, the set $A\{1, 2, 3, 4\}$ which consists of a onlyone element is called the Moore-Penrose inverse of A, will be denoted by $A^\dagger$ (see [10]).

The main motivation behind of this paper originates from the determinantal representation of Moore-Penrose inverse, which is the next theorem:

Theorem 1.1 [2, 4, 6] The element $a^\dagger_{ij}$ in the ith row and the jth column of the Moore-Penrose pseudoinverse of a matrix $A \in \mathbb{C}^{m\times n}_r$ is given by

$$a^\dagger_{ij} = \frac{A^{(\dagger,r)}_{ji}}{N_r(A)} = \frac{\sum\limits_{\substack{1\leqslant\beta_1<\cdots<\beta_r\leqslant n\\ 1\leqslant\alpha_1<\cdots<\alpha_r\leqslant m}} \overline{A}\begin{pmatrix} \alpha_1 & \cdots & j & \cdots & \alpha_r \\ \beta_1 & \cdots & i & \cdots & \beta_r \end{pmatrix} A_{ji}\begin{pmatrix} \alpha_1 & \cdots & j & \cdots & \alpha_r \\ \beta_1 & \cdots & i & \cdots & \beta_r \end{pmatrix}}{\sum\limits_{\substack{1\leqslant\delta_1<\cdots<\delta_r\leqslant n\\ 1\leqslant\gamma_1<\cdots<\gamma_r\leqslant m}} \overline{A}\begin{pmatrix} \gamma_1 & \cdots & \gamma_r \\ \delta_1 & \cdots & \delta_r \end{pmatrix} A\begin{pmatrix} \gamma_1 & \cdots & \gamma_r \\ \delta_1 & \cdots & \delta_r \end{pmatrix}}.$$

We recall that the (i, j)th entry of adjoint matrix $adj^{(\dagger,r)}(A)$, with be defined by $A^{(\dagger,r)}_{ji}$. The next theorem describes the $\{i, j, k\}$-inverse of a rectangular matrix.

Theorem 1.2 [14] If $A \in \mathbb{C}^{m\times n}_r$ has a full-rank factorization $A = PQ, P \in \mathbb{C}^{m\times r}_r, Q \in \mathbb{C}^{r\times n}_r$, $W_1 \in \mathbb{C}^{n\times r}$ and $W_2 \in \mathbb{C}^{r\times m}$ are some matrices such that $\operatorname{rank}(QW_1) = \operatorname{rank}(W_2P) = \operatorname{rank}(A)$, then $A^\dagger = Q^\dagger P^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^*$; and also,
the generalized solution of the equations (1) and (2) is given by

$$A\{1, 2\} = \Big\{W_1(QW_1)^{-1}(W_2P)^{-1}W_2\Big\};$$

the generalized solution of the equations (1), (2) and (3) is given by

$$A\{1,2,3\} = \Big\{W_1(QW_1)^{-1}(P^*P)^{-1}P^*\Big\};$$

the generalized solution of the equations (1), (2) and (4) is given by

$$A\{1,2,4\} = \Big\{Q^*(QQ^*)^{-1}(W_2P)^{-1}W_2\Big\}.$$

Theorem 1.3 [2] Let $A \in \mathbb{C}^{m\times n}$ be a full-rank matrix. If $\text{rank}(A) = m \leqslant n$, then the system

$$AX = I_m; \qquad (XA)^* = XA$$

has a unique solution $X = A^\dagger$. Similarly, if $m > n = \text{rank}(A)$, then the system

$$XA = I_n; \qquad (AX)^* = AX.$$

has a unique solution $X = A^\dagger$.

In this paper, we present a generalization of the weighted determinant and the permanent of rectangular matrices. We first need some definitions and notations.
Let V be a vector space over a field $\mathbb{C}$. The p-th exterior power V, denoted $\bigwedge^p(V)$ is the vector subspace of the exterior algebra $\bigwedge(V)$ spanned by elements of the form $v_1 \wedge \cdots \wedge v_p, v_i \in V, i = 1,\ldots,p$. If the dimension of V is n and $\{e_1,\ldots,e_n\}$ is a basis of V, then the set $\{e_{i_1} \wedge \cdots \wedge e_{i_p} : 1 \leqslant p \leqslant n, 1 \leqslant i_1 < \cdots < i_p \leqslant n\}$ is a basis for $\bigwedge^p(V)$ and $\dim \bigwedge^p(V) = \binom{n}{p}$.

2. Rectangular determinants and induced generalized inverses

In recent years, some researchers have been investigated new versions of the determinant of a rectangular matrices [1, 3, 5, 7, 9, 11–13, 15–17].

Definition 2.1 Suppose $A \in \mathbb{C}^{m\times n}$ is a rectangular matrix with $n \leqslant m$. A weighted determinant of A is a function $\det_{(\tilde{\varepsilon},p)} : \mathbb{C}^{m\times n} \longrightarrow \mathbb{C}$ is defined, as follows:

$$\det_{(\tilde{\varepsilon},p)}(A) = \begin{cases} \displaystyle\sum_{\substack{1\leqslant i_1<\cdots<i_p\leqslant m\\ 1\leqslant j_1<\cdots<j_p\leqslant n}} \varepsilon_{i_1,\ldots,i_p;j_1,\ldots,j_p}\Big\langle \bigwedge_{l=1}^{p} A_{j_l}, \bigwedge_{l=1}^{p} \mathbf{e}_{i_l}\Big\rangle, & \text{if } 1 \leqslant p \leqslant n \leqslant m, \\ 0 & \text{if } p > \min\{m,n\}, \\ 1 & \text{if } p = 0, \end{cases} \tag{1}$$

where $\langle . \, , \, . \rangle$ is the inner product, A_{j_l} is the j_l-th column of the matrix A and

$$\tilde{\varepsilon} = \Big\{\varepsilon_{i_1,\ldots,i_p;j_1,\ldots,j_p} : 1 \leqslant i_1 < \cdots < i_p \leqslant m, 1 \leqslant j_1 < \cdots < j_p \leqslant n\Big\},$$

which $\varepsilon_{i_1,\dots,i_p;j_1,\dots,j_p}$ is an arbitrary constant coefficient. For $n > m \geqslant p \geqslant 1$, we set $\det\limits_{(\tilde{\varepsilon},p)}(A) = \det\limits_{(\tilde{\varepsilon},p)}(A^T)$.

From now on, for simplicity of presentation, we will assume $A \in \mathbb{C}^{m\times n}$ with $n \leqslant m$ and the inner product $\langle . \, , \, . \rangle$ will be considered as the Euclidean inner product over $\bigwedge^p(\mathbb{C}^m)$. Now, in the following lemma, we express the generalized determinant based on its square minors.

Lemma 2.2 Let $A \in \mathbb{C}^{m\times n}$, where $1 \leqslant p \leqslant n \leqslant m$. Then

$$\det_{(\tilde{\varepsilon},p)}(A) = \sum_{\substack{1\leqslant i_1<\cdots<i_p\leqslant m\\ 1\leqslant j_1<\cdots<j_p\leqslant n}} \varepsilon_{i_1,\dots,i_p;j_1,\dots,j_p} A\begin{pmatrix} i_1 & \cdots & i_p \\ j_1 & \cdots & j_p \end{pmatrix}. \tag{2}$$

Proof. According to (1) for $1 \leqslant j_1 < \cdots < j_p \leqslant n$, we obtain

$$\begin{aligned}
\bigwedge_{l=1}^{p} A_{j_l} &= \bigwedge_{l=1}^{p} \sum_{i=1}^{m} a_{i,j_l}\mathbf{e}_i \\
&= \sum_{i_1,\dots,i_p} \bigwedge_{l=1}^{p} a_{i_l,j_l}\mathbf{e}_{i_l} \\
&= \sum_{i_1,\dots,i_p} \prod_{l=1}^{p} a_{i_l,j_l} \bigwedge_{l=1}^{p} \mathbf{e}_{i_l} \\
&= \sum_{1\leqslant i_1<\cdots<i_p\leqslant m} \sum_{\sigma\in S_p} \left(\prod_{l=1}^{p} a_{i_{\sigma(l)},j_l} \bigwedge_{l=1}^{p} \mathbf{e}_{i_{\sigma(l)}}\right) \\
&= \sum_{1\leqslant i_1<\cdots<i_p\leqslant m} \left(\sum_{\sigma\in S_p} \operatorname{sgn}(\sigma) \prod_{l=1}^{p} a_{i_{\sigma(l)},j_l}\right) \bigwedge_{l=1}^{p} \mathbf{e}_{i_l} \\
&= \sum_{1\leqslant i_1<\cdots<i_p\leqslant m} A\begin{pmatrix} i_1 & \cdots & i_p \\ j_1 & \cdots & j_p \end{pmatrix} \bigwedge_{l=1}^{p} \mathbf{e}_{i_l}.
\end{aligned}$$

Using (1), we obtain the formula (2). ■

***Example* 2.3** For $p = 2$ and $A = (a_{i,j})_{3\times 2}$, we have

$$\det_{(\tilde{\varepsilon},2)}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \varepsilon_{1,2;1,2}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \varepsilon_{1,3;1,2}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + \varepsilon_{2,3;1,2}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

For $p = m$, the determinant $\det\limits_{(\tilde{\varepsilon},m)}(A)$ is an alternating multilinear mapping of the column vectors of A. In case of $m = n = p$ and $\varepsilon_{i_1,\cdots,i_p;j_1,\cdots,j_p} = 1$, we obtain the classical determinant of the square matrix A.
In (2), for $\varepsilon_{i_1,\dots,i_p;j_1,\dots,j_p} = 1$, we get the Stojaković determinant [17], which we will denote by $\det\limits_{(S,p)}(A)$. Similarly, for $\varepsilon_{i_1,\dots,i_p;j_1,\dots,j_p} = (-1)^{(i_1+\cdots+i_p)+(j_1+\cdots+j_p)}$, one can obtain

the determinant introduced by Radić [11–13] which we denote it $\det_{(R,p)}(A)$. Now, let ε be an arbitrary but a constant number, for $\varepsilon_{i_1,\ldots,i_p;j_1,\ldots,j_p} = \varepsilon^{(i_1+\cdots+i_p)+(j_1+\cdots+j_p)}$, we will obtain the determinant introduced by Stanimirović [16], which we denote by $\det_{(\varepsilon,p)}(A)$. Moreover, we can also consider some generalized versions of the Stanimirović's determinant by letting $\varepsilon_{i_1,\ldots,i_p;j_1,\ldots,j_p} = \varepsilon^{(i_1+\cdots+i_p)+(j_1+\cdots+j_p)}$, to be written in the following multiplicative forms:

$$\varepsilon_{i_1,\ldots,i_p;j_1,\ldots,j_p} = \alpha_{i_1,\ldots,i_p}\beta_{j_1,\ldots,j_p}$$
$$\varepsilon_{i_1,\ldots,i_p;j_1,\ldots,j_p} = \left(\alpha_1^{i_1}\ldots\alpha_p^{i_p}\right)\left(\beta_1^{j_1}\cdots\beta_p^{j_p}\right)$$
$$\varepsilon_{i_1,\ldots,i_p;j_1,\ldots,j_p} = \alpha^{i_1+\cdots+i_p}\beta^{j_1+\cdots+j_p}$$

In [1], Abhimanyu has consider the weight $\varepsilon_{i_1,\ldots,i_p;j_1,\ldots,j_p} = \overline{A}\begin{pmatrix} i_1 & \cdots & i_p \\ j_1 & \cdots & j_p \end{pmatrix}$, and in [9], Nakagami has defined the determinants of a rectangular matrix $A = (A_1,\ldots,A_n) \in \mathbb{C}^{m\times n}$ with the weight $\varepsilon_{i_1,\ldots,i_p;j_1,\ldots,j_p} = 1$ as follows:

$$\det_{(N,n)}(A) = \sum_{1\leqslant j_1\leqslant\cdots\leqslant j_n\leqslant m} \Big\langle A_1\wedge\cdots\wedge A_n, \mathbf{e}_{j_1}\wedge\cdots\wedge\mathbf{e}_{j_n}\Big\rangle, \tag{3}$$

$$\operatorname*{Det}_{(N,n)}(A) = \sum_{1\leqslant j_1<\cdots<j_n\leqslant m} \Big\langle A_1\wedge\cdots\wedge A_n, \mathbf{e}_{j_1}\wedge\cdots\wedge\mathbf{e}_{j_n}\Big\rangle. \tag{4}$$

In (2), if we replace the determinant with permanent, we immediately obtain a weighted version of the permanent by the following formula:

$$\operatorname*{per}_{(\tilde{\varepsilon},p)}(A) = \sum_{\substack{1\leqslant i_1<\cdots<i_p\leqslant m\\ 1\leqslant j_1<\cdots<j_p\leqslant n}} \varepsilon_{i_1,\,\ldots,\,i_p;j_1,\,\ldots,\,j_p}\operatorname{per}\left(A\begin{bmatrix} i_1 & \cdots & i_p \\ j_1 & \cdots & j_p \end{bmatrix}\right). \tag{5}$$

In particular, for $\varepsilon_{i_1,\,\ldots,\,i_n;j_1,\,\ldots,\,j_n} = 1$, we get the definition of classic permanent (see [8]):

$$\operatorname{per}(A) = \sum_{1\leqslant i_1<\cdots<i_n\leqslant m} \operatorname{per}\left(A\begin{bmatrix} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{bmatrix}\right). \tag{6}$$

3. The generalized Cauchy-Binet formula

A generalization of the multiplicative property of determinants is the well-know Cauchy-Binet formula. In this section, we present several extensions of Cauchy-Binet formula for determinant and permanent of a rectangular matrix. We first need to introduce some notations. Let r and n be positive integers. The set $\Gamma_{r,n}$ consists of all sequences of integers $\omega = (\omega_1,\ldots,\omega_r)$ for which $1\leqslant\omega_i\leqslant n, i = 1,\ldots,r$. If $r\leqslant n$, then

$G_{r,n}$ and $Q_{r,n}$ denote as follows:

$$G_{r,n} = \Big\{(\omega_1, \dots, \omega_r) \in \Gamma_{r,n} : 1 \leqslant \omega_1 \leqslant \cdots \leqslant \omega_r \leqslant n\Big\},$$
$$Q_{r,n} = \Big\{(\omega_1, \dots, \omega_r) \in \Gamma_{r,n} : 1 \leqslant \omega_1 < \cdots < \omega_r \leqslant n\Big\}.$$

If $\omega = (\omega_1, \dots, \omega_r) \in G_{r,n}$ then by the notation $\mu(\omega)$, we mean $\mu(\omega) = \prod_{k=1}^{r} \omega_k!$, where $\omega_k!$ denotes the factorial of the positive integer ω_k.
Let $A = (a_{i,j}) \in \mathbb{C}^{m\times n}$ and $\alpha \in Q_{h,m}$ and $\beta \in Q_{k,n}$. Then $A\big[\alpha|\beta\big]$ denotes the $h \times k$ submatrix of A whose (i, j) entry is $a_{\alpha_i\beta_j}$. Again, if $\alpha \in Q_{h,m}$ and $\beta \in Q_{k,n}$, then $A\big(\alpha|\beta\big)$ denotes the $(m - h) \times (n - k)$ submatrix of A complementary to $A\big[\alpha|\beta\big]$, that is, the submatrix obtained from A by deleting rows α and columns β.

Theorem 3.1 (The generalized Cauchy-Binet formula) Let $A \in \mathbb{C}^{m\times t}$, $B \in \mathbb{C}^{t\times n}$ and $p \leqslant \min\{m, n, t\}$. Then

$$\det_{(\tilde{\varepsilon},p)}(AB) = \sum_{I,J,K} \varepsilon_{I,J} \det\big(A[I|K]\big)\det\big(B[K|J]\big), \tag{7}$$
$$\operatorname{per}_{(\tilde{\varepsilon},p)}(AB) = \sum_{I,J,K} \varepsilon_{I,J} \operatorname{per}\big(A[I|K]\big)\operatorname{per}\big(B[K|J]\big),$$

where $I \in Q_{p,m}$, $J \in Q_{p,n}$ and $K \in Q_{p,t}$.

Proof. According to Definition 2.1, we obtain

$$\det_{(\tilde{\varepsilon},p)}(AB) = \sum_{\substack{1\leqslant i_1<\cdots<i_p\leqslant m,\\ 1\leqslant j_1<\cdots<j_p\leqslant n}} \varepsilon_{i_1,\dots,i_p;j_1,\dots,j_p} \left\langle \bigwedge_{l=1}^{p}(AB)_{j_l}, \bigwedge_{l=1}^{p} e_{i_l} \right\rangle, \tag{8}$$

By a similar calculation as in the proof of Lemma 2.2, it can be seen that

$$\bigwedge_{l=1}^{p}(AB)_{j_l} = \sum_{\substack{1\leqslant s_1<\cdots<s_p\leqslant m\\ 1\leqslant k_1<\cdots<k_p\leqslant t}} \det\big(A[I|K]\big)\det\big(B[K|J]\big) \bigwedge_{l=1}^{p} \mathbf{e}_{s_l}, \tag{9}$$

Thus, considering formulas 8 and 9, we finally get, the formula 7. ■

We note that, in the case $p = 1$, for every $I \in Q_{t,m}$, $J \in Q_{t,n}$ and $K \in Q_{t,t}$, we get the classic Cauchy-Binet formula.

Corollary 3.2 Let $A \in \mathbb{C}^{m\times t}$, $B \in \mathbb{C}^{t\times n}$ and $p \leqslant \min\{m, n, t\}$. Then

$$\det_{(\varepsilon_1,\dots,\varepsilon_p;p)}(AB) = \sum_{I,J,K} \varepsilon_K^{-2} \det_{(\varepsilon_1,\dots,\varepsilon_p;p)}\big(A[I|K]\big) \det_{(\varepsilon_1,\dots,\varepsilon_p;p)}\big(B[K|J]\big),$$
$$\operatorname{per}_{(\varepsilon_1,\dots,\varepsilon_p;p)}(AB) = \sum_{I,J,K} \varepsilon_K^{-2} \operatorname{per}_{(\varepsilon_1,\dots,\varepsilon_p;p)}\big(A[I|K]\big) \operatorname{per}_{(\varepsilon_1,\dots,\varepsilon_p;p)}\big(B[K|J]\big),$$

where $\varepsilon_K = \varepsilon_1^{k_1} \cdots \varepsilon_p^{k_p}$, $I \in Q_{p,m}$, $J \in Q_{p,n}$ and $K \in Q_{p,t}$.

In the special case of $t \leqslant \min\{m,n\}$ and $\varepsilon_1 = \cdots = \varepsilon_t = \varepsilon$,

$$\det_{(\varepsilon,t)}(AB) = \varepsilon^{-t(t+1)} \det_{(\varepsilon,t)}(A) \det_{(\varepsilon,t)}(B),$$

$$\operatorname{per}_{(\varepsilon,t)}(AB) = \varepsilon^{-t(t+1)} \operatorname{per}_{(\varepsilon,t)}(A) \operatorname{per}_{(\varepsilon,t)}(B).$$

Before stating our next theorem, we present the following combinatorial lemma.

Lemma 3.3 [8] Let f be a scalar function defined on the set of m-tuples of integers. Then

$$\sum_{\omega\in\Gamma_{m,n}} f(\omega_1,\ldots,\omega_m) = \sum_{\omega\in G_{m,n}} \frac{1}{\mu(\omega)} \sum_{\sigma\in S_m} f(\omega_{\sigma(1)},\ldots,\omega_{\sigma(m)}),$$

where $\omega = (\omega_1,\ldots,\omega_m)$.

Theorem 3.4 Let $A \in \mathbb{C}^{m\times t}$ and $B \in \mathbb{C}^{t\times n}$ where $p \leqslant \min\{m,n,t\}$. Then

$$\det_{(\tilde{\varepsilon},p)}(AB) = \sum_{I,J,K} \tfrac{1}{\mu(K)}\varepsilon_{I,J} \det\big(A[I|K]\big)\det\big(B[K|J]\big),$$

$$\operatorname{per}_{(\tilde{\varepsilon},p)}(AB) = \sum_{I,J,K} \tfrac{1}{\mu(K)}\varepsilon_{I,J} \operatorname{per}\big(A[I|K]\big)\operatorname{per}\big(B[K|J]\big),$$

where $I \in Q_{p,m}$, $J \in Q_{p,n}$ and $K \in G_{p,t}$.

Proof. Using Lemma 3.3, the proof is similar to the proof of Theorem 3.1. ■

4. The generalized Laplace expansion

One of the fundamental and classic results in the theory of determinants and permanent is the Laplace expansion formula. Next, we obtain some results regarding the Laplace expansion of rectangular matrices.

Theorem 4.1 [3] For $A = (a_{i,j}) \in \mathbb{C}^{m\times n}$ with $1 < n \leqslant m$,

$$\det_{(\varepsilon,n)}(A) = \sum_{1\leqslant i_1<\cdots<i_n\leqslant m} \sum_{\sigma\in S_n} \varepsilon^{\sum_{l=1}^{n}(i_{\sigma(l)}+l)}\operatorname{sgn}(\sigma)\prod_{l=1}^{n} a_{i_{\sigma(l)},l}, \tag{10}$$

$$\operatorname{per}_{(\varepsilon,n)}(A) = \sum_{1\leqslant i_1<\cdots<i_n\leqslant m} \sum_{\sigma\in S_n} \varepsilon^{\sum_{l=1}^{n}(i_{\sigma(l)}+l)}\prod_{l=1}^{n} a_{i_{\sigma(l)},l}.$$

We note that the formula of permanent is different from the formula of determinant A because the sign of permutations is not taken into account.
In [11, 17], it has been shown that the classic Laplace expansion for rectangular matrix ($m \leqslant n$) is valid with respect to each row and for the case of ($n \leqslant m$) is true with respect to each column. Next, we generalize the Laplace expansion formula for an arbitrary partition of rows and columns of rectangular matrices of Radić and Stojaković types.

Theorem 4.2 Let $A = (a_{i,j}) \in \mathbb{C}^{m\times n}$ with $2 \leqslant n \leqslant m$ and $J \in Q_{r,n}$. Then

$$\det_{(R,n)}(A) = \sum_{I\in Q_{r,m}} (-1)^{I+J} \det_{(R,r)}\big(A[I|J]\big) \det_{(R,n-r)}\big(A(I|J)\big), \tag{11}$$

$$\det_{(S,n)}(A) = \sum_{I\in Q_{r,m}} \det_{(S,r)}\big(A[I|J]\big) \det_{(S,n-r)}\big(A(I|J)\big).$$

Proof. Fix arbitrary $j_1, \ldots, j_r$ columns where $J = (j_1, \ldots, j_r) \in Q_{r,n}$. By neglecting the sign of terms, we can imagine that $\det_{(R,n)}(A)$ is the products of $\det_{(R,r)}\big(A[I|J]\big)$ and $\det_{(R,n-r)}\big(A(I|J)\big)$ where $I = (i_1, \ldots, i_r) \in Q_{r,m}$; without no other terms in the expansion of the determinant of A. To compute the signs of these products, let us shuffle the rows and columns so as to replace the term $\det_{(R,r)}\big(A[I|J]\big)$ in the upper left corner. Hence, we have to perform

$$(i_1 - 1) + \cdots + (i_r - r) + (j_1 - 1) + \cdots + (j_r - r) \equiv \textstyle\sum_{l=1}^{r}(i_l + j_l) \pmod 2$$

permutations. ■

Corollary 4.3 [8, 18] For $A = (a_{i,j}) \in \mathbb{C}^{n\times n}$ with $2 \leqslant n$, we have

$$\det(A) = \sum_{I\in Q_{r,n}} (-1)^{I+J} \det\big(A[I|J]\big)\det\big(A(I|J)\big).$$

Our next result is a generalized Laplace expansion for determinant of rectangular matrices based on the generalized cofactors.

Theorem 4.4 For a full-rank matrix $A \in \mathbb{C}^{m\times n}$ the following Laplace's expansion is valid

$$\begin{cases} \det_{(\tilde{\varepsilon},m)}(A) = \sum\limits_{k=1}^{n} a_{ik}A_{ki}^{(\tilde{\varepsilon},m)}, & i = 1, \ldots, m, \quad m \leqslant n; \\ \det_{(\tilde{\varepsilon},n)}(A) = \sum\limits_{k=1}^{m} a_{ik}A_{ki}^{(\tilde{\varepsilon},n)}, & i = 1, \ldots, n, \quad n \leqslant m; \end{cases}$$

where $A_{ij}^{(\tilde{\varepsilon},m)}$, i.e. $A_{ij}^{(\tilde{\varepsilon},n)}$ is the generalized algebraic complement corresponding to the element a_{ji} defined as follows:

$$\begin{cases} A_{ij}^{(\tilde{\varepsilon},m)} = \sum\limits_{1\leqslant j_1<\cdots<j_m\leqslant n} \varepsilon_{1,\ldots,m;j_1,\ldots,j_n} A_{j,i}\begin{pmatrix} 1 & \cdots & i & \cdots & m \\ j_1 & \cdots & j & \cdots & j_m \end{pmatrix}, & m \leqslant n \\ A_{ij}^{(\tilde{\varepsilon},n)} = \sum\limits_{1\leqslant i_1<\cdots<i_n\leqslant m} \varepsilon_{i_1,\ldots,i_n;1,\ldots,n} A_{j,i}\begin{pmatrix} i_1 & \cdots & i & \cdots & i_n \\ 1 & \cdots & j & \cdots & n \end{pmatrix}. & n \leqslant m. \end{cases}$$

Proof. For $1 < n \leqslant m$, by (2) and using Laplace's expansion for the square minors

$A\begin{pmatrix} i_1 \cdots i_n \\ 1 \cdots n \end{pmatrix}$, we conclude that

$$\begin{aligned}
\det_{(\tilde{\varepsilon},n)}(A) &= \sum_{1\leqslant i_1<\cdots<i_n\leqslant m} \varepsilon_{i_1,\ldots,i_n;1,\ldots,n}\left[\sum_{k=1}^{n} a_{i_k,j}A_{i_k,j}\begin{pmatrix} i_1 \cdots i_k \cdots i_n \\ 1 \cdots j \cdots n \end{pmatrix}\right] \\
&= \sum_{l=1}^{n} a_{l,j}\left[\sum_{1\leqslant i_1<\cdots<i_n\leqslant m} \varepsilon_{i_1,\ldots,i_n;1,\ldots,n}A_{l,j}\begin{pmatrix} i_1 \cdots l \cdots i_n \\ 1 \cdots j \cdots n \end{pmatrix}\right] \\
&= \sum_{l=1}^{n} a_{l,j}A_{jl}^{(\tilde{\varepsilon},n)}.
\end{aligned}$$

■

Corollary 4.5 If $A \in \mathbb{C}^{m\times n}$ is a full-rank matrix, then

$$\begin{cases} \sum_{k=1}^{n} a_{ik}A_{kj}^{(\tilde{\varepsilon},m)} = \delta_{ij}\det_{(\tilde{\varepsilon},m)}(A), & m \leqslant n, \\ \sum_{k=1}^{m} a_{ik}A_{kj}^{(\tilde{\varepsilon},n)} = \delta_{ij}\det_{(\tilde{\varepsilon},n)}(A), & n \leqslant m, \end{cases}$$

where δ_{ij} is the Kronecker delta symbol.

Proof. For $m \leqslant n$, in the case that $i \neq j$ the above expansion indicates the rectangular determinant of a matrix which has the identical ith row and jth column. ■

5. The generalized induced inverse of the determinant of rectangular matrices

In this section, we present a definition of generalized inverses of the rectangular matrices based on in tearms of determinant and the generalized cofactors, which we call it the determinantal generalized inverses.

Definition 5.1 Suppose $A \in \mathbb{C}_r^{m\times n}$, the generalized inverse of A denoted by $A^{-1}_{(\tilde{\varepsilon},p)}$ is defined by

$$\left(A^{-1}_{(\tilde{\varepsilon},p)}\right)_{ij} = \frac{A_{ij}^{(\tilde{\varepsilon},p)}}{\det_{(\tilde{\varepsilon},p)}(A)},$$

in which $1 \leqslant p \leqslant \operatorname{rank}(A) \leqslant \min\{m,n\}$ is the greatest integer such that $\det_{(\tilde{\varepsilon},p)}(A) \neq 0$ (where we denote it by $\rho_{\tilde{\varepsilon}}(A)$). Similarly $A_{ij}^{(\tilde{\varepsilon},p)}$ for each p is defined as follows:

$$A_{ij}^{(\tilde{\varepsilon},p)} = \sum_{\substack{1\leqslant j_1<\cdots<j<\cdots<j_p\leqslant n \\ 1\leqslant i_1<\cdots<i<\cdots<i_p\leqslant m}} \varepsilon_{i_1,\ldots,i_p;j_1,\ldots,j_p}A_{j,i}\begin{pmatrix} j_1 \cdots j \cdots j_p \\ i_1 \cdots i \cdots i_p \end{pmatrix}.$$

Now, we define the generalized adjoint of A of the order p as follows:

$$\underset{(\tilde{\varepsilon},p)}{\operatorname{adj}}(A) = \left(A_{ij}^{(\tilde{\varepsilon},p)}\right).$$

Remark 1 *Considering the Corollary 4.5, if* $p = \rho_{\tilde{\varepsilon}}(A) = \min\{m,n\}$*, then the matrix* $A^{-1}_{(\tilde{\varepsilon},p)}$ *with* $m < n$ *has the right inverse and for* $m > n$*, it has the left inverse.*

Our next results are concerned with the properties of the generalized adjoint and determinantal inverse of rectangular matrices.

Lemma 5.2 Let $\varepsilon_{i_1,\cdots,i_r,j_1,\cdots,j_r} = \varepsilon_1^{i_1+j_1}\cdots\varepsilon_r^{i_r+j_r}$, where $1 \leqslant i_1 < \cdots < i_r \leqslant m$, $1 \leqslant j_1 < \cdots < j_r \leqslant n$ and $\varepsilon_1,\ldots,\varepsilon_r$ are arbitrary but fixed non-zero constants. If $A \in \mathbb{C}^{m\times r}$ and $B \in \mathbb{C}^{r\times n}$ are two full-rank matrices such that $\operatorname{rank}(A) = r = \operatorname{rank}(B) = \rho_{(\varepsilon_1,\ldots,\varepsilon_r)}(A) = \rho_{(\varepsilon_1,\ldots,\varepsilon_r)}(B) = \rho_{(\varepsilon_1,\ldots,\varepsilon_r)}(AB)$, then

$$\underset{(\varepsilon_1,\ldots,\varepsilon_r,r)}{\operatorname{adj}}(AB) = \varepsilon_1^{-2}\cdots\varepsilon_r^{-2r}\underset{(\varepsilon_1,\ldots,\varepsilon_r,r)}{\operatorname{adj}}(B)\underset{(\varepsilon_1,\ldots,\varepsilon_r,r)}{\operatorname{adj}}(A).$$

Let $\varepsilon_1 = \cdots = \varepsilon_r = \varepsilon$. Then

$$\underset{(\varepsilon,r)}{\operatorname{adj}}(AB) = \varepsilon^{-r(r+1)}\underset{(\varepsilon,r)}{\operatorname{adj}}(B)\underset{(\varepsilon,r)}{\operatorname{adj}}(A).$$

Proof. The entry in the ith row and jth column of the matrix $\underset{(\varepsilon_1,\cdots,\varepsilon_r,r)}{\operatorname{adj}}(AB)$ is equal to

$$(AB)_{ij}^{(\varepsilon_1,\cdots,\varepsilon_r,r)} = \sum_{\substack{1\leqslant\beta_1<\cdots<i<\cdots<\beta_r\leqslant n\\ 1\leqslant\alpha_1<\cdots<j<\cdots<\alpha_r\leqslant m}} \varepsilon_1^{\alpha_1+\beta_1}\cdots\varepsilon_r^{\alpha_r+\beta_r}(AB)_{j,i}\begin{pmatrix}\alpha_1 & \cdots & j & \cdots & \alpha_r\\ \beta_1 & \cdots & i & \cdots & \beta_r\end{pmatrix}.$$

Using the Cauchy-Binet formula for square matrices (Theorem 3.1), we get

$$\begin{aligned}
(AB)_{ij}^{(\varepsilon_1,\ldots,\varepsilon_r,r)} &= \sum_{\substack{1\leqslant\beta_1<\cdots<i<\cdots<\beta_r\leqslant n\\ 1\leqslant\alpha_1<\cdots<j<\cdots<\alpha_r\leqslant m}} \varepsilon_1^{\alpha_1+\beta_1}\cdots\varepsilon_r^{\alpha_r+\beta_r}\left[\sum_{k=1}^{r} A_{j,k}\begin{pmatrix}\alpha_1 & \cdots & j & \cdots & \alpha_r\\ 1 & \cdots & k & \cdots & r\end{pmatrix}\right.\\
&\qquad \left. .B_{k,i}\begin{pmatrix}1 & \cdots & k & \cdots & r\\ \beta_1 & \cdots & i & \cdots & \beta_r\end{pmatrix}\right]\\
&= \sum_{k=1}^{r}\left[\sum_{1\leqslant\beta_1<\cdots<i<\cdots<\beta_r\leqslant n} \varepsilon_1^{1+\beta_1}\cdots\varepsilon_r^{r+\beta_r}B_{k,i}\begin{pmatrix}1 & \cdots & k & \cdots & r\\ \beta_1 & \cdots & i & \cdots & \beta_r\end{pmatrix}\right]\\
&\quad\times\left[\sum_{1\leqslant\alpha_1<\cdots<j<\cdots<\alpha_r\leqslant m} \varepsilon_1^{1+\alpha_1}\cdots\varepsilon_r^{r+\alpha_r}A_{j,k}\begin{pmatrix}\alpha_1 & \cdots & j & \cdots & \alpha_r\\ 1 & \cdots & k & \cdots & r\end{pmatrix}\right]\\
&= \varepsilon_1^{-2}\cdots\varepsilon_r^{-2r}\sum_{k=1}^{r} B_{i,k}^{(\varepsilon_1,\ldots,\varepsilon_r,r)}A_{k,j}^{(\varepsilon_1,\ldots,\varepsilon_r,r)}.
\end{aligned}$$

■

According to Lemma 2.2 and Lemma 5.2, we can compute the determinantal inverses using the notion of the full-rank matrix factorization.

Corollary 5.3 If $A = PQ$ is a full-rank matrix factorization of $A \in \mathbb{C}_r^{m\times n}$, then the determinantal inverse of A is

$$A^{-1}_{(\varepsilon_1,\ldots,\varepsilon_r,r)} = \varepsilon_1^{-2}\ldots\varepsilon_r^{-2r} Q^{-1}_{(\varepsilon_1,\ldots,\varepsilon_r,r)} P^{-1}_{(\varepsilon_1,\ldots,\varepsilon_r,r)}, \qquad r = \rho_{(\varepsilon_1,\ldots,\varepsilon_r)}(A).$$

Let $\varepsilon_1 = \cdots = \varepsilon_r = \varepsilon$. Then

$$A^{-1}_{(\varepsilon,r)} = \varepsilon^{-r(r+1)} Q^{-1}_{(\varepsilon,r)} P^{-1}_{(\varepsilon,r)} \quad \text{and} \quad r = \rho_\varepsilon(A).$$

Using Theorem 1.2, we can immediately prove the following corollary.

Corollary 5.4 If $A \in \mathbb{C}_r^{m\times n}$ and $r = \rho_{(\varepsilon_1,\ldots,\varepsilon_r)}(A)$, then

$$A^{-1}_{(\varepsilon_1,\ldots,\varepsilon_r,r)} = \varepsilon_1^{-2}\cdots\varepsilon_r^{-2r} Q^{-1}_{(\varepsilon_1,\ldots,\varepsilon_r,r)} P^{-1}_{(\varepsilon_1,\ldots,\varepsilon_r,r)}$$

and also,

- If $P^{-1}_{(\varepsilon_1,\ldots,\varepsilon_r,r)} = P^\dagger$ and $Q^{-1}_{(\varepsilon_1,\ldots,\varepsilon_r,r)} = Q^\dagger$, then $A \in A\{1,2,3,4\}$;
- If $P^{-1}_{(\varepsilon_1,\ldots,\varepsilon_r,r)} = P^\dagger$ and $Q^{-1}_{(\varepsilon_1,\ldots,\varepsilon_r,r)} \neq Q^\dagger$, then $A \in A\{1,2,3\}$;
- If $P^{-1}_{(\varepsilon_1,\ldots,\varepsilon_r,r)} \neq P^\dagger$ and $Q^{-1}_{(\varepsilon_1,\ldots,\varepsilon_r,r)} = Q^\dagger$, then $A \in A\{1,2,4\}$;
- In other cases, we have $A \in A\{1,2\}$.

***Example* 5.5** Let

$$A = \begin{pmatrix} -2 & 4 & 4 \\ -3 & 1 & 6 \\ 2 & 0 & -4 \\ -1 & -1 & 2 \end{pmatrix}.$$

Now, we have $A = PQ$, where

$$P = \begin{pmatrix} 1 & 3 \\ -1 & 2 \\ 1 & -1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 1 & 2 \end{pmatrix}.$$

The right inverse of Q is

$$Q^{-1}_{(R,2)} = \frac{1}{\det\limits_{(R,2)}(Q)} \begin{pmatrix} \det\limits_{(R,1)}\begin{pmatrix}1 & 2\end{pmatrix} & -\det\limits_{(R,1)}\begin{pmatrix}1 & -2\end{pmatrix} \\ -\det\limits_{(R,1)}\begin{pmatrix}-1 & 2\end{pmatrix} & \det\limits_{(R,1)}\begin{pmatrix}1 & -2\end{pmatrix} \\ \det\limits_{(R,1)}\begin{pmatrix}-1 & 1\end{pmatrix} & -\det\limits_{(R,1)}\begin{pmatrix}1 & 1\end{pmatrix} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -1 & -3 \\ 3 & 3 \\ -2 & 0 \end{pmatrix}.$$

The left inverse of P is

$$P^{-1}_{(R,2)} = \frac{1}{\underset{(R,2)}{\det}(P)} \begin{pmatrix} \underset{(R,1)}{\det}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} & -\underset{(R,1)}{\det}\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} & \underset{(R,1)}{\det}\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} & -\underset{(R,1)}{\det}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \\ -\underset{(R,1)}{\det}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} & \underset{(R,1)}{\det}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} & -\underset{(R,1)}{\det}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} & \underset{(R,1)}{\det}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \end{pmatrix}$$
$$= \frac{1}{8}\begin{pmatrix} 3 & -4 & 1 & 0 \\ 3 & -1 & -1 & 3 \end{pmatrix}$$

and the right generalized inverse of A is equal to

$$A^{-1}_{(R,2)} = Q^{-1}_{(R,2)} P^{-1}_{(R,2)} = \frac{1}{48}\begin{pmatrix} -11 & 6 & 3 & -9 \\ 15 & -12 & -3 & 9 \\ -4 & 6 & 0 & 0 \end{pmatrix}.$$

The next result represents a sufficient condition for the equivalence of the determinantal inverse and the Moore-Penrose inverse.

Corollary 5.6 If $r = \rho_{\tilde{\varepsilon}}(A)$ and the matrix A satisfies the condition

$$\overline{A}\begin{pmatrix} i_1 & \cdots & i_r \\ j_1 & \cdots & j_r \end{pmatrix} = k\ \overline{\varepsilon_{i_1,\ldots,i_r;j_1,\ldots,j_r}}, \qquad k \in \mathbb{C} \tag{C1}$$

for all $(i_1, \cdots, i_r) \in Q_{r,m}$ and $(j_1, \cdots, j_r) \in Q_{r,n}$, then $A^{-1}_{(\tilde{\varepsilon},r)} = A^{\dagger}$.

Proof. For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, it can be easily seen that $N_r(A) = k \underset{(\tilde{\varepsilon},r)}{\det}(A)$ and $A^{(\dagger,r)}_{ji} = kA^{(\varepsilon,r)}_{ji}$. Thus, the result follows considering Theorem 1.1. ■

Now, by Corollary 5.6 and Corollary 5.4, the following algorithm is presented for computing the determinantal inverse $A^{-1}_{(\tilde{\varepsilon},\rho_{\tilde{\varepsilon}}(A))}$.

Algorithm 1.

Case 1. If $p = \rho_{\tilde{\varepsilon}}(A) = \min\{m, n\}$, then apply rules 1.1 and 1.2.

Rule 1.1 If A satisfies the codition (C1), then $A^{-1}_{(\tilde{\varepsilon},p)} = A^{\dagger}$.

Rule 1.2 If the condition (C1) does not holds for A, then

(a) For $m \leqslant n$, if $\left(A^{-1}_{(\tilde{\varepsilon},p)}A\right)^* = A^{-1}_{(\tilde{\varepsilon},p)}A$, then $A^{-1}_{(\tilde{\varepsilon},p)} = A^{\dagger}$, else $A^{-1}_{(\tilde{\varepsilon},p)}$ is a right inverse of A;

(b) For $n \leqslant m$, if $\left(AA^{-1}_{(\tilde{\varepsilon},p)}\right)^* = AA^{-1}_{(\tilde{\varepsilon},p)}$, then $A^{-1}_{(\tilde{\varepsilon},p)} = A^{\dagger}$, else $A^{-1}_{(\tilde{\varepsilon},p)}$ is a left inverse of A.

Case 2. If $\rho_{\tilde{\varepsilon}}(A) = ran(A) = r < \min\{m, n\}$, then

Rule 2.1 If A satisfies the codition (C1), then $A^{-1}_{(\tilde{\varepsilon},r)}$ is the Moore-Penrose inverse of A.

Rule 2.2 If the condition (C1) does not holds, compute a full-rank factorization $A = PQ$ and select one of the following two rules.

Rule 2.3 If both P and Q satisfy condition (C1), then $A^{-1}_{(\tilde{\varepsilon},r)} = A^{\dagger}$.

Rule 2.4 If both P or Q satisfy condition (C1), then
(a) $A^{-1}_{(\tilde{\varepsilon},r)}$ satisfies conditions (1), (2) and (3), if $m \leqslant n$;
(b) $A^{-1}_{(\tilde{\varepsilon},r)}$ satisfies conditions (1), (2) and (4), if $m \geqslant n$.
Rule 2.5 If neither P nor Q satisfies condition (C1), use Corollary 5.3.
Case 3. If $\rho_{\tilde{\varepsilon}}(A) < ran(A)$, then $A^{-1}_{(\tilde{\varepsilon},r)} \notin A\{1,2\}$.

***Example* 5.7** The matrix

$$A = \begin{pmatrix} -1 & 1 & 2 \\ -1 & -4 & -3 \end{pmatrix}$$

satisfies the condition (C1) so that

$$A^{-1}_{(R,2)} = A^{\dagger} = \begin{pmatrix} -\frac{1}{5} & \frac{1}{3} \\ -\frac{2}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}.$$

***Example* 5.8** The rank-deficient matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

satisfies condition (C1). According to rule 2.1, $A^{-1}_{(R,2)}$ is the Moore-Penrose inverse of A and

$$A^{(\dagger,2)} = \begin{pmatrix} \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

***Example* 5.9** Consider

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & 3 \\ 2 & 3 & 1 \\ 0 & 2 & 2 \end{pmatrix}.$$

We have rank$(A) = 2$, and $\underset{(R,2)}{\det}(A) = 9$. A full-rank factorization of A is

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 3 \\ 2 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The matrix Q satisfies condition (C1), so that $Q^{-1}_{(R,2)} = Q^{\dagger}$. Also, $P^{-1}_{(R,2)} \neq P^{\dagger}$ so that

$$A^{-1}_{(R,2)} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{6} & \frac{1}{3} & -\frac{2}{3} \\ \frac{7}{6} & -\frac{1}{2} & -\frac{1}{3} & \frac{5}{6} \\ -\frac{2}{3} & \frac{1}{3} & 0 & -\frac{1}{6} \end{pmatrix}$$

satisfies conditions (1), (2) and (4).

***Example* 5.10** Full-rank factorization of

$$A = \begin{pmatrix} 1 & 4 & 6 \\ 3 & 14 & 22 \\ 2 & 10 & 16 \\ 0 & 2 & 4 \end{pmatrix}$$

is

$$P = \begin{pmatrix} 1 & 3 \\ 3 & 11 \\ 2 & 8 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Using $P^{-1}_{(R,2)} \neq P^{\dagger}$ and $Q^{-1}_{(R,2)} \neq Q^{\dagger}$, it is easy to see that

$$A^{-1}_{(R,2)} = \begin{pmatrix} -2 & -1 & 2 & 0 \\ \frac{9}{2} & \frac{5}{2} & -5 & 0 \\ -\frac{5}{2} & -\frac{3}{2} & 3 & 0 \end{pmatrix} \in A\{1,2\}.$$

***Example* 5.11** Consider a matrix of the form $A = \begin{pmatrix} 1 & -2 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 2 & 3 & -3 & -1 \end{pmatrix}$. If we use the Radić definition, then it is easy to verify that $\rho_{\varepsilon}(A) = 2 < \text{rank}(A)$ and $\det_{(R,2)}(A) = -28$. Moreover, $X = A^{-1}_{(R,2)} = \begin{pmatrix} -6 & 6 & 0 \\ 5 & -2 & -3 \\ 3 & -8 & 5 \\ -6 & 9 & -3 \end{pmatrix}$ and $AXA = \begin{pmatrix} -14 & 77 & -35 & -70 \\ 14 & 21 & -35 & -14 \\ -42 & -63 & 105 & 42 \end{pmatrix} \neq A$ and $XAX = X$.

References

[1] P. S. Abhimanyu, Defining the determinant-like function for m by n matrices using the exterior algebra, Adv. Appl. Clifford Algebras. 23 (2013), 787-792.

[2] E. Arghiriade, A. Dragomir, Une nouvelle definition de l'inverse generalisée d'une matrice, Rendiconti Lincei. Scienze Fisiche e Naturali. Serie XXXV. 35 (1963), 158-165.

[3] M. Bayat, A bijective proof of generalized Cauchy-Binet, Laplace, Sylvester and Dodgson formulas, Linear. Multilinear. Algebra., In press.

[4] A. Ben-Israel, Generalized inverses of matrices: a perspective of the work of Penrose, Math. Proc. Camb. Phil. Soc. 100 (1986), 407-425.

[5] C. E. Cullis, Matrices and Determinoids, Cambridge University Press, Vol 3, 1925.

[6] R. Gabril, Extinderea Complementilor Algebrici Generalizati la Matrici Oarecare, Studii si Cercetari Matematice. 17-Nr. 10 (1965), 1566-1581.

[7] V. N. Joshi, A Determinant for Rectangular Matrices, Bull. Austral. Math. Soc. 21 (1980), 137-146.

[8] H. Minc, Permanents, Encyclopedia of Mathematics and its Applications, 6, Addison-Wesley, Reading, Mass, 1978.
[9] Y. Nakagami, H. Yanai, On Cullis' determinant for rectangular matrices, Linear. Algebra. Appl. 422 (2007), 422-441.
[10] R. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos Soc. 51 (1955), 406-413.
[11] M. Radić, A definition of determinant of rectangular matrix, Glasnik Matematički. Ser III. 1 (21) (1966), 17-22.
[12] M. Radić, Areas of certain polygons in connection with determinants of rectangular matrices, Beiträge Algebra Geom. 49 (1) (2008), 71-96.
[13] M. Radić, Inverzije pravokutnih matrica, Doktorska disertacije, 1964.
[14] M. Radić, Some contribution to the inversion of rectangular matricrs, Glasnik Matematički. Ser III. 1 (21) (1966), 23-37.
[15] M. Radić, R. Sušanj, Geometrical meaning of one generalization of the determinant of a square matrix, Glasnik Matematički. Ser III. 29 (49) (1994), 217-233.
[16] P. Stanimirović, M. Stanković, Determinants of rectangular matrices and the Moore-Penrose inverse, Novi Sad J. Math, 27 (1) (1997), 53-69.
[17] M. Stojaković, Determinante Nekvadratnih Matrica, Vesnik DMNRS. 1-2 (1952), 9-12.
[18] L. Tan, Signs in the Laplace expansions and the parity of the distinguished representatives, Discrete Math. 131 (1994), 287-299.