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Solution of some irregular functional equations and their stability

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Abstract. In this note, we study the following functional equations:

$$L(L(p,r)+L(q,r)+p+q,r)+L(L(p,r)+p,r)+L(q,r)=0,$$
$$L(L(p,r)+p+q+e,r)+L(p,r)=L(p+q,r)+pL(q,r),$$

and $L(p,q)=L(\zeta p,q), |\zeta|<1$, without any regularity assumption for all $p,q,r \in A$, where $L: A^2 \to A$ is defined by $L(p,q):=g(p+q)-g(p)-g(q)$ for all $p,q \in A$. Also, we find general solutions of the above functional equations on algebras, unital algebras and real numbers, respectively. Finally, we investigate the stability of those functional equations in algebras and unital algebras, respectively.

Keywords: Additive functional equation, unital algebra, Hyers-Ulam stability.

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1. Introduction

The stability theory of functional equations started from a problem of Ulam [16], concerning the stability of group homomorphisms. Hyers [7] answered Ulam's question on Banach spaces in 1941 about additive mappings, which was an important step towards more solutions in this field. Rassias [14] presented a generalization of Hyers' Theorem which lets the Cauchy difference to be unbounded.

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The theory of stability is an main branch of the theory of differential equations. During the last thirty years interesting results have been investigated on different types of functional equations (see [8, 9, 12]).

The direct method presented by Hyers [7] is the most important and useful tool to study the stability of several functional equations. The other important method is fixed point alternative theorem, that is, the exact solution of the functional equation is explicitly constructed as a fixed point of some certain map (see [11]).

In the 21st century, number of works have been published on different extensions and applications of the stability to a number of functional inequalities and mappings, for instance, Cauchy type functional equation, differential equations and generalized orthogonality mappings (see [1, 10]).

In recent years, the stability of different (others functional, differential and integral) equations and other subjects has been intensively studied (see [2–6, 15]).

Assume that X and Y are vector spaces. A mapping $D : X \to Y$ is named additive if it satisfies $D(p+q) = D(p) + D(q)$ for all $p, q \in X$.

Theorem 1.1 [7] Let X be a real vector space and Y be a Banach space. If a mapping $\sigma : X \to Y$ satisfies

$$\|\sigma(p+q) - \sigma(p) - \sigma(q)\| \leqslant \epsilon$$

for some $\epsilon > 0$ and for all $p, q \in X$, then there exists a unique additive mapping $D : X \to Y$ such that $\|\sigma(p) - D(p)\| \leqslant \epsilon$ for all $p \in X$.

Theorem 1.2 [5] Suppose that H is an Abelian group and X is a Banach space. In addition, assume that $\psi : H \times H \to [0, \infty)$ is a function such that

$$\Psi(p,q) = \sum_{k=0}^{\infty} 2^{-k} \psi(2^k p, 2^k q) < \infty$$

for all $p, q \in H$. If a mapping $\sigma : H \to X$ satisfies

$$\|\sigma(p+q) - \sigma(p) - \sigma(q)\| \leqslant \psi(p,q)$$

for any $p, q \in H$, then there exists a unique additive mapping $D : H \to X$ with

$$\|\sigma(p) - D(p)\| \leqslant \frac{1}{2}\Psi(p,q)$$

for all $p \in H$. Furthermore, if $\sigma(tp)$ is continuous in t for any fixed p in H, then D is a linear mapping.

The essential goal of this note is to present the general solutions of the following functional equations

$$L(L(p,r) + L(q,r) + p + q, r) + L(L(p,r) + p, r) + L(q,r) = 0, \tag{1}$$

$$L(L(p,r) + p + q + e, r) + L(p,r) = L(p+q, r) + pL(q,r) \tag{2}$$

and

$$L(p,q) = L(\zeta p, q), \qquad |\zeta| < 1 \tag{3}$$

without any regularity assumption for all $p, q, r \in A$, where $L : A^2 \to A$ is defined by

$$L(p, q) := g(p+q) - g(p) - g(q)$$

for all $p, q \in A$. Finally, we investigate the stability of (1), (2) and (3) in algebras and unital algebras, respectively.

2. General solution of (1), (2) and (3)

In this section, we study the general solutions of (1), (2) and (3) without any regularity condition.

Theorem 2.1 Let A be an algebra. The mapping $g : A \to A$ satisfies the functional equation (1) for all $p, q, r \in A$ if and only if g is additive.

Proof. It is clear that additivity implies (1). Conversely, we assume that g satisfying (1), and we shall show that g is additive. Suppose that g is any function satisfying (1). Letting $r = 0$ in (1), we obtain $g(0) = 0$, because $L(p, 0) = -g(0)$ for all $p \in A$.

Putting $q = 0$ in (1), we get

$$L(L(p, r) + p, r) = 0 \tag{4}$$

for all $p, r \in A$. Setting $p = 0$ in (1), we have

$$L(L(q, r) + q, r) + L(q, r) = 0 \tag{5}$$

for all $q, r \in A$. Replacing q by p in (5), we obtain

$$L(L(p, r) + p, r) + L(p, r) = 0 \tag{6}$$

for all $p, r \in A$. It follows from (4) and (6) that $L(p, r) = 0$ for all $p, q, r \in A$. Hence $g(p + r) = g(p) + g(r)$ for all $p, r \in A$. ■

Lemma 2.2 [13] Let A be complex Banach algebra and $f : A \to A$ be an additive mapping such that $f(\lambda p) = \lambda f(p)$ for all $\lambda \in \mathbb{T}^1$ and all $p \in A$. Then f is linear over $\mathbb{C}$.

Theorem 2.3 Let A be a unital algebra and $g : A \to A$ be a mapping. Then g is linear if and only if

$$L_\mu(L_\mu(p, r) + L_\mu(q, r) + p + q, r) + L_\mu(L_\mu(p, r) + p, r) + L_\lambda(q, r) = 0 \tag{7}$$

for all $p, q, r \in A$ and all $\lambda, \mu \in \mathbb{T}^1$, where $L_\lambda(p, q) := g(\lambda(p + q)) - \lambda g(p) - \lambda g(q)$.

Proof. It is clear that linearity implies (7). Conversely, assume that g satisfies (7), we shall show that g is linear. Setting $\lambda = \mu = 1$ in (7) and by Theorem 2.1, we conclude that g is additive. Letting $\mu = 1$ in (7), we arrive that

$$L(L(p, r) + L(q, r) + p + q, r) + L(L(p, r) + p, r) + L_\lambda(q, r) = 0 \tag{8}$$

for all $p, q, r \in A$ and all $\lambda \in \mathbb{T}^1$. Suppose that g is any mapping satisfying (8). Then setting $r = 0$ in (8), we obtain $L_\lambda(q, 0) = 0$ for all $q \in A$ and all $\lambda \in \mathbb{T}^1$, because g is additive. Therefore, $g(\lambda q) = \lambda g(q)$ for all $q \in A$ and all $\lambda \in \mathbb{T}^1$. Hence by Lemma 2.2 g is linear. ■

Theorem 2.4 Let A be a unital algebra. The mapping $g : A \to A$ satisfies the functional equation (2) for all $p, q, r \in A$ if and only if g is additive.

Proof. It is clear that additivity implies (2). Conversely, we assume that g satisfying (2), and we shall show that g is additive. Suppose that g is any function satisfies (2). Setting $r = 0$ (2), we obtain $g(0) = 0$, since $L(p, 0) = -g(0)$ for all $p \in A$. Set $q = 0$ in (2), we get

$$L(L(p, r) + p + e, r) = 0 \tag{9}$$

for all $p, r \in A$. Next, let $p = 0$ in (9), we obtain $L(e, r) = 0$ for all $r \in A$. On the other hand, put $p = -q = e$ in (2), we get

$$L(L(e, r) + e, r) + L(e, r) = L(-e, r)$$

and thus, $L(-e, r) = 0$ for all $r \in A$, because $L(e, r) = 0$. Therefore, by letting $p = -e$ in (2), we have

$$L(q - e, r) = 2L(q, r) \tag{10}$$

for all $r \in A$. On the other hand, taking $p = 0$ and replacing q by $q - e$ in (2), we obtain

$$L(q - e, r) = L(q, r) \tag{11}$$

for all $r \in A$. Together, (10) and (11) yield $L(q, r) = 0$ for all $q, r \in A$. Thus g is additive. ■

Theorem 2.5 Let A be a unital algebra and $g : A \to A$ be a mapping. Then g is linear if and only if

$$L(L(p, r) + p + q + e, r) + L_\lambda(p, r) = L(p + q, r) + pL(q, r) \tag{12}$$

for all $p, q, r \in A$ and all $\lambda \in \mathbb{T}^1$, where $L_\lambda(p, q) = g(\lambda(p + q)) - \lambda g(p) - \lambda g(q)$.

Proof. It is clear that linearity implies (12). We assume that g satisfying (12), and we shall show that g is linear. It is clear that for $\lambda = 1$ and using Theorem 2.4 we conclude that g is additive. Suppose that g is any function satisfying (12). Then setting $p = 0$ in (12), we obtain

$$L_\lambda(q, r) = 0 \tag{13}$$

for all $q, r \in A$ and all $\lambda \in \mathbb{T}^1$. Now putting $r = 0$ in (13), we get $g(\lambda q) = \lambda g(q)$ for all $q \in A$ and all $\lambda \in \mathbb{T}^1$, because g is additive. Hence by Lemma 2.2 g is linear. ■

Theorem 2.6 Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous mapping with $g(0) = 0$. Then g is linear if and only if g satisfying (3).

Proof. It is evident that linearity implies (3). Conversely, assume that there exists

$\zeta(|\zeta| < 1)$ such that $L(p,q) = L(\zeta p, q)$ for all $p, q \in \mathbb{R}$. Thus,

$$g(p+q) + g(\zeta p) = g(p) + g(\zeta p + q) \tag{14}$$

for all $p, q \in \mathbb{R}$. Claim: Let n be a natural number. Then

$$g(p+q) + g(\zeta^n p) = g(p) + g(\zeta^n p + q) \tag{15}$$

for all $p, q \in \mathbb{R}$.

Proof of Claim: The proof can be done by applying method induction. By (14) the base of induction for $n = 1$ holds. Now, we suppose that the inequality (15) holds to $n < k + 1$. So

$$g(p+q) + g(\zeta^k p) = g(p) + g(\zeta^k p + q) \tag{16}$$

for all $p, q \in \mathbb{R}$. By replaycing p by $\zeta^k p$ in (14), we get

$$g(\zeta^k p + q) + g(\zeta^{k+1} p) = g(\zeta^k p) + g(\zeta^{k+1} p + q) \tag{17}$$

Upon adding (16) and (17), we obtain

$$g(p+q) + g(\zeta^{k+1} p) = g(p) + g(\zeta^{k+1} p + q)$$

for all $p, q \in \mathbb{R}$. Thus (15) holds for all n and all $p, q \in \mathbb{R}$. Therefore,

$$g(p+q) + \lim_{n\to\infty} g(\zeta^n p) = g(p) + \lim_{n\to\infty} g(\zeta^n p + q)$$

for all $p, q \in \mathbb{R}$. Since $g(0) = 0$ and g is continuous, we have $g(p+q) = g(p) + g(q)$ for all $p, q \in \mathbb{R}$. Hence by Theorem 1.2 g is a linear mapping. ■

3. Stability of (1), (2) and (3)

In this section, we show the Hyers-Ulam stability of equations (1) and (2).

For a given mapping $g : A \to A$, we define

$$\Delta g(p,q,r) := L(L(p,r) + L(q,r) + p + q, r) + L(L(p,r), r) + L(q,r)$$

and for $A = \mathbb{R}$ and if A is a unital algebra define

$$\Gamma g(p,q,r) := L(L(p,r) + p + q + e, r) + L(p,r) - L(p+q,r) - pL(q,r)$$

for all $p, q, r \in A$.

Theorem 3.1 Let A be an algebra and $g : A \to A$ be a mapping with $g(0) = 0$ and satisfies

$$\begin{cases} L(L(p,r) + L(q,r) + p + q, r)) = 0 \\ |\Delta g(p,q,r)| \leqslant \delta. \end{cases}$$

Then there exists a unique additive mapping $D : A \to A$ such that $|g(p) - D(p)| \leqslant \delta$ for all $p \in A$.

Proof. Since $L(L(p,r)+L(q,r)+p+q,r)) = 0$, $g(0) = 0$ and $|\Delta g(p,q,r)| \leqslant \delta$, set $p = 0$ in $|\Delta g(p,q,r)| \leqslant \delta$, we obtain $L(L(0,r),r) = L(-g(0),r) = L(0,r) = -g(0) = 0$ and thus $|L(q,r)| \leqslant \delta$ for all $q,r \in A$. Thus by Hyer's Theorem 1.1, there exists a unique additive map $D : A \to A$ such that $|g(p) - D(p)| \leqslant \delta$ for all $p \in A$. ■

Theorem 3.2 Let $g : \mathbb{R} \to \mathbb{R}$ be a mapping satisfying

$$\begin{cases} L(L(p,r) + p + q + e, r) = 0 \\ |\Gamma g(p,q,r)| \leqslant \delta. \end{cases}$$

Then there exist a unique additive mapping $D : \mathbb{R} \to \mathbb{R}$ such that $|g(p)-D(p)| \leqslant \delta+|g(0)|$ for all $p \in \mathbb{R}$.

Proof. Since $L(L(p,r)+p+q+e,r) = 0$ and $|\Gamma g(p,q,r)| \leqslant \delta$, set $p = 0$ in $|\Gamma g(p,q,r)| \leqslant \delta$, we obtain $|L(q,r) + g(0)| \leqslant \delta$ for all $q,r \in \mathbb{R}$. Therefore, $|L(q,r)| \leqslant \delta + |g(0)|$. Thus, by Hyer's Theorem 1.1, there exists a unique additive map $D : \mathbb{R} \to \mathbb{R}$ such that $|g(p) - D(p)| \leqslant \delta + |g(0)|$ for all $p \in \mathbb{R}$. ■

Theorem 3.3 Let $\zeta \in \mathbb{R}$ with $|\zeta| < 1$, $g : \mathbb{R} \to \mathbb{R}$ be a continuous mapping with $g(0) = 0$ and $\psi : \mathbb{R}^2 \to [0,\infty)$ be a function such that $\Psi(p,q) = \sum\limits_{j,k=0}^{\infty} 2^{-k}\psi(\zeta^j 2^k p, 2^k q) < \infty$ and

$$|g(p+q) + g(\zeta p) - g(p) - g(\zeta p + q)| \leqslant \psi(p,q) \tag{18}$$

for all $p,q \in \mathbb{R}$. Then there exists a linear mapping $D : \mathbb{R} \to \mathbb{R}$ such that

$$|g(p) - D(p)| \leqslant \frac{1}{2}\Psi(p,p) \tag{19}$$

for all $p \in \mathbb{R}$.

Proof. Let n be a natural number. We first claim

$$|g(p+q) + g(\zeta^n p) - g(p) - g(\zeta^n p + q)| \leqslant \sum_{j=0}^{n-1} \psi(\zeta^j p, q) \tag{20}$$

for all $p,q \in \mathbb{R}$. We verify it by induction on n. By (18) the base of induction for $n = 1$ holds. Next, we suppose that the inequality (20) holds for $n < k+1$. So

$$|g(p+q) + g(\zeta^k p) - g(p) - g(\zeta^k p + q)| \leqslant \sum_{j=0}^{k-1} \psi(\zeta^j p, q) \tag{21}$$

for all $p,q \in \mathbb{R}$. Replaycing p by $\zeta^k p$ in (18), we get

$$|g(\zeta^k p + q) + g(\zeta^{k+1} p) - g(\zeta^k p) - g(\zeta^{k+1} p + q)| \leqslant \psi(\zeta^k p, q). \tag{22}$$

Upon adding (21) and (22), we obtain

$$|g(p+q)+g(\zeta^{k+1}p)-g(p)-g(\zeta^{k+1}p+q)| \leqslant \sum_{j=0}^{k}\psi(\zeta^j p,q) \tag{23}$$

for all $p,q\in\mathbb{R}$. Thus (20) holds for all n and all $p,q\in\mathbb{R}$. Therefore,

$$\lim_{n\to\infty}|g(p+q)+g(\zeta^{n}p)-g(p)-g(\zeta^{n}p+q)| \leqslant \lim_{n\to\infty}\sum_{j=0}^{n-1}\psi(\zeta^j p,q)$$

for all $p,q\in\mathbb{R}$. Since $g(0)=0$ and g is continuous, we have

$$|g(p+q)-g(p)-g(q)| \leqslant \sum_{j=0}^{\infty}\psi(\zeta^j p,q)$$

for all $p,q\in\mathbb{R}$. Thus, By Theorem 1.2 there exists a linear mapping $D:\mathbb{R}\to\mathbb{R}$ such that satisfying (19). ■

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