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Convergence, stability and data dependence results for contraction and nonexpansive mappings by a new four step algorithm

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Abstract. Here we show that the UI-iteration scheme (Udofia and Igbokwe, [24]) can be used to approximate the fixed points of contraction and nonexpansive mappings. we prove a strong and weak convergence of the iteration scheme to the fixed point of contraction and nonexpansive mappings. We also prove that the scheme is Γ-stable and data dependent. Analytically and with numerical example we show that the UI-iteration scheme has a faster rate of convergence for contraction and nonexpansive mappings than some well known existing iteration schemes in literature. Finally, we apply the UI-iteration scheme to find the solution of constrained convex minimization problem.

Keywords: Banach space, contraction mapping, nonexpansive mapping, data dependence, stability.

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1. Introduction and preliminaries

Solutions of nonlinear equations can be rigorous and sometimes very difficult to obtain. One of the easiest way to solve such problems is to transform into a fixed point problem:

$$\Gamma\psi = \psi, \tag{1}$$

where Γ is a self map in a well defined space, so that the solution of the fixed point equation (1) of the mapping Γ is considered as the solution to the nonlinear problem. To

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this end, a number of iterative processes have been introduced to study the fixed point of various operators such as:
The Picard iterative process defined by

$$\psi_{k+1} = \Gamma\psi_n \quad \text{for all} \quad k \in \mathbb{N}. \tag{2}$$

In 1992, using the Picard iteration scheme, Banach [3] in his contraction mapping principle showed that if a self map Γ on a complete metric space is such that

$$d(\Gamma\psi, \Gamma\nu) \leqslant \delta d(\psi, \nu) \tag{3}$$

for $\delta \in [0, 1)$ and Γ a contraction, then it has a unique fixed point q.
The Picrad iteration method is known to converge to the fixed points of contraction mappings. However, if Γ is nonexpansive and even if it has a unique fixed point, Picard iteration may fail to converge to its fixed point.
To solve that impending problem, in 1953, Mann [14] introduced the Mann iterative scheme to approximate fixed points of nonexpansive mappings defined by

$$\psi_{k+1} = (1 - a_k)\psi_k + a_k\Gamma\psi_k \quad \text{for all } k \in \mathbb{N}, \tag{4}$$

where $\{a_k\} \subset (0, 1)$.
It is also known that the Mann iteration scheme does not converge to the fixed point of pseudo-contractive mappings.
Again, in 1974, Ishikawa [13] introduced a two-step Mann iterative scheme to approximate fixed point of pseudo-contractive mappings defined by

$$\begin{cases} \psi_1 \in \varphi, \\ \psi_{k+1} = (1 - a_k)\psi_k + a_k\Gamma\nu_k & \text{for all } k \geqslant 1, \\ \nu_k = (1 - b_k)\psi_k + b_k\Gamma\psi_k, \end{cases} \tag{5}$$

where $\{a_k\}, \{b_k\} \subset (0, 1)$ for all $k \in \mathbb{N}$.
Many decades after the introduction of Ishikawa iteration, several authors and researchers in the field of nonlinear analysis have introduced quite a number of iteration schemes with the motivated aim of independently finding the fixed point of some contraction, nonexpansive, pseudocontractive and many other mappings in various spaces such as Noor [16], Agarwal et al. [2] (S-iteration), Abbas-Nazir [1], Thakur et al. [23], Piri et al. [19], M-iteration [25] and many others.
In 2000, Noor [16] introduced the Noor iteration defined by

$$\begin{cases} \psi_1 \in \varphi, \\ \psi_{k+1} = (1 - a_k)\psi_k + a_k\Gamma\nu_k, \\ \nu_k = (1 - b_k)\psi_k + b_k\Gamma\mu_k, \\ \mu_k = (1 - c_k)\psi_k + c_k\Gamma\psi_k \end{cases} \quad \text{for all } k \geqslant 1, \tag{6}$$

where $\{a_k\}, \{b_k\}, \{c_k\} \subset (0, 1)$ for all $k \in \mathbb{N}$.
In 2007, Agarwal et al. [2] introduced the S-iteration defined by

$$\begin{cases} \psi_1 \in \varphi, \\ \psi_{k+1} = (1 - a_k)\Gamma\psi_k + a_k\Gamma\nu_k, & \text{for all } k \geqslant 1, \\ \nu_k = (1 - b_k)\psi_k + b_k\Gamma\psi_k \end{cases} \tag{7}$$

where $\{a_k\}, \{b_k\} \subset (0,1)$ for all $k \in \mathbb{N}$.
They claimed that the iteration (7) converges faster than the Mann iteration for some contractions.
In 2014, Abbas and Nazir [1] introduced the iteration defined by

$$\begin{cases} \psi_1 \in \varphi, \\ \psi_{k+1} = (1-a_k)\Gamma\mu_k + a_k\Gamma\nu_k, \\ \nu_k = (1-b_k)\Gamma\psi_k + b_k\Gamma\mu_k \\ \mu_k = (1-c_k)\psi_k + c_k\Gamma\psi_k \end{cases} \quad \text{for all } k \geqslant 1, \tag{8}$$

where $\{a_k\}, \{b_k\}, \{c_k\} \subset (0,1)$ for all $k \in \mathbb{N}$.
In 2016, Thakur et al. [23] introduced the iteration defined by

$$\begin{cases} \psi_1 \in \varphi, \\ \psi_{k+1} = \Gamma\nu_k, \\ \nu_k = \Gamma((1-a_k)\psi_k + a_k\mu_k), \\ \mu_k = (1-b_k)\psi_k + b_k\Gamma\psi_k \end{cases} \quad \text{for all } k \geqslant 1, \tag{9}$$

where $\{a_k\}, \{b_k\}, \{c_k\} \subset (0,1)$ for all $k \in \mathbb{N}$.
It was asserted that the iteration process (9) converges faster than (4), (5), (6), (7) and (8) for contractive mappings.
In 2018, Piri et al. [19] introduced the iteration defined by

$$\begin{cases} \psi_1 \in \varphi, \\ \psi_{k+1} = (1-a_k)\Gamma\mu_k + a_k\Gamma\nu_k, \\ \nu_k = \Gamma\mu_k, \\ \mu_k = \Gamma((1-b_k)\psi_k + b_k\Gamma\psi_k) \end{cases} \quad \text{for all } k \geqslant 1, \tag{10}$$

where $\{a_k\}, \{b_k\} \subset (0,1)$ for all $k \in \mathbb{N}$.
They proved that the iteration process (10) converges faster than iteration processes (8) and (9) for contractive mappings when $1-a_k < a_k$, $1-b_k < b_k$ and $1-c_k < c_k$ for all $k \in \mathbb{N}$ with numerical example to support the proof.
In 2018, Ullah and Arshad [26] introduced the M-iterations defined by

$$\begin{cases} \psi_1 \in \varphi, \\ \psi_{k+1} = \Gamma\nu_k, \\ \nu_k = \Gamma\mu_k, \\ \mu_k = (1-a_k)\psi_k + a_k\Gamma\psi_k \end{cases} \quad \text{for all } k \geqslant 1, \tag{11}$$

where $\{a_k\} \subset (0,1)$.
In 2020, Garodia and Uddin [8] introduced the iteration defined by

$$\begin{cases} \psi_1 \in \varphi, \\ \psi_{k+1} = \Gamma\nu_k \\ \nu_k = \Gamma((1-b_k)\Gamma\psi_k + b_k\Gamma\mu_k) \\ \mu_k = \Gamma((1-a_k)\psi_k + a_k\Gamma\psi_k) \end{cases} \quad \text{for all } k \geqslant 1, \tag{12}$$

where $\{a_k\}, \{b_k\} \subset (0,1)$ for all $k \in \mathbb{N}$.
They proved that the iteration process (12) converges faster than iteration processes (10)

and other well known existing iteration processes for contractive-like mappings with numerical examples.
The authors showed both analytically and numerically that iteration process (12) converges faster than all of Noor (6), S-iteration (7), Abbas and Nazir (8), Thakur (9), Piri et al. (10), M-iteration (11) processes for contractive-like mappings.
Motivated by the foregoing, the authors (Udofia and Igbokwe [24]) introduced a new four-step iteration algorithm called the UI-iteration scheme to approximate the fixed points of monotone generalized α-nonexpansive mappings and defined by:

$$\begin{cases} \xi_1 \in \varphi, \\ \xi_{k+1} = \mathit{\Gamma} y_k, \\ y_k = \mathit{\Gamma}((1-b_k)\mathit{\Gamma} w_k + b_k \mathit{\Gamma} z_k), \quad \text{for all } k \geqslant 1, \\ z_k = \mathit{\Gamma}((1-a_k)\mathit{\Gamma}\xi_k + a_k \mathit{\Gamma} w_k), \\ w_k = \mathit{\Gamma}((1-c_k)\xi_k + c_k \mathit{\Gamma}\xi_k) \end{cases} \tag{13}$$

where $\{a_k\}, \{b_k\}, \{c_k\} \subset (0,1)$ for all $k \in \mathbb{N}$.
The aim of this paper is to show that the UI-iteration scheme can be used to approximate the fixed points of contraction and nonexpansive mappings. We also prove that the scheme is $\mathit{\Gamma}$-stable and Data dependent. Again, we show that the UI-iteration scheme (13) has a faster rate of convergence for contraction and nonexpansive mappings than Garodia and Uddin iteration scheme (12) and some well known existing iteration schemes in literature. Furthermore, we prove a strong and weak convergence of the iteration scheme to the fixed point of contraction and nonexpansive mappings respectively. Finally, we apply the UI-iteration scheme to the solution of constrained Convex minimization problem.

Throughout this paper, let φ be a nonempty closed convex subset of a uniformly convex Banach space ζ. $F(\mathit{\Gamma}) = \{\psi \in \zeta : \mathit{\Gamma}\psi = \psi\}$ denotes the set of all fixed points of the mapping $\mathit{\Gamma}$.

A mapping $\mathit{\Gamma} : \varphi \to \varphi$ is called

(1) contraction if there exists a constant $\delta \in (0,1)$ such that

$$\|\mathit{\Gamma}\psi - \mathit{\Gamma}\nu\| \leqslant \delta\|\psi - \nu\| \text{ for all } \psi, \nu \in \varphi; \tag{14}$$

(2) nonexpansive if

$$\|\mathit{\Gamma}\psi - \mathit{\Gamma}\nu\| \leqslant \|\psi - \nu\| \text{ for all } \psi, \nu \in \varphi. \tag{15}$$

Remark 1 *Clearly, every contraction map is nonexpansive for $\delta = 1$.*

The following definitions and lemmas will be useful in proving our main results.

Definition 1.1 A Banach space ζ is said to be

(i) strictly convex if $\frac{1}{2}\|\psi + \nu\| < 1$ for all $\psi, \nu \in \zeta$ with $\|\psi\| = \|\nu\| = 1$ and $\psi \neq \nu$.
(ii) uniformly convex if for all $\epsilon \in (0,2]$, there exists $\delta > 0$ such that $\frac{1}{2}\|\psi+\nu\| \leqslant 1-\delta$ for all $\psi, \nu \in E$ with $\|\psi\| \leqslant 1$, $\|\nu\| \leqslant 1$ and $\|\psi - \nu\| \geqslant \epsilon$.

Definition 1.2 [5] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers that converge to a and b respectively, and assume that there exists

$$\ell = \lim_{n\to+\infty} \frac{\|a_n - a\|}{\|b_n - b\|}. \tag{16}$$

Then,

(R_1) If $\ell = 0$, we say that $\{a_n\}$ converges faster to a than $\{b_n\}$ does to b.
(R_2) If $0 < \ell < \infty$, we say that $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Definition 1.3 [5] Let Γ, $\tilde{\Gamma} : \varphi \to \varphi$ be two operators. We say that $\tilde{\Gamma}$ is an approximate operator for Γ if for some $\epsilon > 0$, we have

$$\|\Gamma\psi - \tilde{\Gamma}\nu\| \leqslant \epsilon \ \text{ for all } \ \nu \in \varphi. \tag{17}$$

Definition 1.4 [11] Let $\{t_k\}$ be any sequence in φ. Then, an iteration process $\psi_{k+1} = f(\Gamma, \psi_k)$, converging to fixed point q, is said to be Γ-stable with respect to Γ if for $\varepsilon_k = \|t_{k+1} - f(\Gamma, t_k)\|$ for all $k \in \mathbb{N}$, we have

$$\lim_{k\to+\infty} \varepsilon_k = 0 \ \Leftrightarrow \ \lim_{k\to+\infty} t_k = q. \tag{18}$$

Definition 1.5 [17] A Banach space is said to satisfy the Opial's condition if for each weakly convergent sequence $\{a_k\}$ in ζ, $\{a_k\}$ converges weakly to a point $u \in \zeta$ implies $\limsup ||a_k - u|| < \limsup ||a_k - v||$ for all $v \in \zeta$ with $u \neq v$.

Lemma 1.6 [21] Let $\{a_n\}$ and $\{b_n\}$ be nonnegative real sequences satisfying the following inequalities:

$$a_{k+1} \leqslant (1 - \sigma_k)a_k + \sigma_k b_k, \tag{19}$$

where $\sigma_k \in (0, 1)$ for all $k \in \mathbb{N}$, $\sum\limits_{k=0}^{+\infty} \sigma_k = +\infty$ and $b_k \geqslant 0$ for all $k \in \mathbb{N}$, then

$$0 \leqslant \limsup_{k\to+\infty} a_k \leqslant \limsup_{k\to+\infty} b_k. \tag{20}$$

Lemma 1.7 [27] Let $\{a_k\}$ and $\{b_k\}$ be nonnegative real sequences satisfying the following inequalities:

$$a_{k+1} \leqslant (1 - \sigma_k)a_k + b_k, \tag{21}$$

where $\sigma_k \in (0, 1)$ for all $k \in \mathbb{N}$, $\sum\limits_{k=0}^{+\infty} \sigma_k = +\infty$ and $\lim\limits_{k\to+\infty} \frac{b_k}{\sigma_k} = 0$, then $\lim\limits_{k\to+\infty} a_k = 0$.

Lemma 1.8 [18] Let $\{a_k\}$ and $\{b_k\}$ be nonnegative real sequences such that $\sum\limits_{k=1}^{+\infty} b_k < +\infty$. If

$$a_{k+1} \leqslant a_k + b_k, \tag{22}$$

then $\lim\limits_{k\to+\infty} a_k$ exists. Consequently, if $\liminf\limits_{k\to+\infty} a_k = 0$, then $\lim\limits_{k\to+\infty} a_k = 0$.

Lemma 1.9 (See Goebel and Reich [10], Theorem 5.1) Let φ be a bounded closed convex subset of a uniformly convex Banach space ζ. If $\Gamma : \varphi \to \varphi$ is nonexpansive, then Γ has a fixed point.

Lemma 1.10 (See Goebel and Kirk [9], Lemma 3.4) If φ is a closed and convex subset of a uniformly convex Banach space ζ and if $\Gamma : \varphi \to \varphi$ is nonexpansive, then the set $F(\Gamma))$ of fixed points of Γ is closed and convex.

Lemma 1.11 [28] For any real numbers $q > 1$ and $r > 0$, a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, +\infty) \longrightarrow [0, +\infty)$ with $g(0) = 0$ such that

$$\|t\psi + (1-t)\nu\|^q = t\|\psi\|^2 + (1-t)\|\nu\|^q - \omega(q,t)g(\|\psi - \nu\|) \tag{23}$$

for all $\psi, \nu \in B_r(0) = \{\psi \in \zeta : \|\psi\| \leqslant r\}$ and $t \in [0,1]$, where $\omega(q,t) = t^q(1-t) + t(1-t)^q$. In particular, taking $q = 2$ and $t = \frac{1}{2}$, we have

$$\|\frac{\psi + \nu}{2}\|^2 \leqslant \frac{1}{2}\|\psi\|^2 + \frac{1}{2}\|\nu\|^2 - \frac{1}{4}g(\|\psi - \nu\|). \tag{24}$$

2. Main results

In this section, we prove the strong convergence of UI iterative scheme (13) for contraction and nonexpansive mappings.

2.1 *Convergence of UI iterative scheme for contraction mapping*

We now proof the strong convergence of the UI iterative scheme defined by (13) for contraction mapping.

Theorem 2.1 Let φ be a nonempty closed convex subset of a uniformly convex Banach space ζ and $\Gamma : \varphi \to \varphi$ be a contraction mapping with $F(\Gamma) \neq \emptyset$. Let $\{\xi_k\}$ be the sequence defined by (13) with real sequences $\{a_k\}, \{b_k\}, \{c_k\} \subset (0,1)$ satisfying $\sum\limits_{k=0}^{+\infty} a_k b_k c_k = +\infty$. Then $\{\xi_k\}$ converges strongly to a unique fixed point of Γ.

Proof. Let $q \in F(\Gamma)$. From (13) and (14), we have

$$\begin{aligned} \|w_k - q\| &= \|\Gamma((1-c_k)\xi_k + c_k\Gamma\xi_k) - q\| \\ &\leqslant \delta((1-c_k)\|\xi_k - q\| + c_k\delta\|\xi_k - q\|) \\ &\leqslant \delta(1-(1-\delta)c_k)\|\xi_k - q\|. \end{aligned} \tag{25}$$

Using (13), (14) and (25), we have

$$\begin{aligned} \|z_k - q\| &= \|\Gamma((1-a_k)\Gamma\xi_k + a_k\Gamma w_k) - q\| \\ &\leqslant \delta((1-a_k)\delta\|\xi_k - q\| + a_k\delta\|w_k - q\|) \\ &\leqslant \delta^2((1-a_k)\|\xi_k - q\| + a_k\|w_k - q\|) \\ &\leqslant \delta^2(1-(1-\delta)a_kc_k)\|\xi_k - q\|. \end{aligned} \tag{26}$$

Using (13), (14), (25) and (26), we have

$$\begin{aligned}\|y_k - q\| &= \|\Gamma((1-b_k)\Gamma w_k + b_k \Gamma z_k) - q\| \\ &\leqslant \delta^2((1-b_k)\|w_k - q\| + b_k\|z_k - q\|) \\ &\leqslant \delta^3(\|\xi_k - q\| - (1-\delta)a_k b_k c_k\|\xi_k - q\|) \\ &\leqslant \delta^3(1-(1-\delta)a_k b_k c_k)\|\xi_k - q\|. \end{aligned} \tag{27}$$

Using (13), (14) and (27), we have

$$\begin{aligned}\|\xi_{k+1} - q\| &= \|\Gamma y_k - q\| \\ &\leqslant \delta\|y_k - q\| \\ &\leqslant \delta(\delta^3(1-(1-\delta)a_k b_k c_k)\|\xi_k - q\|) \\ &\leqslant \delta^4(1-(1-\delta)a_k b_k c_k)\|\xi_k - q\|. \end{aligned} \tag{28}$$

From (28), we deduce that

$$\begin{aligned}\|\xi_{k+1} - q\| &\leqslant \delta^4(1-(1-\delta)a_k b_k c_k)\|\xi_k - q\| \\ \|\xi_k - q\| &\leqslant \delta^4(1-(1-\delta)a_{k-1} b_{k-1} c_{k-1})\|\xi_{k-1} - q\| \\ \|\xi_{k-1} - q\| &\leqslant \delta^4(1-(1-\delta)a_{k-2} b_{k-2} c_{k-2})\|\xi_{k-2} - q\| \\ &\cdots \\ \|\xi_1 - q\| &\leqslant \delta^4(1-(1-\delta)a_0 b_0 c_0)\|\xi_0 - q\|. \end{aligned} \tag{29}$$

From (29), we have

$$\|\xi_{k+1} - q\| \leqslant \delta^{4(k+1)} \prod_{n=0}^{k}(1-(1-\delta)a_n b_n c_n)\|\xi_0 - q\|. \tag{30}$$

Since for all $k \in \mathbb{N}$, $\{a_k\}, \{b_k\}, \{c_k\} \subset (0,1)$ and $\delta \in (0,1)$, it follows that $(1-(1-\delta)a_k b_k c_k) < 1$. We know from classical analysis that $1-\xi < \exp^{-\xi}$ for all $\xi \in [0,1]$. Then, from (30), we have

$$\|\xi_{k+1} - q\| \leqslant \frac{\delta^{4(k+1)}\|\xi_0 - q\|}{\exp^{(1-\delta)} \sum_{n=0}^{k} a_k b_k c_k}. \tag{31}$$

Taking the limits of both sides of (31), we obtain that $\lim_{k\to+\infty}\|\xi_k - q\| = 0$. Hence, $\{\xi_k\}$ converges strongly to the fixed point of Γ as required. ■

2.2 *Convergence of UI-iterative scheme for nonexpansive mapping*

We now proof the strong convergence of the UI-iterative scheme defined by (13) for nonexpansive mapping.

Theorem 2.2 Let φ be a nonempty closed convex subset of uniformly convex Banach space ζ. Let $\Gamma : \varphi \to \varphi$ be a nonexpansive self mapping on φ, and $\{\xi_k\}$ a sequence defined by (13) with $F(\Gamma) \neq 0$. Then $\lim_{k\to+\infty} \|\xi_k - q\|$ exists for all $q \in F(\Gamma)$.

Proof. From (13) and (15), we obtain that

$$\begin{aligned}\|w_k - q\| &= \|\Gamma((1-c_k)\xi_k + c_k\Gamma\xi_k) - q\| \\ &\leqslant (1-c_k)\|\xi_k - q\| + c_k\|\xi_k - q\| \\ &\leqslant \|\xi_k - q\|. \end{aligned} \tag{32}$$

Also, from (13), (15) and (32), we have

$$\begin{aligned}\|z_k - q\| &= \|\Gamma((1-a_k)\Gamma\xi_k + a_k\Gamma w_k) - q\| \\ &\leqslant (1-a_k)\|\xi_k - q\| + a_k\|w_k - q\| \\ &\leqslant \|\xi_k - q\|. \end{aligned} \tag{33}$$

Again, from (13), (15), (32) and (33), we have

$$\begin{aligned}\|y_k - q\| &= \|\Gamma((1-b_k)\Gamma w_k + b_k\Gamma z_k) - q\| \\ &\leqslant (1-b_k)\|w_k - q\| + b_k\|z_k - q\| \\ &\leqslant \|\xi_k - q\|. \end{aligned} \tag{34}$$

Further, from (13), (15), (32), (33) and (34), we have

$$\|\xi_{k+1} - q\| = \|\Gamma y_k - q\| \leqslant \|y_k - q\| \leqslant \|\xi_k - q\|. \tag{35}$$

This implies that the sequence $\{\|\xi_k - q\|\}$ is nonincreasing and bounded for all $q \in F(\Gamma)$. Hence, $\lim_{k\to+\infty} \|\xi_k - q\|$ exists as required. ■

Next we show that the UI-iterative scheme defined by (13) is an approximate fixed point sequence of the operator Γ.

Theorem 2.3 Let φ be a nonempty closed convex subset of uniformly convex Banach space ζ and $\Gamma : \varphi \to \varphi$ be a nonexpansive mapping. Suppose that the sequence $\{\xi_k\}$ defined by UI iteration scheme (13) is bounded. If $F(\Gamma) \neq 0$ and for some $q \in F(\Gamma)$, then

(1) $\liminf_{k\to+\infty} \|\xi_k - \Gamma\xi_k\| = 0$, provided $\limsup_{k\to+\infty} b_k(1-b_k) > 0$.

(2) $\lim_{k\to+\infty} \|\xi_k - \Gamma\xi_k\| = 0$, provided $\liminf_{k\to+\infty} b_k(1-b_k) > 0$.

Proof. First, we observe that

$$\|\Gamma w_k - q\| \leqslant \|w_k - q\| \leqslant \|\xi_k - q\| \tag{36}$$

so that $\|\Gamma w_k - q\|^2 \leqslant \|\xi_k - q\|^2$. Also,

$$\begin{aligned}\|\Gamma z_k - q\| &\leqslant \|z_k - q\| \\ &= \|\Gamma((1-a_k)\Gamma\xi_k + a_k\Gamma w_k) - q\| \\ &\leqslant (1-a_k)\|\Gamma\xi_k - q\| + a_k\|\Gamma w_k - q\| \\ &\leqslant (1-a_k)\|\Gamma\xi_k - q\| + a_k\|\xi_k - q\| \\ &\leqslant \|\Gamma\xi_k - q\| \end{aligned} \tag{37}$$

so that $\|\Gamma z_k - q\|^2 \leqslant \|\Gamma\xi_k - q\|^2$. Now, since $\{\xi_k\}$ is bounded and Γ is nonexpansive, it follows from Theorem 2.2 and Lemma 1.11 that

$$\begin{aligned}\|\xi_{k+1} - q\|^2 &= \|\Gamma y_k - q\|^2 \\ &\leqslant \|y_k - q\|^2 \\ &\leqslant \|(1-b_k)(\Gamma w_k - q) + b_k(\Gamma z_k - q)\|^2 \\ &\leqslant (1-b_k)\|\xi_k - q\|^2 + b_k\|\Gamma\xi_k - q\|^2 - b_k(1-b_k)g(\|\xi_k - \Gamma\xi_k\|) \\ &\leqslant \|\xi_k - q\|^2 - b_k\|x_k - q\|^2 + b_k\|\xi_k - q\|^2 - b_k(1-b_k)g(\|\xi_k - \Gamma\xi_k\|) \\ &= \|\xi_k - q\|^2 - b_k(1-b_k)g(\|\xi_k - \Gamma\xi_k\|), \end{aligned} \tag{38}$$

which implies that

$$b_k(1-b_k)g(\|\xi_k - \Gamma\xi_k\|) \leqslant \|\xi_k - p\|^2 - \|\xi_{k+1} - p\|^2. \tag{39}$$

Letting $k \to \infty$, it follows from (35) and (39) that

$$\limsup_{k\to+\infty} b_k(1-b_k)g(\|\xi_k - \Gamma\xi_k\|) = 0. \tag{40}$$

(1) So, by condition (1), $\limsup_{k\to+\infty} b_k(1-b_k) > 0$ and since

$$(\limsup_{k\to+\infty} b_k(1-b_k))(\liminf_{k\to+\infty} g(\|\xi_k - \Gamma\xi_k\|)) \leqslant \limsup_{k\to+\infty} b_k(1-b_k)g(\|\xi_k - \Gamma\xi_k\|),$$

and by (40), we have $\liminf_{k\to+\infty} g(\|\xi_k - \Gamma\xi_k\|) = 0$ and from Lemma (1.11) by the property of g, $\liminf_{k\to+\infty} \|\xi_k - \Gamma\xi_k\| = 0$ as required.

(2) Again from (40), $\limsup_{k\to+\infty} b_k(1-b_k)g(\|\xi_k - \Gamma\xi_k\|) = 0$ and by the assumption of (2), $\liminf_{k\to+\infty} b_k(1-b_k) > 0$. Now, since

$$(\liminf_{k\to+\infty} b_k(1-b_k))(\limsup_{k\to+\infty} g(\|\xi_k - \Gamma\xi_k\|)) \leqslant \limsup_{k\to+\infty} b_k(1-b_k)g(\|\xi_k - \Gamma\xi_k\|),$$

then we have $\lim_{k\to+\infty} g(\|\xi_k - \Gamma\xi_k\|) = \limsup_{k\to+\infty} g(\|\xi_k - \Gamma\xi_k\|) = 0$, and from Lemma 1.11 by property of g, $\lim_{k\to+\infty} \|\xi_k - \Gamma\xi_k\| = 0$.

This completes the proof. ■

Now, we establish a weak convergence of the UI-iterative scheme defined by (13) for nonexpansive mapping.

Theorem 2.4 Let φ be a nonempty closed convex subset of a uniformly convex Banach space ζ and $\Gamma : \varphi \to \varphi$ be a nonexpansive mapping. Assume that ζ satisfies Opials condition [17] and the sequence $\{\xi_k\}$ defined by the UI iteration process (13) is bounded. Let $F(\Gamma) \neq 0$and for some $q \in F(\Gamma)$ and $\liminf_{k\to+\infty} b_k(1-b_k) > 0$, then the sequence $\{\xi_k\}$ converges weakly to a fixed point q of Γ.

Proof. Since $\{\xi_k\}$ is bounded and from Theorem 2.1, $\lim_{k\to+\infty} \|\xi_k - q\|$ exists. From Theorem 2.3(2), we have $\lim_{k\to+\infty} \|\xi_k - \Gamma\xi_k\| = 0$. There exists a subsequence $\{\xi_{k_i}\} \subset \{\xi_k\}$ such that $\{\xi_{k_i}\}$ converges weakly to a point $q \in \varphi$ for all $i \geqslant 1$, then $\lim_{i\to+\infty} \|\xi_{k_i} - T\xi_{k_i}\| = 0$. Now, we claim that $\Gamma q = q$. But suppose that this is not true (i.e $\Gamma q \neq q$). Then since Γ is nonexpansive and by Opial's condition, it follows that

$$\begin{aligned}\limsup_{i\to+\infty} \|\xi_{k_i} - q\| &< \limsup_{i\to+\infty} \|\xi_{k_i} - \Gamma q\| \\ &\leqslant \limsup_{i\to+\infty} \|\xi_{k_i} - \Gamma\xi_{k_i}\| + \limsup_{i\to+\infty} \|\Gamma\xi_{k_i} - \Gamma q\| \\ &\leqslant \limsup_{i\to+\infty} \|\xi_{k_i} - q\|, \end{aligned} \tag{41}$$

which is a contradiction. Thus, by Theorem 2.2, it follows that the limit $\lim_{k\to+\infty} \|\xi_k - q\|$ exists. We now show that $\{\xi_k\}$ converges weakly to the fixed point q. Lets suppose that $\{\xi_k\}$ converges to a point $z \in K$ such that $q \neq z$. Then, there exists a subsequence $\{\xi_{k_j}\} \subset \{\xi_k\}$ which converges weakly to z. Thus, it follows that $z = \Gamma z$ and $\lim_{k\to+\infty} \|\xi_k - q\|$ exists. It follows from Opial's condition that

$$\lim_{k\to+\infty} \|\xi_k - z\| < \limsup_{k\to+\infty} \|\xi_k - q\| = \lim_{k\to+\infty} \|\xi_{k_j} - q\| < \lim_{k\to+\infty} \|\xi_k - z\|.$$

This is a contradiction. Hence, $q = z$ and this completes the proof. ■

Theorem 2.5 Let φ be a nonempty closed convex subset of a uniformly convex Banach space ζ and $\Gamma : \varphi \to \varphi$ be a nonexpansive mapping. Assume the sequence $\{\xi_k\}$ defined by the UI iteration process (13) is bounded. Let $F(\Gamma) \neq 0$ and for some $q \in F(\Gamma)$ and if $\limsup_{k\to+\infty} b_k(1-b_k) > 0$, then the sequence $\{\xi_k\}$ converges strongly to a fixed point q of Γ.

Proof. If $\{\xi_k\}$ converges strongly to a point $q \in F(\Gamma)$, then $\lim_{k\to+\infty} \|\xi_k - q\| = 0$. Since $0 \leqslant d(\xi_k, F(\Gamma)) \leqslant \|\xi_k - q\|$, then $\liminf_{k\to+\infty} d(\xi_k, F(\Gamma)) = 0$. Conversely, suppose that $\liminf_{k\to+\infty} d(\xi_k, F(\Gamma)) = 0$. From (35), $\lim_{k\to+\infty} d(\xi_k, F(\Gamma))$ exists. Thus,

$$\lim_{k\to+\infty} d(\xi_k, F(\Gamma)) = 0. \tag{42}$$

Now, since $\{\xi_k\}$ is bounded, then there exists a subsequence $\{\xi_{k_j}\} \subseteq \{\xi_k\}$ such that $\|\xi_{k_j} - q_j\| \leqslant 1/2^j$ for all $j \geqslant 1$, where $\{q_j\}$ is a sequence in $F(\Gamma)$. Combining with (35), we have

$$\|\xi_{k_{j+1}} - q_j\| \leqslant \|\xi_{k_j} - q_j\| \leqslant 1/2^j. \tag{43}$$

It follows from (43) that

$$\|q_{j+1} - q_j\| \leqslant \|q_{j+1} - \xi_{k_{j+1}}\| + \|\xi_{k_{j+1}} - q_j\| \leqslant \frac{1}{2^{j+1}} + \frac{1}{2^j} \leqslant \frac{1}{2^{j-1}} \to 0 \; as \; j \to +\infty. \quad (44)$$

This shows that $\{q_j\}$ is a Cauchy sequence in $F(\Gamma)$. By Lemma 1.10, $F(\Gamma)$ is closed. So, $\{q_j\}$ converges to some $\xi^* \in F(\Gamma)$. Moreover, by the triangle inequality, we have

$$\|\xi_{k_{j+1}} - \xi^*\| \leqslant \|\xi_{k_j} - q_j\| + \|q_j - \xi^*\|. \quad (45)$$

Taking $j \longrightarrow +\infty$ implies that ξ_{k_j} converges strongly to ξ^*. Again from (35), $\lim_{k\to+\infty} \|\xi_k - \xi^*\|$ exists and the sequence $\{\xi_k\}$ converges strongly to $\xi^* \in F(\Gamma)$. ■

3. Stability results

3.1 *Stability of UI-iteration for contraction mappings*

Theorem 3.1 Let φ be a nonempty closed convex subset of a uniformly convex Banach space ζ and $\Gamma : \varphi \to \varphi$ be a contraction mapping satisfying (1). Let $\{\xi_k\}$ be the UI iterative scheme defined by (13) with the real sequences $\{a_k\}$, $\{b_k\}$, $\{c_k\} \subset (0,1)$ satisfying $\sum_{s=0}^{+\infty}(1 - a_k)b_k = +\infty$. Then the iterative scheme (13) is Γ-stable.

Proof. Let $\{\theta_n\} \subset \zeta$ be an arbitrary sequence in φ. Also, suppose that the UI-iterative sequence generated by (13) is $\xi_{k+1} = f(\Gamma, \xi_k)$ converges to a unique fixed point q and that $\varepsilon_k = \|\theta_{k+1} - f(\Gamma, \theta_k)\|$. To prove that Γ is stable, we have to show that $\lim_{k\to+\infty} \varepsilon_k = 0$ if and only if $\lim_{k\to+\infty} \theta_k = q$. Let $\lim_{k\to+\infty} \varepsilon_k = 0$. Then, from (28), we have

$$\begin{aligned}
\|\theta_{k+1} - q\| &= \|\theta_{k+1} - f(\Gamma, \theta_k) + f(\Gamma, \theta_k) - q\| \\
&\leqslant \|\theta_{k+1} - f(\Gamma, \theta_k)\| + \|f(\Gamma, \theta_k) - q\| \\
&= \|\theta_{k+1} - \Gamma y_k\| + \|\Gamma y_k - q\| \leqslant \varepsilon_k + \|\Gamma y_k - q\| \leqslant \varepsilon_k + \delta\|y_k - q\| \\
&\leqslant \varepsilon_k + \delta^2\|(1 - b_k)(\Gamma w_k - q) + b_k(\Gamma z_k - q)\| \\
&\leqslant \varepsilon_n + \delta^3((1 - b_k)\|w_k - q\| + b_k\|z_k - q\|) \\
&\leqslant \varepsilon_n + \delta^4((1 - b_k)((1 - c_k)\|\theta_k - q\| + c_k\|\Gamma\theta_k - q\|) \\
&\quad + b_k((1 - a_k)\|\Gamma\theta_k - q\| + a_k\|\Gamma w_k - q\|)) \\
&\leqslant \varepsilon_k + \delta^4((1 - b_k)(1 - (1 - \delta)c_k)\|\theta_k - q\| \\
&\quad + a_k b_k \delta^2(1 - (1 - \delta)c_k)\|\theta_k - q\| + b_k\delta(1 - a_k)\|\theta_k - q\|) \\
&\leqslant \varepsilon_k + \delta^4((1 - b_k)(1 - (1 - \delta)c_k)\|\theta_k - q\| \\
&\quad + a_k b_k(1 - (1 - \delta)c_k)\|\theta_k - q\| + b_k\delta(1 - a_k)\|\theta_k - q\|) \\
&\leqslant \varepsilon_k + \delta^4((1 - (1 - a_k)b_k)\|\theta_k - q\| + b_k\delta(1 - a_k)\|\theta_k - q\|) \\
&= \delta^4(1 - (1 - \delta)(1 - a_k)b_k)\|\theta_k - q\| + \varepsilon_k. \quad (46)
\end{aligned}$$

Now, let $p_k = \|\theta_k - q\|$, $\sigma_k = (1 - \delta)(1 - a_k)b_k$ in $(0,1)$ and $r_k = \varepsilon_k$. Since by our hypothesis that $\lim\limits_{k\to+\infty} \varepsilon_k = 0$, it follows that $\lim\limits_{k\to+\infty} \frac{r_k}{\sigma_k} = \lim\limits_{k\to+\infty} \frac{\varepsilon_k}{(1-\delta)(1-a_k)b_k} = 0$. By Lemma 1.7 and since all the conditions are satisfied by inequality (46), we obtain that $\lim\limits_{k\to+\infty} \theta_k = q$. Conversely, suppose that $\lim\limits_{k\to+\infty} \theta_k = q$, then we have

$$\begin{aligned}
\varepsilon_k &= \|\theta_{k+1} - f(\Gamma, \theta_k)\| \\
&= \|(\theta_{k+1} - q) + (q - f(\Gamma, \theta_k)\| \leqslant \|\theta_{k+1} - q\| + \|f(\Gamma, \theta_k) - q\| \\
&= \|\theta_{k+1} - q\| + \|\Gamma y_k - q\| \leqslant \|\theta_{k+1} - q\| + \delta\|y_k - q\| \\
&\leqslant \|\theta_{k+1} - q\| + \delta^2\|(1-b_k)(\Gamma w_k - q) + b_k(\Gamma z_k - q)\| \\
&\leqslant \|\theta_{k+1} - q\| + \delta^3((1-b_k)\|w_k - q\| + b_k\|z_k - q\|) \\
&\leqslant \|\theta_{k+1} - q\| + \delta^4((1-b_k)((1-c_k)\|\theta_k - q\| + c_k\delta\|\theta_k - q\|) \\
&\quad + b_k((1-a_k)\delta\|\theta_k - q\| + a_k\delta\|w_k - q\|)) \\
&\leqslant \|\theta_{k+1} - q\| + \delta^4((1-b_k)(1-(1-\delta)c_k)\|\theta_k - q\| \\
&\quad + a_k b_k \delta^2(1-(1-\delta)c_k)\|\theta_k - q\| + b_k\delta(1-a_k)\|\theta_k - q\|) \\
&\leqslant \|\theta_{k+1} - q\| + \delta^4((1-b_k)(1-(1-\delta)c_k)\|\theta_k - q\| \\
&\quad + a_k b_k(1-(1-\delta)c_k)\|\theta_k - q\| + b_k\delta(1-a_k)\|\theta_k - q\|) \\
&\leqslant \|\theta_{k+1} - q\| + \delta^4((1-(1-\delta)c_k)(1-(1-a_k)b_k)\|\theta_k - q\| + b_k\delta(1-a_k)\|\theta_k - q\|) \\
&\leqslant \|\theta_{k+1} - q\| + \delta^4(1-(1-\delta)(1-a_k)b_k)\|\theta_k - q\|. \qquad (47)
\end{aligned}$$

Now, since $\lim\limits_{k\to+\infty} \theta_k = q$ and taking limits as $k \to +\infty$ in (47), we have that $\lim\limits_{k\to+\infty} \varepsilon_k = 0$. Hence, UI-iteration defined by (13) is stable with respect to T. ■

3.2 *Stability of UI-iteration for nonexpansive mappings*

Theorem 3.2 Let φ be a nonempty closed convex subset of a uniformly convex Banach space ζ and $\Gamma : \varphi \to \varphi$ be a nonexpansive mapping satisfying (1). Let $\{\xi_k\}$ be the UI iterative scheme defined by (13) with the real sequences $\{a_k\}$, $\{b_k\}$, $\{c_k\} \subset (0,1)$. Then the iterative scheme (13) is Γ-stable.

Proof. To prove that Γ is stable, we have to show that $\lim\limits_{k\to+\infty} \varepsilon_k = 0$ if and only if $\lim\limits_{k\to+\infty} \theta_k = q$. Let $\lim\limits_{k\to+\infty} \varepsilon_k = 0$. Then, from (28), we have

$$\begin{aligned}
\|\theta_{k+1} - q\| &= \|\theta_{k+1} - f(\Gamma, \theta_k) + f(\Gamma, \theta_k) - q\| \\
&\leqslant \|\theta_{k+1} - f(\Gamma, \theta_k)\| + \|f(\Gamma, \theta_k) - q\| \\
&= \|\theta_{k+1} - \Gamma y_k\| + \|\Gamma y_k - q\| \leqslant \varepsilon_k + \|\Gamma y_k - q\| \leqslant \varepsilon_k + \|y_k - q\| \\
&\leqslant \varepsilon_k + (1-b_k)\|w_k - q\| + b_k\|z_k - q\| \\
&\leqslant \|\theta_k - q\| + \varepsilon_k. \qquad (48)
\end{aligned}$$

Now, let $a_k = \|\theta_k - q\|$ and $b_k = \varepsilon_k$. Since $\sum\limits_{k=1}^{+\infty} \varepsilon_k < +\infty$ and by Lemma 1.8, $\lim\limits_{k\to+\infty} \|\theta_k - q\|$ exists. Consequently, the conclusion follows from Lemma 1.8. Thus, $\lim\limits_{k\to+\infty} \theta_k = q$.

Conversely, suppose that $\lim_{k\to+\infty} \theta_k = q$, then we have

$$\begin{aligned}
\varepsilon_k &= \|\theta_{k+1} - f(\Gamma, \theta_k)\| = \|(\theta_{k+1} - q) + (q - f(\Gamma, \theta_k)\| \\
&\leqslant \|\theta_{k+1} - q\| + \|f(\Gamma, \theta_k) - q\| = \|\theta_{k+1} - q\| + \|\Gamma y_k - q\| \\
&\leqslant \|\theta_{k+1} - q\| + \|y_k - q\| \\
&\leqslant \|\theta_{k+1} - q\| + \|\theta_k - q\|. \qquad (49)
\end{aligned}$$

Now, since $\lim_{k\to+\infty} \theta_k = q$ and taking limits as $k \to +\infty$ in (47), we have that $\lim_{k\to+\infty} \varepsilon_k = 0$. Hence, UI-iteration defined by (13) is stable with respect to Γ. ■

4. Data dependence result

Theorem 4.1 Let $\tilde{\Gamma}$ be an approximate operator of a mapping Γ satisfying the contraction mapping (14). Let $\{\xi_k\}$ be the UI- iterative sequence generated by (13) for Γ and define an iterative sequence $\{\tilde{\xi}_k\}$ as follows:

$$\begin{cases}
\tilde{\xi}_1 \in \varphi, \\
\tilde{\xi}_{k+1} = \tilde{\Gamma}\tilde{y}_k, \\
\tilde{y}_k = \tilde{\Gamma}((1-b_k)\tilde{\Gamma}\tilde{w}_k + b_k\tilde{\Gamma}\tilde{z}_k), \qquad \text{for all } k \geqslant 1, \\
\tilde{z}_k = \tilde{\Gamma}((1-a_k)\tilde{\Gamma}\tilde{\xi}_k + a_k\tilde{\Gamma}\tilde{w}_k), \\
\tilde{w}_k = \tilde{\Gamma}((1-c_k)\tilde{\xi}_k + c_k\tilde{\Gamma}\tilde{\xi}_k)
\end{cases} \qquad (50)$$

where $\{a_k\}, \{b_k\}, \{c_k\} \subset (0,1)$ for all $k \in \mathbb{N}$ satisfying the following conditions:

(i) $\frac{1}{2} \leqslant a_k b_k c_k$, for all $k \in \mathbb{N}$,

(ii) $\sum_{k=0}^{+\infty} a_k b_k c_k = +\infty$.

If $\Gamma q = q$ and $\tilde{\Gamma}\tilde{q} = \tilde{q}$ such that $\lim_{k\to+\infty} \tilde{\xi}_k = \tilde{q}$, then we have

$$\|q - \tilde{q}\| \leqslant \frac{15\epsilon}{1-\delta}, \qquad (51)$$

where $\epsilon > 0$ is a fixed number.

Proof. From (13), (50) and (14), we have

$$\begin{aligned}
\|w_k - \tilde{w}_k\| &= \|\Gamma((1-c_k)\xi_k + c_k\Gamma\xi_k) - \tilde{\Gamma}((1-c_k)\tilde{\xi}_k + c_k\tilde{\Gamma}\tilde{\xi}_k)\| \\
&\leqslant \|\Gamma((1-c_k)\xi_k + c_k\Gamma\xi_k) - \Gamma((1-c_k)\tilde{\xi}_k + c_k\tilde{\Gamma}\tilde{\xi}_k)\| \\
&\quad + \|\Gamma((1-c_k)\tilde{\xi}_k + c_k\tilde{\Gamma}\tilde{\xi}_k) - \tilde{\Gamma}((1-c_k)\tilde{\xi}_k + c_k\tilde{\Gamma}\tilde{\xi}_k)\| \\
&\leqslant \delta(1-c_k)\|\xi_k - \tilde{\xi}_k\| + c_k\delta\|\Gamma\xi_k - \tilde{\Gamma}\tilde{\xi}_k\| + \epsilon \\
&\leqslant \delta(1-c_k)\|\xi_k - \tilde{\xi}_k\| + c_k\delta^2\|\xi_k - \tilde{\xi}_k\| + c_k\delta\|\Gamma\tilde{\xi}_k - \tilde{\Gamma}\tilde{\xi}_k\| + \epsilon \\
&\leqslant \delta(1-(1-\delta)c_k)\|\xi_k - \tilde{\xi}_k\| + \delta c_k\epsilon + \epsilon. \qquad (52)
\end{aligned}$$

From (13), (50) and (52), we have

$$\begin{aligned}\|z_k - \tilde{z}_k\| &= \|\Gamma((1-a_k)\Gamma\xi_k + a_k\Gamma w_k) - \tilde{\Gamma}((1-a_k)\tilde{\Gamma}\tilde{\xi}_k + a_k\tilde{\Gamma}\tilde{w}_k)\| \\ &\leqslant \|\Gamma((1-a_k)\Gamma\xi_k + a_k\Gamma w_k) - \Gamma((1-a_k)\tilde{\Gamma}\tilde{\xi}_k + a_k\tilde{\Gamma}\tilde{w}_k)\| \\ &\quad + \|\Gamma((1-a_k)\tilde{\Gamma}\tilde{\xi}_k + a_k\tilde{\Gamma}\tilde{w}_k) - \tilde{\Gamma}((1-a_k)\tilde{\Gamma}\tilde{\xi}_k + a_k\tilde{\Gamma}\tilde{w}_k)\| \\ &\leqslant \delta(1-a_k)(\|\Gamma\xi_k - \Gamma\tilde{\xi}_k\| + \|\Gamma\tilde{\xi}_k - \tilde{\Gamma}\tilde{\xi}_k\|) \\ &\quad + a_k\delta(\|\Gamma w_k - \Gamma\tilde{w}_k\| + \|\Gamma\tilde{w}_k - \tilde{\Gamma}\tilde{w}_k\|) + \epsilon \\ &\leqslant \delta^2(1-a_k)\|\xi_k - \tilde{\xi}_k\| + a_k\delta^2\|w_k - \tilde{w}_k\| + \delta\epsilon + \epsilon. \end{aligned} \tag{53}$$

Substituting (52) into (53), we have

$$\begin{aligned}\|z_k - \tilde{z}_k\| &\leqslant \delta^2(1-a_k)\|\xi_k - \tilde{\xi}_k\| + a_k\delta^2(\delta(1-(1-\delta)c_k)\|\xi_k - \tilde{\xi}_k\| \\ &\quad + \delta c_k\epsilon + \epsilon) + \delta\epsilon + \epsilon \\ &\leqslant \delta^2(1-a_k)\|\xi_k - \tilde{\xi}_k\| + a_k\delta^3(1-(1-\delta)c_k)\|\xi_k - \tilde{\xi}_k\| \\ &\quad + \delta^3 a_k c_k\epsilon + \delta^2 a_k\epsilon + \delta\epsilon + \epsilon \\ &\leqslant \delta^2(1-(1-\delta)a_k c_k)\|\xi_k - \tilde{\xi}_k\| + \delta^3 a_k c_k\epsilon + \delta^2 a_k\epsilon + \delta\epsilon + \epsilon. \end{aligned} \tag{54}$$

From (13), (50), (52) and (54), we have

$$\begin{aligned}\|y_k - \tilde{y}_k\| &= \|T((1-b_k)\Gamma w_k + b_k\Gamma z_k) - \tilde{\Gamma}((1-b_k)\tilde{\Gamma}\tilde{w}_k + b_k\tilde{\Gamma}\tilde{z}_k)\| \\ &\leqslant \|\Gamma((1-b_k)\Gamma w_k + b_k\Gamma z_k) - \Gamma((1-b_k)\tilde{\Gamma}\tilde{w}_k + b_k\tilde{\Gamma}\tilde{z}_k)\| \\ &\quad + \|\Gamma((1-b_k)\tilde{\Gamma}\tilde{w}_k + b_k\tilde{\Gamma}\tilde{z}_k) - \tilde{\Gamma}((1-b_k)\tilde{\Gamma}\tilde{w}_k + b_k\tilde{\Gamma}\tilde{z}_k)\| \\ &\leqslant \delta(1-b_k)\|\Gamma w_k - \tilde{\Gamma}\tilde{w}_k\| + b_k\delta\|\Gamma z_k - \tilde{\Gamma}\tilde{z}_k\| + \epsilon \\ &\leqslant \delta(1-b_k)(\|\Gamma w_k - \Gamma\tilde{w}_k\| + \|\Gamma\tilde{w}_k - \tilde{\Gamma}\tilde{w}_k\|) \\ &\quad + \beta_k\delta(\|\Gamma z_k - \Gamma\tilde{z}_k\| + \|\Gamma\tilde{z}_k - \tilde{\Gamma}\tilde{z}_k\|) + \epsilon \\ &\leqslant \delta^2(1-b_k)\|w_k - \tilde{w}_k\| + \delta(1-b_k)\epsilon + b_k\delta^2\|z_k - \tilde{z}_k\| + \delta\epsilon + \epsilon. \end{aligned} \tag{55}$$

Substituting (52) and (54) into (55), we have

$$\begin{aligned}\|y_k - \tilde{y}_k\| &\leqslant \delta^3(1-b_k)(1-(1-\delta)c_k)\|\xi_k - \tilde{\xi}_k\| + \delta^4 b_k(1-(1-\delta)a_k c_k)\|\xi_k - \tilde{\xi}_k\| \\ &\quad + \delta^3(1-b_k)c_k\epsilon + \delta^2(1-b_k)\epsilon + \delta^5 a_k b_k c_k\epsilon + \delta^4 a_k b_k\epsilon + \delta^3 b_k\epsilon \\ &\quad + \delta^2 b_k\epsilon + \delta\epsilon + \epsilon \\ &\leqslant \delta^3(1-b_k)\|\xi_k - \tilde{\xi}_k\| + \delta^3 b_k(1-(1-\delta)a_k c_k)\|\xi_k - \tilde{\xi}_k\| \\ &\quad + \delta^3(1-b_k)c_k\epsilon + \delta^5 a_k b_k c_k\epsilon + \delta^4 a_k b_k\epsilon + \delta^3 b_k\epsilon + \delta^2\epsilon + \delta\epsilon + \epsilon \\ &\leqslant \delta^3(1-(1-\delta)a_k b_k c_k)\|\xi_k - \tilde{\xi}_k\| \\ &\quad + \delta^3(1-b_k)c_k\epsilon + \delta^5 a_k b_k c_k\epsilon + \delta^4 a_k b_k\epsilon + \delta^3 b_k\epsilon + \delta^2\epsilon + \delta\epsilon + \epsilon. \end{aligned} \tag{56}$$

From (13), (50) and (56), we have

$$\|\xi_{k+1} - \tilde{\xi}_{k+1}\| \leqslant \|\Gamma y_k - \tilde{\Gamma}\tilde{y}_k\| \leqslant \|\Gamma y_k - \Gamma\tilde{y}_k\| + \|\Gamma\tilde{y}_k - \tilde{\Gamma}\tilde{y}_k\| \leqslant \delta\|y_k - \tilde{y}_k\| + \epsilon. \tag{57}$$

Substituting (56) into (57), we have

$$\|\xi_{k+1}-\tilde{\xi}_{k+1}\| \leqslant \delta^4(1-(1-\delta)a_kb_kc_k)\|x_n-\tilde{\xi}_k\| + \delta^4(1-b_k)c_k\epsilon \\ +\delta^6 a_kb_kc_k\epsilon + \delta^5 a_kb_k\epsilon + \delta^4 b_k\epsilon + \delta^3\epsilon + \delta^2\epsilon + \delta\epsilon + \epsilon. \tag{58}$$

Now, since $\delta, \delta^2, \delta^3, \delta^4, \delta^5, \delta^6 \in (0,1)$ and $a_k, b_k, c_k \in (0,1)$, then (58) becomes

$$\|\xi_{k+1}-\tilde{\xi}_{k+1}\| \leqslant \delta^4(1-(1-\delta)a_kb_kc_k)\|\xi_n-\tilde{\xi}_k\| + \delta^6 a_kb_kc_k\epsilon + 7\epsilon. \tag{59}$$

Using our assumption of Theorem 4.1 (i) that $\frac{1}{2} \leqslant a_kb_kc_k$ for all $k \in \mathbb{N}$, we have $1 - a_kb_kc_k \leqslant a_kb_kc_k$, which implies that

$$1 = 1 - a_kb_kc_k + a_kb_kc_k \leqslant a_kb_kc_k + a_kb_kc_k = 2a_kb_kc_k. \tag{60}$$

So, using (60), (59) yields

$$\|\xi_{k+1}-\tilde{\xi}_{k+1}\| \leqslant (1-(1-\delta)a_kb_kc_k)\|\xi_n-\tilde{\xi}_k\| + a_kb_kc_k\epsilon + 14\epsilon \\ = (1-(1-\delta)a_kb_kc_k)\|\xi_k-\tilde{\xi}_k\| + \frac{(1-\delta)15\epsilon}{(1-\delta)}. \tag{61}$$

Now, we set

$$p_k = \|\xi_k-\tilde{\xi}_k\|, \quad \sigma_k = (1-\delta)a_kb_kc_k, \quad \text{and} \quad r_n = \frac{15\epsilon}{(1-\delta)}. \tag{62}$$

From Theorem 2.1, we have that $\lim\limits_{k\to+\infty} \xi_k = q$. Thus, by Lemma 1.6, $\sum\limits_{k=0}^{+\infty} a_kb_kc_k = +\infty$ and $\frac{15\epsilon}{(1-\delta)} > 0$. So, we have

$$0 \leqslant \limsup_{k\to+\infty} \|\xi_k-\tilde{\xi}_k\| \leqslant \limsup_{k\to+\infty} \frac{15\epsilon}{(1-\delta)}. \tag{63}$$

Since, by Theorem 2.1, $\lim\limits_{k\to+\infty} \xi_k = q$ and by our assumption, $\lim\limits_{k\to+\infty} \tilde{\xi}_k = \tilde{q}$, then from (63), we have

$$\|z-\tilde{z}\| \leqslant \frac{15\epsilon}{(1-\delta)}. \tag{64}$$

Thus, the UI-iterative scheme (13) is data dependent. This completes the proof. ■

5. Rate of convergence

5.1 *Rate of convergence of UI-iteration scheme for contraction map*

In this section, we show that UI-iterative scheme (13) converges faster than Garodia and Uddin iterative scheme (12) for contraction mapping.

Theorem 5.1 Let Γ be a contraction mapping defined on a nonempty closed convex subset φ of a uniformly convex Banach space ζ with a contraction factor $\delta \in (0,1)$ and

$F(\Gamma) \neq \phi$. If $\{x_k\}$ is a sequence defined by (13), then $\{x_k\}$ converges faster than the iteration process (12).

Proof. Let $q \in F(\Gamma)$. From (13) and (14), we have

$$\begin{aligned} \|w_k - q\| &= \|\Gamma((1-c_k)\xi_k + c_k\Gamma\xi_k) - q\| \\ &\leqslant \delta((1-c_k)\|\xi_k - q\| + c_k\delta\|\xi_k - q\|) \\ &\leqslant \delta\|\xi_k - q\|. \end{aligned} \tag{65}$$

Since $1-(1-\delta)c_k < 1$, by using (13), (14) and (65), we have

$$\begin{aligned} \|z_k - q\| &= \|\Gamma((1-a_k)\Gamma\xi_k + a_k\Gamma w_k) - q\| \\ &\leqslant \delta((1-a_k)\delta\|\xi_k - q\| + a_k\delta\|w_k - q\|) \\ &\leqslant \delta^2(1-(1-\delta)a_k)\|\xi_k - q\| \\ &\leqslant \delta^2\|\xi_k - q\|. \end{aligned} \tag{66}$$

Using (13), (14), (65) and (66), we have

$$\begin{aligned} \|y_k - q\| &= \|\Gamma((1-b_k)\Gamma w_k + b_k\Gamma z_k) - q\| \\ &\leqslant \delta((1-b_k)\delta\|w_k - q\| + b_k\delta\|z_k - q\|) \\ &\leqslant \delta^2((1-b_k)\delta\|\xi_k - q\| + b_k\delta^2\|\xi_k - q\|) \\ &\leqslant \delta^3(1-(1-\delta)b_k)\|\xi_k - q\| \\ &\leqslant \delta^3\|\xi_k - q\|. \end{aligned} \tag{67}$$

Using (13), (14) and (67), we have

$$\|\xi_{k+1} - q\| = \|\Gamma y_k - q\| \leqslant \delta\|y_k - q\| \leqslant \delta^4\|\xi_k - q\| \cdots \leqslant \delta^{4k}\|\xi_1 - q\|. \tag{68}$$

Let

$$p_n = \delta^{4k}\|\xi_1 - q\|. \tag{69}$$

Also, from (12) and (14), we have

$$\begin{aligned} \|z_k - q\| &= \|\Gamma((1-a_k)\xi_k + a_k\Gamma\xi_k) - q\| \\ &\leqslant \delta((1-a_k)\|\xi_k - q\| + a_k\|\Gamma\xi_k - q\|) \\ &\leqslant \delta(1-(1-\delta)a_k)\|\xi_k - q\| \\ &\leqslant \delta\|\xi_k - q\|. \end{aligned} \tag{70}$$

Using (12), (14) and (70), we have

$$\begin{aligned}\|y_k - q\| &= \|\Gamma((1-b_k)\Gamma\xi_k + b_k\Gamma z_k) - q\| \\ &\leqslant \delta((1-b_k)\delta\|\xi_k - q\| + b_k\delta\|z_k - q\|) \\ &\leqslant \delta^2(1-(1-\delta)b_k)\|\xi_k - q\| \\ &\leqslant \delta^2\|\xi_k - q\|. \end{aligned} \tag{71}$$

Using (12), (14) and (71), we have

$$\|\xi_{k+1} - q\| = \|\Gamma y_k - q\| \leqslant \delta\|y_k - q\| = \delta^3\|\xi_k - q\| \cdots \leqslant \delta^{3k}\|\xi_1 - q\|. \tag{72}$$

Let

$$r_k = \delta^{2k}\|\xi_1 - q\|. \tag{73}$$

So, from (69) and (73), we have $\frac{p_k}{r_k} = \frac{\delta^{4k}\|\xi_1 - q\|}{\delta^{3k}\|\xi_1 - q\|} = \delta^k \longrightarrow 0$ as $k \to +\infty$. Hence, (13) converges faster than (12). ■

Remark 2 *Since every contraction map is nonexpansive for $\delta = 1$, the result holds for nonexpansive mappings.*

5.2 *Numerical example*

We now show the comparison between the rate of convergence of the UI iteration process (13) and other well known iteration algorithms in literature.

***Example* 5.2** Let Γ be a contraction given by $\Gamma : [1, \infty) \longrightarrow [1, \infty)$ and defined by $\Gamma(\upsilon) = \frac{2}{3}\upsilon$. For Table 1, choose $a_k = \frac{1}{2}$, $b_k = \frac{1}{\sqrt{8}}$, $c_k = \frac{2}{7}$, and the initial value $t_1 = 30$. Obviously, the fixed point of T is $p = 0$ with a contraction constant $\delta = \frac{2}{3}$. Table 1 shows the behavior of the UI iteration process (13) in comparison with the iteration processes of Picard, Mann (4), Ishikawa (5), Noor (6), Agarwal et al. (S-iteration) (7), Abbas and Nazir (8), Thakur et al. (9), Piri et al. (10) and Garodia-Uddin (12) to the fixed point of Γ in 150-iterations with $\|t_n - p\| < 10^{-15}$ as the stop criterion.

Clearly, the tabulation below in Table 1 shows that the UI-iteration scheme (13) has a faster rate of convergence for contraction mapping than the Garodia and Uddin iterative scheme (12) and some well known iteration schemes in literature.

***Example* 5.3** Let Γ be a nonexpansive mapping given by $\Gamma : \varphi \longrightarrow \varphi$, where $\varphi = [0, 1]$ and defined by $\Gamma(\upsilon) = 1 - \upsilon$ for all $\upsilon \in [0, 1]$. For Table 2, choose $a_k = b_k = c_k = \frac{2}{3}$ and the initial value $t_1 = 30$. Obviously, the fixed point of T is $p = \frac{1}{2}$. Table 2 shows the behavior of the UI iteration process (13) in comparison with the iteration processes of Mann (4), Ishikawa (5), Noor (6), Agarwal et al. (S-iteration) (7), Abbas and Nazir (8), Thakur et al. (9), Piri et al. (10), M-iteration (11) and Garodia-Uddin (12) to the fixed point of Γ in 108-iterations with $\|t_n - p\| < 10^{-15}$ as the stop criterion.

Again, the tabulation in Table 2 below shows that the UI iteration scheme (13) has a faster rate of convergence for nonexpansive mapping than the Garodia and Uddin iteration scheme (12) and some well known iteration schemes in literature.

Tabel 1.

n	PICARD	MANN	ISHIKAWA	NOOR	AGARWAL
0	30.000000000	30.000000000	30.000000000	30.00000000	30.000000000
1	20.000000000	25.000000000	23.821333333	23.59682539	18.821333333
2	13.333333333	20.833333333	18.915197392	18.56033896	11.808086281
3	8.8888888889	17.361111111	15.019507405	14.59883592	7.4081309311
4	5.9259259259	14.467592592	11.926156413	11.48287274	4.6476967210
5	3.9506172840	12.056327160	9.4698982436	9.031978106	2.9158616406
6	2.6337448560	10.046939300	7.5195200898	7.104200345	1.8293467964
7	1.7558299040	8.3724494170	5.9708331522	5.587885837	1.1476915279
8	1.1705532693	6.9770411808	4.7411068932	4.395212215	0.7200361604
9	0.7803688462	5.8142009840	3.7646495891	3.457101840	0.4517346862
10	0.5202458975	4.8451674867	2.9892990915	2.719220950	0.2834083036
11	0.3468305983	4.0376395722	2.3736363364	2.138832732	0.1778040717
12	0.2312203989	3.3646996435	1.8847727460	1.682322085	0.1115503234
13	0.1541469326	2.8039163696	1.4965933280	1.323248683	0.0699841940
14	0.1027646217	2.3365969747	1.1883616177	1.040815604	0.0439065281
15	0.0685097478	1.9471641456	0.9436119405	0.818664803	0.0275459800
16	0.0456731652	1.6226367880	0.7492698191	0.643929680	0.0172817357
17	0.0304487768	1.3521973233	0.5949535372	0.506489874	0.0108421770
18	0.0202991845	1.1268311028	0.4724195509	0.398385104	0.0068021409
19	0.0135327897	0.9390259190	0.3751221199	0.313354125	0.0042675120
20	0.0090218598	0.7825215991	0.2978636353	0.246472085	0.0026773422
21	0.0060145732	0.6521013326	0.2365169648	0.193865292	0.0016797050
22	0.0040097155	0.5434177772	0.1878049819	0.152486848	0.0010538096
23	0.0026731436	0.4528481476	0.1491255025	0.119940184	0.0006611367
24	0.0017820958	0.3773734564	0.1184122768	0.094340253	0.0004147825
25	0.0011880638	0.3144778803	0.0940246106	0.074204349	0.0002602253
26	0.0007920426	0.2620649003	0.0746597196	0.058366235	0.0001632596
27	0.0005280284	0.2183874169	0.0592831356	0.045908595	0.0001024254
28	0.0003520189	0.1819895141	0.0470734445	0.036109904	0.0000642594
29	0.0002346793	0.1516579284	0.0373784071	0.028402636	0.0000403149
30	0.0001564529	0.1263816070	0.0296801165	0.022340402	0.0000252927
31	0.0001043019	0.1053180058	0.0235673316	0.017572085	0.0000158681
32	0.0000695346	0.0877650049	0.0187135087	0.013821514	0.0000099553
33	0.0000463564	0.0731375040	0.0148593576	0.010871462	0.0000062457
34	0.0000309043	0.0609479200	0.0117989904	0.008551066	0.0000039184
35	0.0000206028	0.0507899334	0.0093689228	0.006725934	0.0000024583
36	0.0000137352	0.0423249445	0.0074393411	0.005290356	0.0000015423
37	0.0000091568	0.0352707871	0.0059071674	0.004161187	0.0000009676
38	0.0000061045	0.0293923225	0.0046905535	0.003273026	0.0000006071
39	0.0000040697	0.0244936021	0.0037245079	0.002574434	0.0000003809
40	0.0000027131	0.0204113351	0.0029574248	0.002024949	0.0000002389
41	0.0000018088	0.0170094459	0.0023483268	0.001592746	0.0000001499

Tabel 1 CONTD.

n	PICARD	MANN	ISHIKAWA	NOOR	AGARWAL
42	0.0000012058	0.0141745383	0.0018646758	0.001252791	0.0000000940
43	0.0000008039	0.0118121152	0.0014806355	0.000985396	0.0000000590
44	0.0000005359	0.0098434294	0.0011756904	0.000775074	0.0000000370
45	0.0000003573	0.0082028578	0.0009335504	0.000609643	0.0000000232
46	0.0000002382	0.0068357148	0.0007412805	0.000479521	0.0000000146
47	0.0000001588	0.0056964290	0.0005886097	0.000377172	0.0000000091
48	0.0000001059	0.0047470242	0.0004673822	0.000296669	0.0000000057
49	0.0000000706	0.0039558535	0.0003711223	0.000233348	0.0000000036
50	0.0000000470	0.0032965446	0.0002946876	0.000183542	0.0000000023
51	0.0000000314	0.0027471205	0.0002339950	0.000144367	0.0000000014
52	0.0000000209	0.0022892671	0.0001858025	0.000113553	0.0000000009
53	0.0000000139	0.0019077226	0.0001475354	0.000089317	0.0000000006
54	0.0000000093	0.0015897688	0.0001171497	0.000070253	0.0000000003
55	0.0000000062	0.0013248073	0.0000930220	0.000055258	0.0000000002
56	0.0000000041	0.0011040061	0.0000738636	0.000043464	0.0000000001
57	0.0000000028	0.0009200051	0.0000586510	0.000034187	0.0000000001
58	0.0000000018	0.0007666709	0.0000465715	0.000026890	0.0000000001
59	0.0000000012	0.0006388924	0.0000369798	0.000021150	0.0000000000
60	0.0000000008	0.0005324104	0.0000293636	0.000016636	0.0000000000
61	0.0000000005	0.0004436753	0.0000233160	0.000013085	0.0000000000
62	0.0000000004	0.0003697294	0.0000185140	0.000010292	0.0000000000
63	0.0000000002	0.0003081078	0.0000147009	0.000008095	0.0000000000
64	0.0000000002	0.0002567565	0.0000116732	0.000006367	0.0000000000
65	0.0000000001	0.0002139638	0.0000092690	0.000005008	0.0000000000
66	0.0000000001	0.0001783031	0.0000073600	0.000003939	0.0000000000
67	0.0000000000	0.0001485860	0.0000058442	0.000003098	0.0000000000
68	0.0000000000	0.0001238216	0.0000046405	0.000002437	0.0000000000
69	0.0000000000	0.0001031847	0.0000036848	0.000001917	0.0000000000
70	0.0000000000	0.0000859872	0.0000029259	0.000001507	0.0000000000
71	0.0000000000	0.0000716560	0.0000023233	0.000001186	0.0000000000
72	0.0000000000	0.0000597134	0.0000018448	0.000000932	0.0000000000
73	0.0000000000	0.0000497611	0.0000014648	0.000000733	0.0000000000
74	0.0000000000	0.0000414676	0.0000011632	0.000000577	0.0000000000
75	0.0000000000	0.0000345563	0.0000009236	0.000000454	0.0000000000
76	0.0000000000	0.0000287970	0.0000007334	0.000000357	0.0000000000
77	0.0000000000	0.0000239975	0.0000005823	0.000000280	0.0000000000
78	0.0000000000	0.0000199979	0.0000004624	0.000000220	0.0000000000
79	0.0000000000	0.0000166649	0.0000003672	0.000000173	0.0000000000
80	0.0000000000	0.0000138874	0.0000002915	0.000000136	0.0000000000
81	0.0000000000	0.0000115729	0.0000002315	0.000000107	0.0000000000
82	0.0000000000	0.0000096440	0.0000001838	0.000000084	0.0000000000
83	0.0000000000	0.0000080367	0.0000001460	0.000000066	0.0000000000
84	0.0000000000	0.0000066971	0.0000001159	0.000000052	0.0000000000

Tabel 1 CONTD.

n	PICARD	MANN	ISHIKAWA	NOOR	AGARWAL
85	0.0000000000	0.000005581	0.0000000920	0.000000041	0.000000000
86	0.0000000000	0.000004650	0.0000000731	0.000000032	0.000000000
87	0.0000000000	0.000003875	0.0000000580	0.000000025	0.000000000
88	0.0000000000	0.000003229	0.0000000461	0.000000020	0.000000000
89	0.0000000000	0.000002691	0.0000000366	0.000000015	0.000000000
90	0.0000000000	0.000002242	0.0000000291	0.000000012	0.000000000
91	0.0000000000	0.000001869	0.0000000231	0.000000009	0.000000000
92	0.0000000000	0.000001557	0.0000000183	0.000000007	0.000000000
93	0.0000000000	0.000001298	0.0000000145	0.000000006	0.000000000
94	0.0000000000	0.000001081	0.0000000115	0.000000004	0.000000000
95	0.0000000000	0.000000901	0.0000000092	0.000000003	0.000000000
96	0.0000000000	0.000000751	0.0000000073	0.000000002	0.000000000
97	0.0000000000	0.000000626	0.0000000058	0.000000002	0.000000000
98	0.0000000000	0.000000521	0.0000000046	0.000000001	0.000000000
99	0.0000000000	0.000000434	0.0000000036	0.000000001	0.000000000
100	0.0000000000	0.000000362	0.0000000029	0.000000001	0.000000000
101	0.0000000000	0.000000301	0.0000000023	0.000000000	0.000000000
102	0.0000000000	0.000000251	0.0000000018	0.000000000	0.000000000
103	0.0000000000	0.000000209	0.0000000014	0.000000000	0.000000000
104	0.0000000000	0.000000174	0.0000000012	0.000000000	0.000000000
105	0.0000000000	0.000000145	0.0000000009	0.000000000	0.000000000
106	0.0000000000	0.000000121	0.0000000007	0.000000000	0.000000000
107	0.0000000000	0.000000101	0.0000000006	0.000000000	0.000000000
108	0.0000000000	0.000000084	0.0000000005	0.000000000	0.000000000
109	0.0000000000	0.000000070	0.0000000004	0.000000000	0.000000000
110	0.0000000000	0.000000058	0.0000000003	0.000000000	0.000000000
111	0.0000000000	0.000000048	0.0000000002	0.000000000	0.000000000
112	0.0000000000	0.000000040	0.0000000002	0.000000000	0.000000000
113	0.0000000000	0.000000033	0.0000000001	0.000000000	0.000000000
114	0.0000000000	0.000000028	0.0000000001	0.000000000	0.000000000
115	0.0000000000	0.000000023	0.0000000001	0.000000000	0.000000000
116	0.0000000000	0.000000019	0.0000000001	0.000000000	0.000000000
117	0.0000000000	0.000000016	0.0000000001	0.000000000	0.000000000
118	0.0000000000	0.000000013	0.0000000000	0.000000000	0.000000000
119	0.0000000000	0.000000011	0.0000000000	0.000000000	0.000000000
120	0.0000000000	0.000000009	0.0000000000	0.000000000	0.000000000
121	0.0000000000	0.000000007	0.0000000000	0.000000000	0.000000000
122	0.0000000000	0.000000006	0.0000000000	0.000000000	0.000000000
123	0.0000000000	0.000000005	0.0000000000	0.000000000	0.000000000
..	....	....	....	....	....
148	0.0000000000	0.000000001	0.0000000000	0.000000000	0.000000000
149	0.0000000000	0.000000000	0.0000000000	0.000000000	0.000000000
150	0.0000000000	0.000000000	0.0000000000	0.000000000	0.000000000

Tabel 1 CONTD.

n	ABBAS	THAKUR	PIRI	GAR-UDDIN	NEW UI
0	30.000000000	30.000000000	30.000000000	30.000000000	30.000000000
1	15.489777777	12.547555555	9.8014814815	7.4919506173	4.5854752107
2	7.9977738535	5.2480383473	3.2023013077	1.8709774684	0.7008860969
3	4.1294579903	2.1950017574	1.0462432322	0.4672423600	0.1071298608
4	2.1321462204	0.9180635498	0.3418244555	0.1166852229	0.0163747107
5	1.1008823715	0.3839817798	0.1116795357	0.0291399976	0.0025028610
6	0.5684141098	0.1606010905	0.0364874967	0.0072771808	0.0003825602
7	0.2934869415	0.0671717035	0.0119210508	0.0018173426	0.0000584740
8	0.1515349168	0.0280946894	0.0038947986	0.0004538480	0.0000089377
9	0.0782414062	0.0117506559	0.0012724932	0.0001133402	0.0000013661
10	0.0403980665	0.0049147336	0.0004157440	0.0000283046	0.0000002088
11	0.0208585691	0.0020555964	0.0001358302	0.0000070686	0.0000000319
12	0.0107698200	0.0008597570	0.0000443779	0.0000017652	0.0000000049
13	0.0055607373	0.0003595950	0.0000144990	0.0000004408	0.0000000007
14	0.0028711528	0.0001504013	0.0000047370	0.0000001101	0.0000000001
15	0.0014824506	0.0000629056	0.0000015477	0.0000000275	0.0000000000
16	0.0007654277	0.0000263104	0.0000005056	0.0000000069	0.0000000000
17	0.0003952102	0.0000110044	0.0000001652	0.0000000017	0.0000000000
18	0.0002040573	0.0000046026	0.0000000540	0.0000000004	0.0000000000
19	0.0001053601	0.0000019250	0.0000000176	0.0000000001	0.0000000000
20	0.0000544001	0.0000008052	0.0000000058	0.0000000000	0.0000000000
21	0.0000280882	0.0000003368	0.0000000019	0.0000000000	0.0000000000
22	0.0000145027	0.0000001408	0.0000000006	0.0000000000	0.0000000000
23	0.0000074881	0.0000000589	0.0000000002	0.0000000000	0.0000000000
24	0.0000038663	0.0000000246	0.0000000001	0.0000000000	0.0000000000
25	0.0000019963	0.0000000103	0.0000000000	0.0000000000	0.0000000000
26	0.0000010307	0.0000000043	0.0000000000	0.0000000000	0.0000000000
27	0.0000005322	0.0000000018	0.0000000000	0.0000000000	0.0000000000
28	0.0000002748	0.0000000008	0.0000000000	0.0000000000	0.0000000000
29	0.0000001419	0.0000000003	0.0000000000	0.0000000000	0.0000000000
30	0.0000000733	0.0000000001	0.0000000000	0.0000000000	0.0000000000
31	0.0000000378	0.0000000001	0.0000000000	0.0000000000	0.0000000000
32	0.0000000195	0.0000000000	0.0000000000	0.0000000000	0.0000000000
33	0.0000000101	0.0000000000	0.0000000000	0.0000000000	0.0000000000
34	0.0000000052	0.0000000000	0.0000000000	0.0000000000	0.0000000000
35	0.0000000027	0.0000000000	0.0000000000	0.0000000000	0.0000000000
36	0.0000000014	0.0000000000	0.0000000000	0.0000000000	0.0000000000
37	0.0000000007	0.0000000000	0.0000000000	0.0000000000	0.0000000000
38	0.0000000004	0.0000000000	0.0000000000	0.0000000000	0.0000000000
39	0.0000000002	0.0000000000	0.0000000000	0.0000000000	0.0000000000
40	0.0000000001	0.0000000000	0.0000000000	0.0000000000	0.0000000000
41	0.0000000001	0.0000000000	0.0000000000	0.0000000000	0.0000000000
42	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000

Tabel 2.

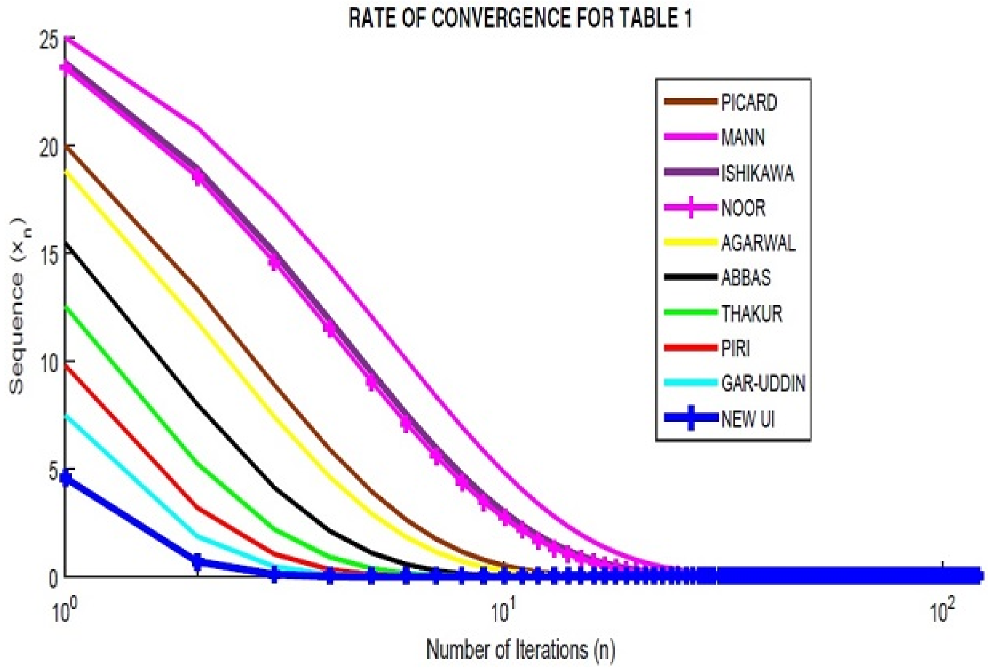
n	THAKUR	AGARWAL	GAR-UDDIN	ISHIKAWA	NEW UI
0	30.000000000	30.000000000	30.000000000	30.000000000	30.000000000
1	23.444444444	-22.444444444	-15.888888888	16.888888888	1.5925925926
2	18.345679012	18.345679012	9.6049382716	9.6049382716	0.5404663923
3	14.379972565	-13.379972565	-4.5582990398	5.5582990398	0.5014987553
4	11.295534217	11.295534217	3.3101661332	3.3101661332	0.5000555095
5	8.8965266135	-7.8965266135	-1.0612034073	2.0612034073	0.5000020559
6	7.0306318105	7.0306318105	1.3673352263	1.3673352263	0.5000000761
7	5.5793802970	-4.5793802970	0.0181470965	0.9818529035	0.5000000028
8	4.4506291199	4.4506291199	0.7676960575	0.7676960575	0.5000000001
9	3.5727115377	-2.5727115377	0.3512799681	0.6487200319	0.5000000000
10	2.8898867516	2.8898867516	0.5826222400	0.5826222400	0.5000000000
11	2.3588008068	-1.3588008068	0.4540987556	0.5459012444	0.5000000000
12	1.9457339608	1.9457339608	0.5255006913	0.5255006913	0.5000000000
13	1.6244597473	-0.6244597473	0.4858329493	0.5141670507	0.5000000000
14	1.3745798035	1.3745798035	0.5078705837	0.5078705837	0.5000000000
15	1.1802287360	-0.1802287360	0.4956274535	0.5043725465	0.5000000000
16	1.0290667947	1.0290667947	0.5024291925	0.5024291925	0.5000000000
17	0.9114963959	0.0885036041	0.4986504486	0.5013495514	0.5000000000
18	0.8200527523	0.8200527523	0.5007497508	0.5007497508	0.5000000000
19	0.7489299185	0.2510700815	0.4995834718	0.5004165282	0.5000000000
20	0.6936121588	0.6936121588	0.5002314046	0.5002314046	0.5000000000
21	0.6505872346	0.3494127654	0.4998714419	0.5001285581	0.5000000000
22	0.6171234047	0.6171234047	0.5000714212	0.5000714212	0.5000000000
23	0.5910959814	0.4089040186	0.4999603216	0.5000396784	0.5000000000
24	0.5708524300	0.5708524300	0.5000220436	0.5000220436	0.5000000000
25	0.5551074456	0.4448925544	0.4999877536	0.5000122464	0.5000000000
26	0.5428613466	0.5428613466	0.5000068036	0.5000068036	0.5000000000
27	0.5333366029	0.4666633971	0.4999962202	0.5000037798	0.5000000000
28	0.5259284689	0.5259284689	0.5000020999	0.5000020999	0.5000000000
29	0.5201665869	0.4798334131	0.4999988334	0.5000011666	0.5000000000
30	0.5156851232	0.5156851232	0.5000006481	0.5000006481	0.5000000000
31	0.5121995402	0.4878004598	0.4999996399	0.5000003601	0.5000000000
32	0.5094885313	0.5094885313	0.5000002000	0.5000002000	0.5000000000
33	0.5073799688	0.4926200312	0.4999998889	0.5000001111	0.5000000000
34	0.5057399757	0.5057399757	0.5000000617	0.5000000617	0.5000000000
35	0.5044644256	0.4955355744	0.4999999657	0.5000000343	0.5000000000
36	0.5034723310	0.5034723310	0.5000000191	0.5000000191	0.5000000000
37	0.5027007019	0.4972992981	0.4999999894	0.5000000106	0.5000000000
38	0.5021005459	0.5021005459	0.5000000059	0.5000000059	0.5000000000
39	0.5016337579	0.4983662421	0.4999999967	0.5000000033	0.5000000000
40	0.5012707006	0.5012707006	0.5000000018	0.5000000018	0.5000000000
41	0.5009883227	0.4990116773	0.4999999990	0.5000000010	0.5000000000
42	0.5007686954	0.5007686954	0.5000000006	0.5000000006	0.5000000000

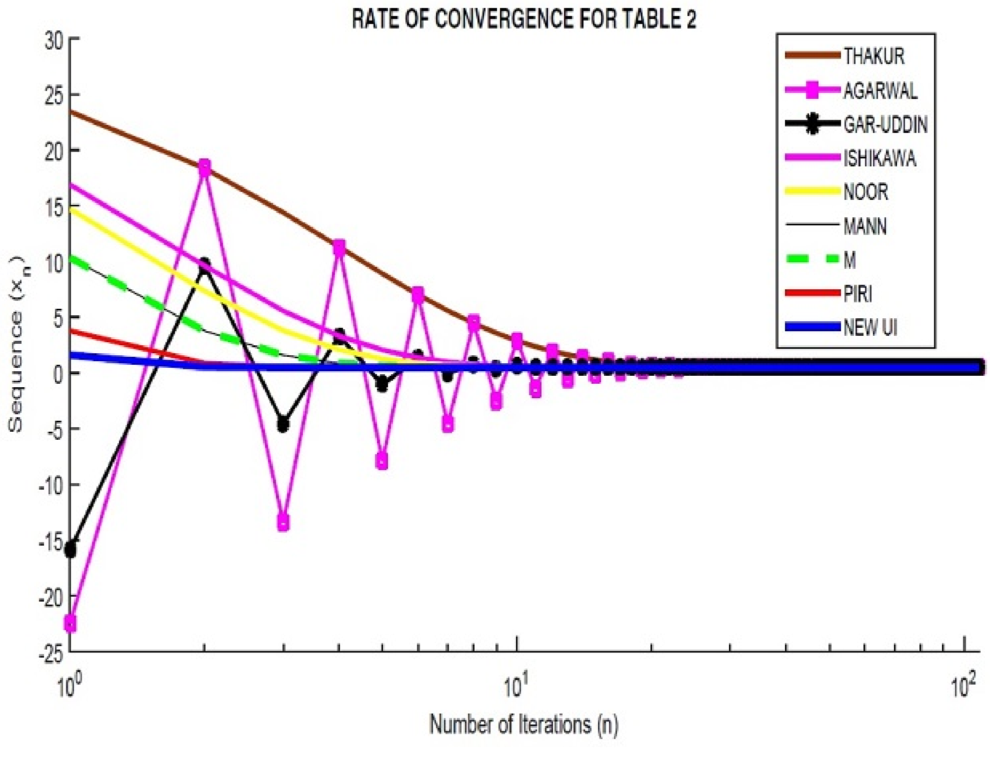
Tabel 2. CONTD.

n	THAKUR	AGARWAL	GAR-UDDIN	ISHIKAWA	NEW UI
43	0.5005978742	0.4994021258	0.4999999997	0.5000000003	0.5000000000
44	0.5004650133	0.5004650133	0.5000000002	0.5000000002	0.5000000000
45	0.5003616770	0.4996383230	0.4999999999	0.5000000001	0.5000000000
46	0.5002813043	0.5002813043	0.5000000001	0.5000000000	0.5000000000
47	0.5002187923	0.4997812077	0.5000000000	0.5000000000	0.5000000000
48	0.5001701718	0.5001701718	0.5000000000	0.5000000000	0.5000000000
49	0.5001323558	0.4998676442	0.5000000000	0.5000000000	0.5000000000
50	0.5001029434	0.5001029434	0.5000000000	0.5000000000	0.5000000000
51	0.5000800671	0.4999199329	0.5000000000	0.5000000000	0.5000000000
52	0.5000622744	0.5000622744	0.5000000000	0.5000000000	0.5000000000
53	0.5000484357	0.4999515643	0.5000000000	0.5000000000	0.5000000000
54	0.5000376722	0.5000376722	0.5000000000	0.5000000000	0.5000000000
55	0.5000293006	0.4999706994	0.5000000000	0.5000000000	0.5000000000
56	0.5000227893	0.5000227893	0.5000000000	0.5000000000	0.5000000000
57	0.5000177250	0.4999822750	0.5000000000	0.5000000000	0.5000000000
58	0.5000137861	0.5000137861	0.5000000000	0.5000000000	0.5000000000
59	0.5000107226	0.4999892774	0.5000000000	0.5000000000	0.5000000000
60	0.5000083398	0.5000083398	0.5000000000	0.5000000000	0.5000000000
61	0.5000064865	0.4999935135	0.5000000000	0.5000000000	0.5000000000
62	0.5000050450	0.5000050450	0.5000000000	0.5000000000	0.5000000000
63	0.5000039239	0.4999960761	0.5000000000	0.5000000000	0.5000000000
64	0.5000030519	0.5000030519	0.5000000000	0.5000000000	0.5000000000
65	0.5000023737	0.4999976263	0.5000000000	0.5000000000	0.5000000000
66	0.5000018462	0.5000018462	0.5000000000	0.5000000000	0.5000000000
67	0.5000014360	0.4999985640	0.5000000000	0.5000000000	0.5000000000
68	0.5000011169	0.5000011169	0.5000000000	0.5000000000	0.5000000000
69	0.5000008687	0.4999991313	0.5000000000	0.5000000000	0.5000000000
70	0.5000006756	0.5000006756	0.5000000000	0.5000000000	0.5000000000
71	0.5000005255	0.4999994745	0.5000000000	0.5000000000	0.5000000000
72	0.5000004087	0.5000004087	0.5000000000	0.5000000000	0.5000000000
73	0.5000003179	0.4999996821	0.5000000000	0.5000000000	0.5000000000
74	0.5000002472	0.5000002472	0.5000000000	0.5000000000	0.5000000000
75	0.5000001923	0.4999998077	0.5000000000	0.5000000000	0.5000000000
76	0.5000001496	0.5000001496	0.5000000000	0.5000000000	0.5000000000
77	0.5000001163	0.4999998837	0.5000000000	0.5000000000	0.5000000000
78	0.5000000905	0.5000000905	0.5000000000	0.5000000000	0.5000000000
79	0.5000000704	0.4999999296	0.5000000000	0.5000000000	0.5000000000
80	0.5000000547	0.5000000547	0.5000000000	0.5000000000	0.5000000000
81	0.5000000426	0.4999999574	0.5000000000	0.5000000000	0.5000000000
..	....	....	....	....	....
106	0.5000000002	0.5000000001	0.5000000000	0.5000000000	0.5000000000
107	0.5000000001	0.4999999999	0.5000000000	0.5000000000	0.5000000000
108	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000

Tabel 2 CONTD.

n	NOOR	MANN	M	PIRI	NEW UI
0	30.000000000	30.000000000	30.000000000	30.000000000	30.000000000
1	14.703703703	10.333333333	10.333333333	3.7777777778	1.5925925926
2	7.3388203018	3.7777777778	3.7777777778	0.8641975309	0.5404663923
3	3.7927653305	1.5925925926	1.5925925926	0.5404663923	0.5014987553
4	2.0854055295	0.8641975309	0.8641975309	0.5044962658	0.5000555095
5	1.2633434031	0.6213991770	0.6213991770	0.5004995851	0.5000020559
6	0.8675357126	0.5404663923	0.5404663923	0.5000555095	0.5000000761
7	0.6769616394	0.5134887974	0.5134887974	0.5000061677	0.5000000028
8	0.5852037523	0.5044962658	0.5044962658	0.5000006853	0.5000000001
9	0.5410240289	0.5014987553	0.5014987553	0.5000000761	0.5000000000
10	0.5197523102	0.5004995851	0.5004995851	0.5000000085	0.5000000000
11	0.5095103716	0.5001665284	0.5001665284	0.5000000009	0.5000000000
12	0.5045790678	0.5000555095	0.5000555095	0.5000000001	0.5000000000
13	0.5022047363	0.5000185032	0.5000185032	0.5000000000	0.5000000000
14	0.5010615397	0.5000061677	0.5000061677	0.5000000000	0.5000000000
15	0.5005111117	0.5000020559	0.5000020559	0.5000000000	0.5000000000
16	0.5002460908	0.5000006853	0.5000006853	0.5000000000	0.5000000000
17	0.5001184882	0.5000002284	0.5000002284	0.5000000000	0.5000000000
18	0.5000570499	0.5000000761	0.5000000761	0.5000000000	0.5000000000
19	0.5000274685	0.5000000254	0.5000000254	0.5000000000	0.5000000000
20	0.5000132256	0.5000000085	0.5000000085	0.5000000000	0.5000000000
21	0.5000063679	0.5000000028	0.5000000028	0.5000000000	0.5000000000
22	0.5000030660	0.5000000009	0.5000000009	0.5000000000	0.5000000000
23	0.5000014762	0.5000000003	0.5000000003	0.5000000000	0.5000000000
24	0.5000007108	0.5000000001	0.5000000001	0.5000000000	0.5000000000
25	0.5000003422	0.5000000000	0.5000000000	0.5000000000	0.5000000000
26	0.5000001648	0.5000000000	0.5000000000	0.5000000000	0.5000000000
27	0.5000000793	0.5000000000	0.5000000000	0.5000000000	0.5000000000
28	0.5000000382	0.5000000000	0.5000000000	0.5000000000	0.5000000000
29	0.5000000184	0.5000000000	0.5000000000	0.5000000000	0.5000000000
30	0.5000000089	0.5000000000	0.5000000000	0.5000000000	0.5000000000
31	0.5000000043	0.5000000000	0.5000000000	0.5000000000	0.5000000000
32	0.5000000021	0.5000000000	0.5000000000	0.5000000000	0.5000000000
33	0.5000000010	0.5000000000	0.5000000000	0.5000000000	0.5000000000
34	0.5000000005	0.5000000000	0.5000000000	0.5000000000	0.5000000000
35	0.5000000002	0.5000000000	0.5000000000	0.5000000000	0.5000000000
36	0.5000000001	0.5000000000	0.5000000000	0.5000000000	0.5000000000
37	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
38	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
39	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
40	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
41	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
42	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000





6. Application to constrained convex minimization problem

In this section, we will use UI-iterative scheme (13) to find the solution of a constrained convex minimization problem. Let C be a nonempty closed convex subset of a real Hilbert space H. For any $\xi \in H$, we define the map $P_C : H \to C$ satisfying

$$\|\xi - P_C\xi\| \leqslant \|\xi - \vartheta\| \text{ for all } \vartheta \in C. \tag{74}$$

P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive. Lets consider the following constrained convex minimization problem:

$$\text{minimize } \{f(\xi) : \xi \in C\}, \tag{75}$$

where $f : C \to \Re$ is a real-valued function. The minimization problem (75) is consistent if it has a solution. In this section, we shall use Ω to stand for the solution set of the problem (75). It is worthy of note that if f is (Frechet) differentiable, then the gradient-projection method (GPM) generates a sequence $\{\xi_k\}$ by using the recursive formula:

$$\begin{cases} \xi_0 \in C, \\ \xi_{k+1} = P_C(\xi_k - \lambda \nabla f(\xi_k)), \text{ for all } k \geqslant 1 \end{cases}. \tag{76}$$

In more general form, (76) can be written as

$$\begin{cases} \xi_0 \in C, \\ \xi_{k+1} = P_C(\xi_k - \lambda_k \nabla f(\xi_k)), \text{ for all } k \geqslant 1 \end{cases}, \tag{77}$$

where λ and λ_k are positive real numbers. It is well known that if ∇f is μ-strongly monotone and L-Lipschitzian with $\mu, L > 0$, then the operator $\Gamma = P_C(I - \lambda\nabla f)$ is a contraction. Thus, the sequence $\{\xi_k\}$ in (76) converges in norm to the unique minimizer of (75). From [7, 20], we know that $q \in C$ solve the minimization problem (75) if and only if q solves the following fixed point equation:

$$q = P_C(I - \lambda\nabla f)q, \tag{78}$$

where $\lambda > 0$ is any fixed positive number. The operator $\Gamma = P_C(I - \lambda\nabla f)$ is well known to be nonexpansive (see [7, 20]) and the references therein). Several authors have considered different iterative algorithm for constrained convex minimization problems (see [4, 6, 12, 22] and the references therein). We now give our main results.

Theorem 6.1 Let C be a nonempty closed convex subset of a real Hilbert space H. Supposed that the minimization problem (75) is consistent and let Ω denote the solution set. Supposed that the gradient ∇f is L-Lipschitzian with constant $L > 0$. Let $\{\xi_k\}$ be the sequence generated iteratively by

$$\begin{aligned} \xi_{k+1} &= P_C(I - \lambda\nabla f)y_k, \\ y_k &= P_C(I - \lambda\nabla f)((1 - b_k)P_C(I - \lambda\nabla f)w_k + b_kP_C(I - \lambda\nabla f)z_k), \\ z_k &= P_C(I - \lambda\nabla f)((1 - a_k)P_C(I - \lambda\nabla f)\xi_k + a_kP_C(I - \lambda\nabla f)w_k), \\ w_k &= P_C(I - \lambda\nabla f)((1 - c_k)\xi_k + c_kP_C(I - \lambda\nabla f)\xi_k), \end{aligned}$$

where $\{a_k\}, \{b_k\}, \{c_k\} \subset (0, 1)$ for $\lambda \in (0, \frac{L}{2})$. Then the sequence $\{\xi_k\}$ converges strongly to a minimizer q of (75).

References

[1] M. Abbas, T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, Mat. Vesnik. 66 (2) (2014), 223-234.
[2] R. P. Agarwal, D. ORegan, D. R. Sahu, Iterative construction of fixed points of nearly asymptotically non-expansive mappings, J. Nonlinear Convex Anal. 8 (1) (2007), 61-79.
[3] S. Banach, Fixed point iteration for local strictly pseudocontractive mapping, Proc. Amer. Math. Soc. 113 (1991), 727-731.
[4] A. Bejenaru, M. Postolache, Partially projective algorithm for the split feasibility problem with visualization of the solution set, Symmetry. 12 (4) (2020), 12:608.
[5] V. Berinde, Picard iteration converges faster than Mann iteration for a class of quasicontractive operators, Fixed Point Theory Appl. 2 (2004), 97-105.
[6] G. Cai, Y. Shehu, An iterative algorithm for fixed point problem and convex minimization problem with applications, Fixed Point Theory Appl. (2015), 2015:7.
[7] C. D. Enyi, M. E. Soh, Modified gradient-projection algorithm for solving convex minimization problem in Hilbert spaces, Inter. J. Appl. Math. 44 (2014), 44:3.
[8] C. Garodia, I. Uddin, A new iterative method for solving split feasibility problem, J. Appl. Anal. Compt. 10 (3) (2020), 986-1004.
[9] K. Goebel, W. A. Kirk, Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.
[10] K. Goebel, S. Reich, Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings, Marcel Dekker Inc, New York, 1984.
[11] M. A. Harder, Fixed point theory and stability results for fixed point iteration procedures, PhD thesis, University of Missouri-Rolla, Missouri, United States, 1987.
[12] S. He, Z. Zhao, Strong convergence of a relaxed CQ algorithm for the split feasibility problem, J. Inequal. Appl. (2013), 2013:197.
[13] S. Ishikawa, Fixed point by a new iteration method, Proc. Amer. Math. Soc. 4 (1) (1974), 147-150.
[14] W. R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953), 506-510.
[15] A. A. Mebawondu, O. T. Mewomo, Fixed point results for a new three steps iteration process, Annals. Uni. Craiova. Math. Comput. Sci. Ser. 46 (2) (2019), 298-319.
[16] M. A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251 (2000), 217-229.
[17] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967), 591-597.
[18] M. O. Osilike, D. I. Igbokwe, Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations, Comput. Math. Appl. 40 (2000), 559-567.
[19] H. Piri, B. Daraby, S. Rahrovi, M. Ghasemi, Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces by new faster iteration process, Numer. Algor. 81 (2019), 1129-1148.
[20] Y. Shehu, O. S. Iyiola, C. D. Enyi, Iterative approximation of solutions for constrained convex minimization problem, Arab J. Math. 2 (2013), 393-402.
[21] S. M. Soltuz, T. Grosan, Data dependence for Ishikawa iteration when dealing with contractive like operators, Fixed Point Theory Appl. (2008), 2008:242916.
[22] K. K. Tan, H. K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993), 301-308.
[23] D. Thakur, B. S. Thakur, M. Postolache, A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings, Appl. Math. Comput. 275 (2016), 147-155.
[24] U. E. Udofia, D. I. Igbokwe, Convergence theorems for monotone generalized α-nonexpansive mappings in ordered Banach space by a new four-step iteration process with application, Commun. Nonlinear Anal. 1 (2021), 1-18.
[25] K. Ullah, M. Arshad, New iteration process and numerical reckoning fixed points in Banach spaces, Scientific Bull. (Series A). 79 (2017), 113-122.
[26] K. Ullah, M. Arshad, Numerical reckoning fixed points for Suzuki generalized nonexpansive mappings via new iteration process, Filomat. 32 (2018), 187-196.
[27] X. Weng, Fixed point iteration for local strictly pseudocontractive mapping, Proc. Amer. Math. Soc. 113 (1991), 727-731.
[28] H. K. Xu, Inequality in Banach spaces with applications, Nonlinear Anal. 16 (1991), 1127-1138.