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On equality of complete positivity and complete copositivity of positive map

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Abstract. In this paper we construct a 2-positive map from $\mathcal{M}_4(\mathbb{C})$ to $\mathcal{M}_5(\mathbb{C})$ and state the conditions under which the map is positive and completely positive (copositivity of positive). The construction allows us to create a decomposable map, where the Choi matrix of complete positivity is equal to the Choi matrix of complete copositivity.

Keywords: 2-positivity, Choi matrix, completely positivity.

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1. Introduction

Positive maps are essential in the description of quantum systems. However, characterization of the structure of the set of all positive maps is a challenge in mathematics and mathematical physics. The famous Choi result in [1] affirms that a map ϕ is completely positive if and only if it's Choi matrix C_ϕ is positive definite. The positive map ϕ is completely positive if and only if C_ϕ is positive, otherwise it is not completely positive.

The construction of Choi's map [1–3] and Robertson's map [8, 9] among other indecomposable maps have been used to justify the importance of these maps in their application in quantum mechanics. A family of indecomposable maps for an arbitrary finite dimension $n = 3$ was constructed in [6]. Other construction of indecomposable maps have been given in [5, 7, 11] are in the context of quantum entanglement.

We construct a linear map $\phi_{(\mu,c_1,c_2,c_3)}$ from $\mathcal{M}_4$ to $\mathcal{M}_5$, where $\mu, c_1, c_2, c_3 \in \mathbb{R}^+$ and study its properties of positivity, completely positivity and decomposability.

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By $\mathcal{M}_n$ we denote the set of positive semidefinite matrices of order n; that is, $A \in \mathcal{M}_n$. The identity map on $\mathcal{M}_n(\mathbb{C})$ and the transpose map on $\mathcal{M}_n(\mathbb{C})$ are denoted by $\mathcal{I}_n$ and τ_n respectively. Let A be a $n \times n$ square matrix. A is positive semidefinite if, for any vector x with real components, $\langle x, Ax\rangle \geqslant 0$ for all $x \in \mathbb{R}^n$ or equivalently, A is Hermitian and all its eigenvalues are nonnegative and positive definite if $\langle x, Ax\rangle > 0$ for all $x \neq 0$. A linear map ϕ is from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$ is called positive if $\phi(\mathcal{M}_n(\mathbb{C}))^+ \subseteq \mathcal{M}_m(\mathbb{C})^+$. A linear map ϕ form $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$ is *$k-$positive* if $\mathcal{I}_k \otimes \phi : \mathcal{M}_k \otimes \mathcal{M}_n \longrightarrow \mathcal{M}_k \otimes \mathcal{M}_m$ is positive. On the other hand, a linear map ϕ form $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$ is *k-copositive* if the map $\tau_k \otimes \phi : \mathcal{M}_k \otimes \mathcal{M}_n \longrightarrow \mathcal{M}_k \otimes \mathcal{M}_n$ is positive. A linear map ϕ from A to $\mathbf{B}(\mathcal{H})$ is k-decomposable if there are maps $\phi_1, \phi_2 : A \longrightarrow \mathbf{B}(\mathcal{H})$ such that ϕ_1 is k-positive, ϕ_2 is k-copositive and $\phi = \phi_1 + \phi_2$.

Let $X \in \mathcal{M}_n(\mathbb{C})$ be a positive semidefinite matrix written, $X = \left(x_i x_j^*\right)$, where $x_i = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$ is a column vector and x_j^* is the transpose conjugate(row vector) of x_i. The diagonal elements of the positive semidefinite matrix X given by $x_n\bar{x}_n = |x_n|$ are positive real numbers.

Definition 1.1 Let X be a 4×4 positive semidefinite matrix with complex entries. Let $c_1, c_2, c_3 \in \mathbb{R}^+$, $0 < \mu < 1$ and $r \in \mathbb{N}$. Then we define the positive map $\phi_{(\mu,c_1,c_2,c_3)}$ as follows:

$$\phi_{(\mu,c_1,c_2,c_3)} : \mathcal{M}_4(\mathbb{C}) \longrightarrow \mathcal{M}_5(\mathbb{C})$$

$$X \mapsto \begin{pmatrix} P_1 & -c_1x_1\bar{x}_2 & -c_2x_1\bar{x}_3 & 0 & -\mu x_1\bar{x}_4 \\ -c_1x_2\bar{x}_1 & P_2 & -c_2x_2\bar{x}_3 & -c_3x_2\bar{x}_4 & 0 \\ -c_2x_3\bar{x}_1 & -c_2x_3\bar{x}_2 & P_3 & -c_3x_3\bar{x}_4 & 0 \\ 0 & -c_3x_4\bar{x}_2 & -c_3x_4\bar{x}_3 & P_4 & 0 \\ -\mu x_4\bar{x}_1 & 0 & 0 & 0 & P_5 \end{pmatrix}, \tag{1}$$

where

$$\begin{aligned}
P_1 &= \mu^{-r}(|x_1| + c_1|x_2|\mu^r + c_2|x_3|\mu^r + c_3|x_4|\mu^r),\\
P_2 &= \mu^{-r}(|x_2| + c_1|x_3|\mu^r + c_2|x_4|\mu^r + c_3|x_1|\mu^r),\\
P_3 &= \mu^{-r}(|x_3| + c_1|x_1|\mu^r + c_2|x_2|\mu^r + c_3|x_3|\mu^r),\\
P_4 &= \mu^{-r}(|x_1| + |x_2| + |x_3| + |x_4|),\\
P_5 &= \mu^{-r}(|x_4| + c_1|x_1|\mu^r + c_2|x_2|\mu^r + c_3|x_4|\mu^r).
\end{aligned}$$

2. Positivity

A linear map ϕ from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$ preserving symmetry is positive if the matrices $\phi(X)$ are positive semidefinite for all positive semidefinite matrices $X \in \mathcal{M}_n(\mathbb{C})$. The linear map ϕ is the image of positive semidefinite matrices of rank 1 if the matrix $x_i x_j^*$ has rank 1. By definition of positive semidefinite matrices, positivity of the map ϕ gives the biquadratic polynomials of $\phi(X)$. The linear map ϕ is uniquely determined by the polynomial function $F(z, x) := z\phi(x_i x_j^*)z^T$ as a biquadratic function in $x := (x_1, ..., x_n)$ and $z := (z_1, ..., z_m)$. The map ϕ is positive if and only if the biquadratic form $F(z, x)$ is a biquadratic function.

Lemma 2.1 Let $0 < \mu < 1$ and $c_1, c_2, c_3 \geqslant 0$. Then the function

$$\begin{aligned}
&F(z_1, z_2, z_3, z_4, z_5, t)\\
&= c_3|t|z_1^2 + (c_3 + c_2|t| - 2\mu^r c_2^2)z_2^2 + (c_1|t| + c_3)z_3^2 + (3\mu^{-r} + \mu^{-r}|t| - 3\mu^r c_3\mathrm{Re}(t)^2)z_4^2\\
&\quad + (c_1 + c_2 + c_3 + |t|\mu^{-r} - \mu^{2+r}\mathrm{Re}(t)^2)z_5^2 + c_1(z_1 - z_2)^2 + c_2(z_1 - z_3)^2 + \frac{\mu^{-r}}{2}(z_3 - 2\mu^r c_2 z_2)^2\\
&\quad + \mu^{-r}(z_2 - 2\mu^r c_3\mathrm{Re}(t)z_4)^2 + \mu^{-r}(z_3 - 2\mu^r c_3\mathrm{Re}(t)z_4)^2 + \mu^{-r}(z_1 - \mu^{1+r}\mathrm{Re}(t)z_5)^2
\end{aligned}$$

is positive semidefinite for every z_1, z_2, z_3, z_4, z_5 and $t \in \mathbb{C}$ whenever it satisfy the inequalities

$$\mu^{-r} \geqslant 2c_3, \tag{2}$$

$$\mu^{-r} \geqslant 2c_1, \tag{3}$$

$$c_1 \geqslant c_2, \tag{4}$$

$$c_1\mu^{-r} \geqslant c_2^2. \tag{5}$$

Proof. If $z_1 = 0$, then

$$\begin{aligned}
&F(0, z_2, z_3, z_4, z_5, t)\\
&= \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r + c_3\mu^r)z_2^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r + c_3\mu^r)z_3^2 + \mu^{-r}(3 + |t|)z_4^2\\
&\quad + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)z_5^2 - 2c_2z_2z_3 - 2c_3\mathrm{Re}(t)z_2z_4 - 2c_3\mathrm{Re}(t)z_3z_4\\
&= \mu^{-r}(1 + c_1\mu^r)z_2^2 + \mu^{-r}(1 + c_1|t|\mu^r)z_3^2 + 3\mu^{-r}z_4^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)z_5^2\\
&\quad + c_2(z_3 - c_2)^2 + c_2(|t| - 1)z_2^2 + c_3(z_2 - \mathrm{Re}(t)z_4)^2 + (\mu^{-r}\frac{|t|}{2} - c_3\mathrm{Re}(t)^2)z_4^2\\
&\quad + c_3(z_3 - \mathrm{Re}(t)z_4)^2 + (\mu^{-r}\frac{|t|}{2} - c_3\mathrm{Re}(t)^2)z_4^2.
\end{aligned}$$

From the coefficients of z_2^2 and z_4^2, we have

$$\begin{aligned}
\mu^{-r} + c_1 + c_2|t| - c_2 &= \mu^{-r} + (c_1 - c_2) + c_2|t|,\\
3\mu^{-r} + \mu^{-r}|t| - 2c_3\mathrm{Re}(t)^2 &= 3\mu^{-r} + \mu^{-r}(|x|^2 + |y|^2) - 2c_3|x|^2,
\end{aligned}$$

respectively. The function $F(0, z_2, z_3, z_4, z_5, t)$ is positive whenever it satisfy the inequalities $\mu^{-r} \geqslant c_2$ and $\mu^{-r} \geqslant 2c_3$.

If $z_2 = 0$, then

$$\begin{aligned}
&F(z_1, 0, z_3, z_4, z_5, t)\\
&= \mu^{-r}(1 + c_1\mu^r + c_2\mu^r + c_3|t|\mu^r)z_1^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r + c_3\mu^r)z_3^2 + \mu^{-r}(3 + |t|)z_4^2\\
&\quad + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)z_5^2 - 2c_2z_1z_3 - 2c_3\mathrm{Re}(t)z_3z_4 - 2\mu\mathrm{Re}(t)z_1z_5\\
&= (c_1 + c_3|t|)z_1^2 + (c_1|t| + c_3)z_3^2 + 3\mu^{-r}z_4^2 + (c_1 + c_2 + c_3)z_5^2 + c_2(z_1 - z_3)^2\\
&\quad + \mu^{-r}(z_3 - \mu^r c_3\mathrm{Re}(t)z_4)^2 + (\mu^{-r}|t| - \mu^r c_3^2\mathrm{Re}(t)^2)z_4^2\\
&\quad + \mu^{-r}(z_1 - \mu^{1+r}\mathrm{Re}(t)z_5)^2 + (\mu^{-r}|t| - \mu^{2+r}\mathrm{Re}(t)^2)z_5^2 \geqslant 0.
\end{aligned}$$

The coefficients of z_4^2 satisfy the inequality

$$\mu^{-2r}(3+|t|) - c_3^2\mathrm{Re}(t)^2 = 3\mu^{-2r} + \mu^{-2r}(|x|^2+|y|^2) - c_3^2|x|^2 \geqslant 0$$

whenever (2) hold.

If $z_3 = 0$, then

$$\begin{aligned}
&F(z_1, z_2, 0, z_4, z_5, t)\\
&= \mu^{-r}(1 + c_1\mu^r + c_2\mu^r + c_3|t|\mu^r)z_1^2 + \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r + c_3\mu^r)z_2^2 + \mu^{-r}(3 + |t|)z_4^2\\
&\quad + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)z_5^2 - 2c_1z_1z_2 - 2c_3\mathrm{Re}(t)z_2z_4 - 2\mu\mathrm{Re}(t)z_1z_5\\
&= (c_2 + c_3|t|)z_1^2 + (c_1 + c_2|t|)z_2^2 + 3\mu^{-r}z_4^2 + (c_1 + c_2 + c_3)z_5^2 + c_1(z_1 - z_2)^2\\
&\quad + (c_3 - c_1)z_2^2 + \mu^{-r}(z_2 - \mu^r c_3\mathrm{Re}(t)z_4)^2 + (\mu^{-r}|t| - \mu^r c_3^2\mathrm{Re}(t)^2)z_4^2\\
&\quad + \mu^{-r}(z_1 - \mu^{1+r}\mathrm{Re}(t)z_5)^2 + (\mu^{-r}|t| - \mu^{2+r}\mathrm{Re}(t)^2)z_5^2 \geqslant 0.
\end{aligned}$$

The coefficients of z_4^2 satisfy the inequality (2).

If $z_4 = 0$, then

$$\begin{aligned}
&F(z_1, z_2, z_3, 0, z_5, t)\\
&= \mu^{-r}(1 + c_1\mu^r + c_2\mu^r + c_3|t|\mu^r)z_1^2 + \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r + c_3\mu^r)z_2^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r\\
&\quad + c_3\mu^r)z_3^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)z_5^2 - 2c_1z_1z_2 - 2c_2z_1z_3 - 2c_2z_2z_3 - 2\mu\mathrm{Re}(t)z_1z_5\\
&= c_3|t|z_1^2 + c_1z_2^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_3\mu^r)z_3^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)z_5^2\\
&\quad + c_1(z_1 - z_2)^2 + c_2(z_1 - z_3)^2 + \mu^{-r}(z_3 - \mu^r c_2z_2)^2 + (c_2|t| - \mu^r c_2^2)z_2^2\\
&\quad + \mu^{-r}(z_1 - \mu^{1+r}\mathrm{Re}(t)z_5)^2 + (\mu^{-r}|t| - \mu^{2+r}\mathrm{Re}(t)^2)z_5^2 \geqslant 0.
\end{aligned}$$

The function $F(z_1, z_2, z_3, 0, z_5, t)$ is positive if the coefficients of z_2^2 satisfy the inequality (5).

If $z_5 = 0$, then

$$\begin{aligned}
&F(z_1, z_2, z_3, z_4, 0, t)\\
&= \mu^{-r}(1 + c_1\mu^r + c_2\mu^r + c_3|t|\mu^r)z_1^2 + \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r + c_3\mu^r)z_2^2\\
&\quad + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r + c_3\mu^r)z_3^2 + \mu^{-r}(3 + |t|)z_4^2\\
&\quad - 2c_1z_1z_2 - 2c_2z_1z_3 - 2c_2z_2z_3 - 2c_3\mathrm{Re}(t)z_2z_4 - 2c_3Re(t)z_3z_4\\
&= \mu^{-r}(1 + c_3|t|\mu^r)z_1^2 + c_3z_2^2 + \mu^{-r}z_3^2 + 3\mu^{-r}z_4^2 + c_1(z_1 - z_2)^2 + c_2(z_1 - z_3)^2\\
&\quad + c_2(|t|z_2 - z_3)^2 + (c_1|t| - c_2)z_3^2 + \mu^{-r}(z_2 - c_1\mu^r\mathrm{Re}(t)z_4)^2 + (\mu^{-r}\frac{|t|}{2} - \mu^r c_1^2\mathrm{Re}(t)^2)z_4^2\\
&\quad + c_3(z_3 - \mathrm{Re}(t)z_4)^2 + (\mu^{-r}\frac{|t|}{2} - c_3\mathrm{Re}(t)^2)z_4^2.
\end{aligned}$$

The function $F(z_1, z_2, z_3, z_4, 0, t)$ is positive whenever the coefficients of z_3^2 satisfy (4) while the coefficients z_4^2 satisfy the inequalities (2) and (3).

Now let $z_i \neq 0, i = 1, 2, 3, 4, 5$ and assume that there exist $z_1, z_2, z_3, z_4, z_5 \in \mathbb{R}$ and $t \in \mathbb{C}$ such that $z_1 \neq 0$ and $F(z_1, z_2, z_3, z_4, z_5, t) < 0$. Since $0 < \mu < 1$ and $c_1, c_2 \geqslant 0$, then

$$
\begin{aligned}
&F(z_1, z_2, z_3, z_4, z_5, t)\\
&= \mu^{-r}(1 + c_1\mu^r + c_2\mu^r + c_3|t|\mu^r)z_1^2 + \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r + c_3\mu^r)z_2^2\\
&\quad + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r + c_3\mu^r)z_3^2 + \mu^{-r}(3 + |t|)z_4^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r + c_3\mu^r)z_5^2\\
&\quad - 2c_1z_1z_2 - 2c_2z_1z_3 - 2c_2z_2z_3 - 2c_3\mathrm{Re}(t)z_2z_4 - 2c_3Re(t)z_3z_4 - 2\mu\mathrm{Re}(t)z_1z_5\\
&= c_3|t|z_1^2 + \mu^{-r}z_2^2 + \mu^{-r}z_3^2 + 3\mu^{-r}z_4^2 + (c_1 + c_2 + c_3)z_5^2 + c_1(z_1 - z_2)^2 + c_2(c_1 - c_3)^2\\
&\quad + c_2(|t|z_2 - z_3)^2 + (c_1|t| - c_2)z_3^2 + c_3(z_2 - \mathrm{Re}(t)z_4)^2 + (\mu^{-r}\frac{|t|}{2} - c_3\mathrm{Re}(t)^2)z_4^2\\
&\quad + c_3(z_3 - \mathrm{Re}(t)z_4)^2 + (\mu^{-r}\frac{|t|}{2} - c_3\mathrm{Re}(t)^2)z_4^2\\
&\quad + \mu^{-r}(z_1 - \mu^{1+r}\mathrm{Re}(t)z_5)^2 + (|t|\mu^{-r} - \mu^{2+r}\mathrm{Re}(t)^2)z_5^2 < 0.
\end{aligned}
$$

The function $F(z_1, z_2, z_3, z_4, z_5, t) < 0$ is a contradiction when the inequalities (2) and (4) hold. Thus, $F(z_1, z_2, z_3, z_4, z_5, t) \geqslant 0$ for every $z_1, z_2, z_3, z_4, z_5 \in \mathbb{R}$ and $t \in \mathbb{C}$ ■

Proposition 2.2 Let $\phi_{(\mu,c_1,c_2,c_3)}$ satisfy the conditions in Lemma 2.1. The linear map $\phi_{(\mu,c_1,c_2,c_3)}$ is positive if $c_1 \geqslant c_2$.

Proof. We need to show that

$$
\phi_{(\mu,c_1,c_2,c_3)}\left(\begin{pmatrix} q\\ s\\ u\\ t\end{pmatrix}\begin{pmatrix}\bar{q} & \bar{s} & \bar{u} & \bar{t}\end{pmatrix}\right) \in \mathcal{M}_5^+
$$

for every $q, s, u, t \in \mathbb{C}$; that is,

$$
\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\\ z_5\end{pmatrix}^T
\begin{pmatrix}
p_1 & -c_1q\bar{s} & -c_2q\bar{u} & 0 & -\mu q\bar{t}\\
-c_1s\bar{q} & p_2 & -c_2s\bar{u} & -c_3s\bar{t} & 0\\
-c_2u\bar{q} & -c_2u\bar{s} & p_3 & -c_3u\bar{t} & 0\\
0 & -c_3t\bar{s} & -c_3t\bar{u} & p_4 & 0\\
-\mu t\bar{q} & 0 & 0 & 0 & p_5
\end{pmatrix}
\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\\ z_5\end{pmatrix} \geqslant 0, \tag{6}
$$

where

$$
\begin{aligned}
p_1 &= \mu^{-r}(|q| + |s|c_1\mu^r + |u|c_2\mu^r + c_3|t|\mu^r),\\
p_2 &= \mu^{-r}(|s| + |u|c_1\mu^r + c_2|t|\mu^r + |q|c_3\mu^r),\\
p_3 &= \mu^{-r}(|u| + c_1|t|\mu^r + |q|c_2\mu^r + |s|c_3\mu^r),\\
p_4 &= \mu^{-r}(|q| + |s| + |u| + |t|),\\
p_5 &= \mu^{-r}(|t| + |q|c_1\mu^r + |s|c_2\mu^r + |u|c_3\mu^r)
\end{aligned}
$$

for every $z_1, z_2, z_3, z_4, z_5 \in \mathbb{R}$ and $q, s, u, t \in \mathbb{C}$.

For q, s and u are not equal to zero, assume that $q = s = u = 1$. Then, by Lemma 2.1,

$$z^T \phi_{(\mu,c_1,c_2,c_3)} \left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ t \end{pmatrix} \begin{pmatrix} 1 \; 1 \; 1 \; \bar{t} \end{pmatrix} \right) z$$

is positive for every $z = (z_1, z_2, z_3, z_4, z_5) \in \mathbb{R}^5$ and $t \in \mathbb{C}$. Taking $q = s = u = 0$, we have

$$c_3|t|z_1^2 + c_2|t|z_2^2 + c_1|t|z_3^2 + \mu^{-r}|t|z_4^2 + \mu^{-r}|t|z_5^2 \geqslant 0.$$

If $u = 0$ and $0 < \mu < 1$, then

$$\begin{aligned} &\mu^{-r}(1 + c_1\mu^r + c_3|t|\mu^r)z_1^2 + \mu^{-r}(1 + c_2|t|\mu^r + c_3\mu^r)z_2^2 + (c_1|t| + c_2 + c_3)z_3^2 \\ &\quad + \mu^{-r}(2 + |t|)z_4^2 + \mu^{-r}(|t| + c_1\mu^r + c_2\mu^r)z_5^2 - 2c_1z_1z_2 - 2c_3\mathrm{Re}(t)z_2z_4 - 2\mu\mathrm{Re}(t)z_1z_5 \\ &= c_3|t|z_1^2 + \mu^{-r}(1 + c_2|t|)z_2^2 + (c_1|t| + c_2 + c_3)z_3^2 + 2\mu^{-r}z_4^2 + (c_1 + c_2)z_5^2 \\ &\quad + c_1(z_1 - z_2)^2 + (\mu^{-r} - c_1)z_2^2 + c_3(z_2 - \mathrm{Re}(t)z_4)^2 + (\mu^{-r}|t| - c_3\mathrm{Re}(t)^2)z_4^2 \\ &\quad + \mu^{-r}(z_1 - \mu^{1+r}\mathrm{Re}(t)z_5)^2 + (|t|\mu^{-r} - \mu^{2+r}\mathrm{Re}(t)^2)z_5^2 \end{aligned}$$

is positive when the inequalities (2) and (3) are satisfied.

Let $s = 0$. Since $0 < \mu < 1$, then

$$\begin{aligned} &\mu^{-r}(1 + c_2\mu^r + c_3|t|\mu^r)z_1^2 + (c_1 + c_2|t| + c_3)z_2^2 + \mu^{-r}(1 + c_1|t|\mu^r + c_2\mu^r)z_3^2 \\ &\quad + \mu^{-r}(2 + |t|)z_4^2 + \mu^{-r}(|t| + c_1\mu^r + c_3\mu^r)z_5^2 - 2z_2z_3c_2 - 2z_3z_4c_3\mathrm{Re}(t) - 2z_1z_5\mu\mathrm{Re}(t) \\ &= c_3|t|z_1^2 + (c_1 + c_2|t| + c_3)z_2^2 + c_1|t|z_3^2 + 2\mu^{-r}z_4^2 + (c_1 + c_3)z_5^2 + c_2(z_1 - z_3)^2 \\ &\quad + \mu^{-r}(z_3 - \mu^r c_3\mathrm{Re}(t)z_4)^2 + (\mu^{-r}|t| - \mu^r c_3^2\mathrm{Re}(t)^2)z_4^2 \\ &\quad + \mu^{-r}(z_1 - \mu^{1+r}\mathrm{Re}(t)z_5)^2 + (|t|\mu^{-r} - \mu^{2+r}\mathrm{Re}(t)^2)z_5^2 \end{aligned}$$

is positive when the inequality (4) hold.

If $q = 0$ and $0 < \mu < 1$, then

$$\begin{aligned} &(c_1 + c_2 + c_3|t|)z_1^2 + \mu^{-r}(1 + c_1\mu^r + c_2|t|\mu^r)z_2^2 + \mu^{-r}(1 + c_1\mu^r + c_3\mu^r)z_3^2 \\ &\quad + \mu^{-r}(2 + |t|)z_4^2 + \mu^{-r}(|t| + c_2\mu^r + c_3\mu^r)z_5^2 - 2c_2z_2z_3 - 2c_3\mathrm{Re}(t)z_2z_4 - 2c_3\mathrm{Re}(t)z_3z_4 \\ &= (c_1 + c_2 + c_3|t|)z_1^2 + c_1z_2^2 + \mu^{-r}z_3^2 + 2\mu^{-r}z_4^2 + \mu^{-r}(|t| + c_2\mu^r + c_3\mu^r)z_5^2 \\ &\quad + c_2(|t|z_2 - z_3)^2 + (\frac{c_1}{c_2} - 1)z_2^2 + \mu^{-r}(z_2 - \mu^r c_3\mathrm{Re}(t)z_4)^2 \\ &\quad + (\mu^{-r}\frac{|t|}{2} - \mu^r c_3^2\mathrm{Re}(t)^2)z_4^2 + c_3(z_3 - \mathrm{Re}(t)z_4)^2 + (\mu^{-r}\frac{|t|}{2} - c_3\mathrm{Re}(t)^2)z_4^2 \end{aligned}$$

is positive when inequalities (2) and $c_1 \geqslant c_2$ hold. ■

3. Completely positivity

The tensor product of positive semidefinite matrices $\mathcal{M}_n$ and $\mathcal{M}_{n+1}$ is isomorphic to the block matrices $\mathcal{M}_n(\mathcal{M}_{n+1})$.

$$\mathcal{M}_n \otimes \mathcal{M}_{n+1} \cong \mathcal{M}_n(\mathcal{M}_{n+1}) \cong \mathcal{M}_2(\mathcal{M}_{m+1}) \text{ for some } n \in \mathbb{N}.$$

By the isomorphism and canonical shuffling we present the structure of the Choi matrix $C_{\phi_{(\mu,c_1,\ldots,c_n)}}$ as

$$C_\phi = \left(\begin{array}{cc|cc} a & C_{1\times m} & 0 & Y_{1\times m} \\ C^*_{m\times 1} & B_{m\times m} & Z^*_{m\times 1} & T_{m\times m} \\ \hline 0 & Z_{1\times m} & d & F_{1\times m} \\ Y^*_{m\times 1} & T^*_{m\times m} & F^*_{m\times 1} & U_{m\times m} \end{array}\right), \tag{7}$$

where a, d are positive real numbers while B, U and T are positive semidefinite matrices in $\mathcal{M}_m$ and C, Y, Z are vectors in $\mathbb{C}^m$. By $\bar{c}_{ij}$ we denote the conjugate of $c_{ij} \in \mathbb{C}$ while conjugate transpose of a matrix C is denoted by C^*. Recall a classical result,

Theorem 3.1 [4] Let S be an invertible matrix. The self-adjoint block matrix $M = \begin{pmatrix} S & P \\ P^* & Q \end{pmatrix}$

(i) is positive if and only if S is positive and $P^*S^{-1}P \leqslant Q$.
(ii) $\det M = (\det S)\det(Q - P^*S^{-1}P)$.

Remark 1 *For* $M = \begin{pmatrix} s & \vec{p} \\ \vec{p}^* & Q \end{pmatrix}$,

$$\det M = (\det S)\det(Q - P^*S^{-1}P) = s\det(Q - \vec{p}^* s^{-1}\vec{p}) \geqslant 0$$

if and only if $sQ - \vec{\vec{p}}^*\vec{p} \geqslant 0$.

The Choi result in [1] shows that a positive map is completely positive if and only if the Choi matrix is positive semidefinite. We look at the conditions for 2-positive, complete positivity and complete copositivity of this map by applying Remark *1* the matrix (7). Then Propositions 3.1 and 3.2 in [12] can be stated as the following:

Proposition 3.2 Let $\phi : \mathcal{M}_n \longrightarrow \mathcal{M}_{n+1}$ be a 2-positive map with the Choi matrix of the form (7). ϕ is completely positive if the following conditions hold:

(i) $Z = 0$;
(ii) $C^*C \leqslant aB$;
(iii) $Y^*Y \leqslant aU$;
(iv) if B is invertible, then $T^*B^{-1}T \leqslant U$.

Proposition 3.3 Let $\phi : \mathcal{M}_n \longrightarrow \mathcal{M}_{n+1}$ be a 2-positive map with the Choi matrix of the form (7). ϕ is completely copositive if the following conditions hold.

(i) $Y = 0$;
(ii) $CC^* \leqslant aB$;
(iii) $ZZ^* \leqslant aU$;
(iv) if B is invertible, then $TB^{-1}T^* = U$.

Remark 2

(1) The transposition in this case implies the partial positive transpose of the Choi

matrix $C_\phi \in \mathcal{M}_n(\mathcal{M}_{n+1})$. The transposition is operated with respect to the blocks $\mathcal{M}_n$. This leads to the partial positive transpose Choi matrix $C_\phi^\Gamma \in \mathcal{M}_n(\mathcal{M}_{n+1})$ with the structure

$$C_\phi^\Gamma = \left(\begin{array}{cc|cc} a & C^*_{1\times m} & 0 & Z^*_{1\times m} \\ C_{m\times 1} & B_{m\times m} & Y_{m\times 1} & T^*_{m\times m} \\ \hline 0 & Y^*_{1\times m} & d & F_{1\times m} \\ Z_{m\times 1} & T_{m\times m} & F^*_{m\times 1} & U_{m\times m} \end{array}\right) \in \mathcal{M}_n(\mathcal{M}_{n+1}). \tag{8}$$

(2) *The proof of $F^*F \geqslant dU$ which we include in our paper follows from the proof of Remark 1 of Theorem 3.1.*

3.1 *Completely positivity of $\phi_{(\mu,c_1,c_2,c_3)}$*

Proposition 3.4 Let $\phi_{(\mu,c_1,c_2,c_3)}$ be a positive map given by (1). Then the following conditions are equivalent:

(i) $\phi_{(\mu,c_1,c_2,c_3)}$ is completely positive.
(ii) $\phi_{(\mu,c_1,c_2,c_3)}$ is 2-positive.

Proof. *(i)* $\Rightarrow$ *(ii)*. Assume $\phi_{(\mu,c_1,c_2,c_3)}$ is 2-positive. Consider a rank one matrix $P = [x_ix_j]$ a positive element in $\mathcal{M}_2(\mathcal{M}_5(\mathbb{C}))$, where $x_i = (1,1,0,0,1,1,0,0,1,1)^T$. We have

$$\mathcal{I}_2 \otimes \phi_{(\mu,c_1,c_2,c_3)}(P) = \left(\begin{array}{ccccc|ccccc} \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & -c_1 & -c_2 & \cdot & -\mu \\ \cdot & c_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot \\ -c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & -c_2 & -c_3 & \cdot \\ -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \mu^{-r} & -c_3 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & -c_3 & \mu^{-r} & \cdot \\ -\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} \end{array}\right) \tag{9}$$

in $\mathcal{M}_2(\mathcal{M}_5(\mathbb{C}))$, where zeros are replaced by dots.

Since $\phi_{(\mu,c_1,c_2,c_3)}$ is 2-positive, the above matrix is positive definite. Therefore,

$$\begin{vmatrix} \mu^{-r} & -c_1 & -c_2 & \cdot & \mu \\ -c_1 & \mu^{-r} & -c_2 & -c_3 & \cdot \\ -c_2 & -c_2 & \mu^{-r} & -c_3 & \cdot \\ \cdot & -c_3 & -c_3 & \mu^{-r} & \cdot \\ \mu & \cdot & \cdot & \cdot & \mu \end{vmatrix} \geqslant 0 \tag{10}$$

and

$$\mu^{-r} > c_1,\ \mu^{-r} > c_2 \text{ and } \mu^{-r} \geqslant 2c_3 \tag{11}$$

hold.

The Choi matrix $C_{\phi_{(\mu,c_1,c_2,c_3)}}$ is

$$\left(\begin{array}{c|ccccccccc||c|ccccccccc}
\mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu \\
\hline
0 & c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_1 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_2 & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\
-\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r}
\end{array}\right).$$

Clearly, Z is a zero matrix and $a, d \in \mathbb{R}^+$ while B and U are positive matrices. Since $a \geqslant 0$, we have

$$aB - C^*C = \begin{pmatrix}
c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu^{-2r} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_1\mu^{-r} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_1\mu^{-r} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu^{-2r} - c_1^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_3\mu^{-r} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-2r} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2\mu^{-r}
\end{pmatrix},$$

which inequality $aB - C^*C \geqslant 0$ hold when $\mu^{-r} > c_1$. Next,

$$dU = c_2 \begin{pmatrix}
c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\
0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 \\
0 & -c_3 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r}
\end{pmatrix}.$$

The inequality $dU \geqslant 0$ holds when $\mu^{-r} > c_3$. For

$$aU - Y^*Y = \begin{pmatrix} c_1\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-2r} - c_2^2 & 0 & 0 & 0 & 0 & 0 & -c_3\mu^{-r} & 0 \\ 0 & 0 & \mu^{-2r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3\mu^{-r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2\mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1\mu^{-r} & 0 & 0 \\ 0 & -c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 & \mu^{-2r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-2r} - \mu^2 \end{pmatrix}.$$

the matrix $aU - Y^*Y$ is positive when $\mu^{-2r} > c_2^2 + c_3^2$ holds. Since $c_3 \geqslant c_1 \geqslant c_2$, the maximum value of $c_2^2 + c_3^2$ is attained when $c_3 = c_2$. Therefore, $\mu^{-2r} > c_2^2 + c_3^2 \leqslant c_3^2 + c_3^2 = 2c_3^2$, where $\mu^{-r} \geqslant 2c_3$.

$$U - T^*BT = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1 - c_2^2\mu^r - c_3^2\mu^r & 0 & 0 \\ 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix}.$$

All the principal minors of $U - TB^{-1}T$ are positive when $c_1\mu^{-r} - (c_2^2 + c_3^2) > 0$. It is clear that $c_3 \geqslant c_1 \geqslant c_2$, so

$$c_1\mu^{-r} - (c_2^2 + c_3^2) < c_3\mu^{-r} - (c_3^2 + c_3^2) = c_3\mu^{-r} - 2c_3^2 > 0.$$

This implies that $\mu^{-r} > 2c_3$. Hence the set of inequalities (11) are satisfied, consequently $C_{\phi_{(\mu,c_1,c_2,c_3)}}$ is positive semidefinite. Hence, complete positivity of $\phi_{(\mu,c_1,c_2,c_3)}$ follows. ■

3.2 *Completely copositivity of* $\phi_{(\mu,c_1,c_2,c_3)}$

Proposition 3.5 Let $\phi_{(\mu,c_1,c_2,c_3)}$ be a positive map given by (1). The positive map $\phi_{(\mu,c_1,c_2,c_3)}$ is completely copositive if the following conditions holds:

(i) $\phi_{(\mu,c_1,c_2,c_3)}$ is 2-copositive.
(ii) $\phi_{(\mu,c_1,c_2,c_3)}$ is completely copositive.

Proof. Assume $\phi_{(\mu,c_1,c_2,c_3)}$ is 2-copositive. Consider a rank one matrix P an element in

$\mathcal{M}_2(\mathcal{M}_5(\mathbb{C}))$ where $x_i = (1,1,0,0,1,1,0,0,1,1)^T$, we have

$$\tau_2 \otimes \phi_{(\mu,c_1,c_2,c_3)}(P) = \left(\begin{array}{ccccc|ccccc}
\mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & c_3 & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & c_2 & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & c_1 & -\mu & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & -c_1 & -c_2 & \cdot & -\mu & c_1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & -c_2 & -c_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \mu^{-r} & -c_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & -c_3 & \mu^{-r} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r}
\end{array}\right) \tag{12}$$

in $\mathcal{M}_2(\mathcal{M}_5(\mathbb{C}))$. By computation of the minors, $\tau_2 \otimes \phi_{(\mu,c_1,c_2,c_3)}(P)$ is positive semidefinite on condition that

$$\mu^{-r} > c_1, \ \mu^{-r} > c_2, \ \mu^{-r} \geqslant 2c_3, \ c_3 \geqslant c_1 \ \textit{and} \ c_1 \geqslant c_2 \tag{13}$$

hold. The choi matrix $C^{\Gamma}_{\phi_{(\mu,c_1,c_2,c_3)}}$ is

$$\left(\begin{array}{c|ccccccccc||c|ccccccccc}
\mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & c_3 & \cdot & \cdot & \cdot & -c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\mu & \cdot & \cdot & \cdot & \cdot \\
\cdot & -c_1 & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_3 & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline\hline
\cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_2 & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_3 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_3 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -c_3 & \cdot & \cdot & \cdot & c_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mu^{-r}
\end{array}\right).$$

Since $a \geqslant 0$ and $C = 0$, we have

$$aB - CC^* = \mu^{-r} \begin{pmatrix}
c_3 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 \\
0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 \\
-c_1 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2
\end{pmatrix}.$$

The inequality $aB - CC^* \geqslant 0$ hold when $c_3 \geqslant c_1$.

Since F is a zero matrix, $dU - FF^*$ is positive when the inequality $c_1\mu^{-r} > c_3^2$ hold.

$$aU - ZZ^* = \begin{pmatrix} \mu^{-r}c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-2r} - c_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^{-2r} & 0 & 0 & 0 & -c_3\mu^{-r} & 0 & 0 \\ 0 & 0 & 0 & \mu^{-r}c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu^{-r}c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^{-r}c_2 & 0 & 0 & 0 \\ 0 & 0 & -c_3\mu^{-r} & 0 & 0 & 0 & \mu^{-r}c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-2r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-2r} \end{pmatrix}.$$

The matrix is positive when the inequality $c_1\mu^{-r} > c_2^2$ holds.

Finally,

$$U - TB^{-1}T^* = \begin{pmatrix} c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^{-r} & 0 & 0 & 0 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & c_3 & 0 & 0 & \frac{-c_2\mu}{c_1+c_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & -c_3 & 0 & 0 & 0 & \frac{c_1^2+c_1c_3-c_2^2}{c_1+c_3} & 0 & 0 \\ 0 & 0 & 0 & -c_3\mu^{1+r} & 0 & 0 & 0 & \mu^{-r} - c_3^2\mu^r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} \end{pmatrix}.$$

The minors of $U - TB^{-1}T^*$ are nonnegative when the inequalities $\mu^{-r} > c_3$, $c_1 \geqslant c_2$ and $c_1\mu^{-r} > c_3^2$ hold. ■

***Example* 3.6** The linear map $\phi_{(\frac{1}{2},\frac{2}{3},\frac{1}{5},\frac{3}{4})}$ is completely positive and completely copositive for $r \geqslant 1$.

4. Construction of a decomposable map

Given the positive map $\phi : \mathcal{M}_n \longrightarrow \mathcal{M}_{n+1}$. Our aim is to construct a new map $\Phi : \mathcal{M}_n \longrightarrow \mathcal{M}_{n+1}$ by means of the given map ϕ such that $\Phi_1 = \phi(X)$ and $\Phi_2 = \phi(X^T)$. It is clear that the Hermitian conjugation and transposition transform column-vectors into row-vectors and vice-versa.

Definition 4.1 By a merging of the maps Φ_1 and Φ_2, we can define

$$\Phi_{(\mu,c_1,c_2,c_3)} : \mathcal{M}_4(\mathbb{C}) \longrightarrow \mathcal{M}_5(\mathbb{C}),$$

$$X \mapsto \begin{pmatrix} 2P_1 & -c_1(x_1\bar{x}_2+x_2\bar{x}_1) & -c_2(x_1\bar{x}_3+x_3\bar{x}_1) & 0 & -\mu(x_1\bar{x}_4+x_4\bar{x}_1) \\ -c_1(x_2\bar{x}_1+x_1\bar{x}_2) & 2P_2 & -c_2(x_2\bar{x}_3+x_3\bar{x}_2) & -c_3(x_2\bar{x}_4+x_4\bar{x}_2) & 0 \\ -c_2(x_3\bar{x}_1+x_1\bar{x}_3) & -c_2(x_3\bar{x}_2+x_2\bar{x}_3) & 2P_3 & -c_3(x_3\bar{x}_4+x_4\bar{x}_3) & 0 \\ 0 & -c_3(x_4\bar{x}_2+x_2\bar{x}_4) & -c_3(x_4\bar{x}_3+x_3\bar{x}_4) & 2P_4 & 0 \\ -\mu(x_4\bar{x}_1+x_1\bar{x}_4) & 0 & 0 & 0 & 2P_5 \end{pmatrix}, \quad (14)$$

where

$$
\begin{aligned}
P_1 &= \mu^{-r}(|x_1| + c_1|x_2|\mu^r + c_2|x_3|\mu^r + c_3|x_4|\mu^r),\\
P_2 &= \mu^{-r}(|x_2| + c_1|x_3|\mu^r + |x_4|c_2\mu^r + |x_1|c_3\mu^r),\\
P_3 &= \mu^{-r}(|x_3| + c_1|x_1|\mu^r + |x_2|c_2\mu^r + |x_3|c_3\mu^r),\\
P_4 &= \mu^{-r}(|x_1| + |x_2| + |x_3| + |x_4|),\\
P_5 &= \mu^{r}(|x_4| + c_1|x_1|\mu^r + c_2|x_2|\mu^r + c_3|x_4|\mu^r).
\end{aligned}
$$

Proposition 4.2 The linear map $\Phi_{(\mu,c_1,c_2,c_3)}$ is positive.

Proof. Let $b_{ij} = x_i\bar{x}_j + x_j\bar{x}_i$. Then the map (14) reduces to

$$
\Phi(X) = \begin{pmatrix}
2P_1 & -c_1b_{12} & -c_2b_{12} & 0 & -\mu b_{14}\\
-c_1b_{21} & 2P_2 & -c_2b_{23} & -c_3b_{24} & 0\\
-c_2b_{31} & -c_2b_{32} & 2P_3 & -c_3b_{34} & 0\\
0 & -c_3b_{42} & -c_3b_{43} & 2P_4 & 0\\
-\mu b_{41} & 0 & 0 & 0 & 2P_5
\end{pmatrix}.
$$

The proof follows from Proposition 2.2. ■

Proposition 4.3 The linear map $\phi_{(\mu,c_1,c_2,c_3)}$ is 2-positive (2-copositive).

Proof. Let $\Phi_{(\mu,c_1,c_2,c_3)}$ be positive. We have that $\mathcal{I}_2 \otimes \Phi_{(\mu,c_1,c_2,c_3)}(X)$ is the matrix

$$
\left(\begin{array}{ccccc|ccccc}
2\mu^{-r} & . & . & . & . & . & -c_1 & -c_2 & . & -\mu\\
. & 2c_3 & . & . & . & -c_1 & . & . & . & .\\
. & . & 2c_2 & . & . & -c_2 & . & . & . & .\\
. & . & . & 2\mu^{-r} & . & . & . & . & . & .\\
. & . & . & . & 2c_1 & -\mu & . & . & . & .\\
\hline
. & -c_1 & -c_2 & . & -\mu & 2c_1 & . & . & . & .\\
-c_1 & . & . & . & . & . & 2\mu^{-r} & -2c_2 & -2c_3 & .\\
-c_2 & . & . & . & . & . & -2c_2 & 2\mu^{-r} & -2c_3 & .\\
. & . & . & . & . & . & -2c_3 & -2c_3 & 2\mu^{-r} & .\\
-\mu & . & . & . & . & . & . & . & . & 2\mu^{-r}
\end{array}\right). \tag{15}
$$

Since $\phi_{(\mu,c_1,c_2,c_3)}$ is 2-positive, the matrix $\mathcal{I}_2 \otimes \Phi_{(\mu,c_1,c_2,c_3)}(X)$ is positive definite. Therefore,

$$
\mu^{-r} > c_1,\;\; \mu^{-r} > c_2,\;\; \mu^{-r} > c_3,\;\; c_3 \geqslant c_1,\;\; c_1 \geqslant \mu,\;\; c_1 \geqslant c_2
$$

hold. ■

Proposition 4.4 The linear map $\phi_{(\mu,c_1,c_2,c_3)}$ is completely positive (completely copositive).

Proof. The computation of the Choi matrix of the linear map $\Phi_{(\mu,c_1,c_2,c_3)}$ gives

$C_{\Phi_{(\mu,c_1,c_2,c_3)}}$ as

$$\left(\begin{array}{c|ccccccccc||c|ccccccccc}
2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu \\ \hline
0 & 2c_3 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\
0 & -c_1 & 0 & 0 & 0 & 2c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_1 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_3 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\hline
0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 2c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_2 & 0 & 0 & 0 & 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & -c_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 & 2c_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} & 0 \\
-\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r}
\end{array}\right)$$

and

$$aB - C^*C = \begin{pmatrix}
4c_3\mu^{-r} & 0 & 0 & 0 & -2c_1\mu^{-r} & 0 & 0 & 0 & 0 \\
0 & 4c_2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4\mu^{-2r} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4c_1\mu^{-r} & 0 & 0 & 0 & 0 & 0 \\
-2c_1\mu^{-r} & 0 & 0 & 0 & 4c_1\mu^{-r} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4\mu^{-2r} - c_1^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4c_3\mu^{-r} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4\mu^{-2r} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4c_2\mu^{-r}
\end{pmatrix}.$$

The matrix is positive when $2\mu^{-r} > c_1$ and $4c_3 \geqslant c_1$.

$$dU = 2c_2 \begin{pmatrix}
2c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\
0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & -c_3 & 0 & 0 \\
0 & 0 & 0 & 2c_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2c_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2c_2 & 0 & 0 & 0 \\
0 & 0 & -c_3 & 0 & 0 & 0 & 2c_1 & 0 & 0 \\
0 & -c_3 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r}
\end{pmatrix}.$$

$dU \geqslant 0$ is positive when $4\mu^{-r} > c_3$ and $4c_1\mu^{-r} > c_3^2$.

$$aU - Y^*Y = \begin{pmatrix}
4c_1\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4\mu^{-2r} - c_2^2 & 0 & 0 & 0 & 0 & 0 & -2c_3\mu^{-r} & -c_2\mu \\
0 & 0 & 4\mu^{-2r} & 0 & 0 & 0 & -2c_3\mu^{-r} & 0 & 0 \\
0 & 0 & 0 & 4c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4c_3\mu^{-r} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2c_4\mu^{-r} & 0 & 0 & 0 \\
0 & 0 & -2c_3\mu^{-r} & 0 & 0 & 0 & 4c_1\mu^{-r} & 0 & 0 \\
0 & -2c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 4\mu^{-2r} & 0 \\
0 & -c_2\mu & 0 & 0 & 0 & 0 & 0 & 0 & 4\mu^{-2r} - \mu^2
\end{pmatrix}$$

is positive when $2\mu^{-r} > c_2$, $4\mu^{-2r} > c_2^2 + c_3^2$ and $4\mu^{-2r} > c_2^2 + \mu^2$ holds.

$$aU - Z^*Z = \begin{pmatrix} 4c_1\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4\mu^{-2r} - c_2^2 & 0 & 0 & 0 & 0 & 0 & -2c_3\mu^{-r} & 0 \\ 0 & 0 & 4\mu^{-2r} & 0 & 0 & 0 & -2c_3\mu^{-r} & 0 & 0 \\ 0 & 0 & 0 & 4c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4c_3\mu^{-r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_4\mu^{-r} & 0 & 0 & 0 \\ 0 & 0 & -2c_3\mu^{-r} & 0 & 0 & 0 & 4c_1\mu^{-r} & 0 & 0 \\ 0 & -2c_3\mu^{-r} & 0 & 0 & 0 & 0 & 0 & 4\mu^{-2r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4\mu^{-2r} - \mu^2 \end{pmatrix}$$

is positive when $2\mu^{-r} > c_2$ and $4\mu^{-2r} > c_2^2 + c_3^2$ holds.

$$U - TB^{-1}T^* = \begin{pmatrix} 2c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\mu^{-r} & 0 & 0 & 0 & 0 & 0 & -c_3 & 0 \\ 0 & 0 & 2\mu^{-r} & 0 & 0 & 0 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 2c_3 & 0 & 0 & \frac{c_2\mu}{c_1+c_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_2 - \frac{1}{2}(c_2 + c_3\mu^r) & 0 & 0 & 0 \\ 0 & 0 & -c_3 & 0 & 0 & 0 & 2c_1 - \frac{c_2^2}{c_1+c_3} & 0 & 0 \\ 0 & -c_3 & 0 & \frac{1}{2}c_3\mu^{r+1} & 0 & 0 & 0 & 2\mu^{-r} - \frac{1}{2}c_3^2\mu^r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu^{-r} \end{pmatrix}.$$

The minors of $U - TB^{-1}T^*$ are nonnegative when the inequalities $2\mu^{-r} > c_3$, $c_1 \geqslant c_2$ and $4c_1\mu^{-r} > c_3^2$ hold. ■

Proposition 4.5 The linear map $\Phi_{(\mu,c_1,c_2,c_3)}$ is decomposable.

Proof. From Proposition 4.5, $\Phi_{(\mu,c_1,c_2,c_3)}$ is 2-positive (2-copositive) and complete positivity is equivalent to complete copositivity. Observe that the sum of $C_{\phi_{(\mu,c_1,c_2,c_3)}}$ in Proposition 3.4 and $C^{\Gamma}_{\phi_{(\mu,c_1,c_2,c_3)}}$ in Proposition 3.5 is given by $C_{\Phi_{(\mu,c_1,c_2,c_3)}}$. That is, $C^{\Gamma}_{\Phi_{(\mu,c_1,c_2,c_3)}} = C_{\phi_{(\mu,c_1,c_2,c_3)}} + C^{\Gamma}_{\phi_{(\mu,c_1,c_2,c_3)}}$. Therefore, $\Phi_{(\mu,c_1,c_2,c_3)}$ is decomposable with $\phi_{1(\mu,c_1,c_2,c_3)}$ 2-positive and $\phi_{2(\mu,c_1,c_2,c_3)}$ 2-copositive. ■

Proposition 4.6 Let be ϕ linear map in $\mathbf{B}(\mathcal{M}_n(\mathbb{C}), \mathcal{M}_m(\mathbb{C}))$. If the matrix transpose of $[\phi(x_{ij})]$ is equal to $[\phi(x_{ji})]$, then ϕ decomposable.

Proof. Assume that $\mathcal{M}_n \subset \mathbf{B}(\mathcal{S})$ for some Hilbert space $\mathcal{S}$. Let

$$S = \{\begin{pmatrix} x & 0 \\ 0 & x^T \end{pmatrix} \in \mathcal{M}_2(\mathbf{B}(\mathcal{S})) : x \in \mathcal{M}_n\}, \tag{16}$$

where T represents the transposition map with respect to some orthonormal basis in $\mathcal{S}$. Then S is a self-adjoint subspace of $\mathcal{M}_2(\mathbf{B}(\mathcal{S}))$ with the identity. One can observe that both $[x_{ij}]$ and $[x_{ji}]$ are in $\mathcal{M}_k(\mathcal{M}_n)^+$ if and only if

$$\begin{pmatrix} \begin{bmatrix} x11 & 0 \\ 0 & x_{11}^T \end{bmatrix} \cdots \begin{bmatrix} x_{1k} & 0 \\ 0 & x_{1k}^T \end{bmatrix} \\ \vdots \qquad\qquad \vdots \\ \begin{bmatrix} x_{k1} & 0 \\ 0 & x_{k1}^T \end{bmatrix} \cdots \begin{bmatrix} x_{kk} & 0 \\ 0 & x_{kk}^T \end{bmatrix} \end{pmatrix} \in \mathcal{M}_K(\mathcal{S})^+.$$

Therefore, ϕ is k-positive. Since $[\phi(x_{ij})] = [x_{ij}]^T = [x_{ji}] = [\phi(x_{ji})] \in \mathcal{M}_k(\mathcal{A})^+$. By [10, Theorem 1], ϕ is decomposable. ■

Conjecture 4.7 If ϕ is a linear map in $\mathbf{B}(\mathcal{A}, \mathbf{B}(\mathcal{H}))$ and is decomposable to a 2-positive map ϕ_1 and a 2-copositive map ϕ_2 such that $\phi_1, \phi_2 : \mathcal{A} \longrightarrow \mathbf{B}(\mathcal{H})$. Then ϕ_1 is also 2-decomposable whenever C_Φ is equal to C_Φ^Γ.

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