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On lifting acts over monoids

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Abstract. Let A be an S-act where S is a monoid. Then A is called lifting if every proper subact L of A lies over a direct summand, that is, L contains a direct summand K of A such that $K \subset L$ is co-small in A. In this paper, characterizations of lifting S-acts and co-closed subacts are presented. We show that the class of supplemented acts are strictly larger than that of lifting ones.

Keywords: S-act, supplement, supplemented, lifting, co-closed.

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1. Introduction and preliminaries

The study of lifting modules was initiated by Oshiro [7] and then continued in many papers (see, for example, [1–3, 7]). A module M is called lifting if every submodule N of M lies over a direct summand, that is, N contains a direct summand X of M such that N/X is small in M/X. We refer to [1, 6] for basic terminology on lifting modules. Here we extend the notion of lifting to S-acts over a monoid S and give a characterization for such kinds of S-acts. Thereafter, we consider co-closed subacts and characterize them. It is also proved that lifting acts are supplemented but not vice versa.

Let us first recall some preliminaries from [4] about S-acts needed in the sequel.

Throughout S is a monoid unless otherwise stated. A (non-empty) set A is called a (*right*) S*-act* if there is a mapping $\lambda : A \times S \to A$, denoting $\lambda(a, s)$ by as, satisfying $a(st) = (as)t$ and $a1 = a$ for all $a \in A$ and $s, t \in S$. An element $\theta \in A$ is said to be a

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zero element if $\theta s = \theta$ for all $s \in S$. The singleton S-act $\{\theta\}$ is denoted by Θ. Also, for an S-act A, A^θ denotes the S-act $A \cup \Theta$ in which a zero element θ is externally adjoint to A. A (non-empty) subset B of A is called a *subact* of A, denoted as $B \leqslant A$, if $bs \in B$ for every $b \in B$ and $s \in S$. If B is a proper subact of A, then we write $B < A$. Clearly, S is an S-act with its operation as the action. Any set A can be made into an S-act by setting $as = a$ for all $a \in A, s \in S$; this action of S is said to be *trivial*. By a *simple* S-act we mean an S-act with no proper subact. An S-act A is said to be *decomposable* if it is a disjoint union of two proper subacts; otherwise, it is called *indecomposable*.

2. Lifting acts and co-closed subacts

In this section we introduce and characterize the concepts of lifting S-acts and co-closed subacts which are based on the notion of superfluous subacts.

Recall from [5] that a subact B of an S-act A is called *superfluous* if $B \cup C \neq A$ for each proper subact C of A, and it is denoted by $B \leqslant_s A$. Also an S-act A is said to be *hollow* if any proper subact of A is superfluous.

Definition 2.1 Given subacts $K < N < A$ of an S-act A, the inclusion $K \subset N$ is called *co-small* in A if $N/K \leqslant_s A/K$.

We say that a subact A_1 of an S-act A is a *direct summand* of A in case there is a subact A_2 of A with $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. In this case, we write $A = A_1 \dot{\cup} A_2$.

Definition 2.2 An S-act A is called *lifting* if every proper subact L of A lies over a direct summand, that is, L contains a direct summand K of A such that $K \subset L$ is co-small in A.

The following observations are directly obtained from the definition of a lifting S-act:
(1) Every simple act is lifting. In particular, every group is a lifting act over itself.
(2) Any non-simple lifting act is decomposable.
(3) A lifting S-act A contains no minimal proper subact and so A has no zero element. As a consequence, all proper subacts of a non-simple lifting act are infinite.

Theorem 2.3 (Characterization of lifting S-acts) An S-act A is lifting if and only if every proper subact N of A can be written as $N = N_1 \dot{\cup} N_2$ with N_1 a direct summand of A and $N_2 \leqslant_s A$.

Proof. Let A be a lifting S-act and N be a proper subact of A. Then there exists a proper subact X of N with $A = X \dot{\cup} X'$ for some $X' \leqslant A$, and $N/X \leqslant_s A/X$. Let $\varphi : A/X \to X' \dot{\cup} \Theta$ be the obvious isomorphism. Then it follows from [5, Lemma 2.4(iii)] that

$$\varphi(N/X) = (N \cap X') \dot{\cup} \Theta \leqslant_s X' \dot{\cup} \Theta.$$

Now, by [5, Lemma 2.3(ii)], we have $(N \cap X')^\theta \leqslant_s A^\theta$ and hence $\emptyset \neq N \cap X' \leqslant_s A$. Since $N = X \dot{\cup} (N \cap X')$, taking $N_1 = X$ and $N_2 = N \cap X'$ gives the assertion. For the converse, let N be a proper subact of A. By the assumption, $N = N_1 \dot{\cup} N_2$ such that $A = N_1 \dot{\cup} X$ for some subact X of A and $N_2 \leqslant_s A$. We claim that $N_1 \subset N$ is co-small in A. To this end, first note that $N/N_1 = (N_1 \dot{\cup} N_2)/N_1 = \Theta \dot{\cup} N_2$ and $A/N_1 = (N_1 \dot{\cup} X)/N_1 = \Theta \dot{\cup} X$. It must be shown that $\Theta \dot{\cup} N_2 \leqslant_s \Theta \dot{\cup} X$. Clearly, it suffices to show that $N_2 \leqslant_s X$. Let $N_2 \cup Y = X$ for $Y \leqslant X$. Then $(N_2 \cup Y) \dot{\cup} N_1 = X \dot{\cup} N_1 = A$. Since $N_2 \leqslant_s A$, $Y \dot{\cup} N_1 = A$. This implies that $Y = X$, as desired. Hence, A is lifting. ■

Corollary 2.4 A is a lifting S-act if and only if for every proper subact N of A, $A = A_1 \dot{\cup} A_2$ for some $A_1 < N$ and $A_2 < A$ with $\emptyset \neq N \cap A_2 \leqslant_s A$.

Proof. Let A be a lifting S-act and N be a proper subact of A, then there exists a proper subact X of N such that $A = X \dot{\cup} X'$, for some $X' < A$, and $N/X \leqslant_s A/X$. Now by the proof of Theorem 2.3, $\emptyset \neq N \cap X' \leqslant_s A$. By setting $A_1 := X$ and $A_2 := X'$, the result follows. Conversely, let N be a proper subact of A. Using the assumption, $A = A_1 \dot{\cup} A_2$ for some $A_1 < N$ and $A_2 < A$ with $\emptyset \neq N \cap A_2 \leqslant_s A$. Putting $N_1 = A_1$ and $N_2 = N \cap A_2$, we have $N = N_1 \dot{\cup} N_2$, N_1 is a direct summand of A and $N_2 \leqslant_s A$. Now, applying Theorem 2.3, A is lifting. ■

Definition 2.5 A subact N of an S-act A is said to be *co-closed* in A provided that N has no proper subact K which $K \subset N$ is co-small in A.

Clearly, a lifting S-act has no co-closed proper subact.

Remark 1 *Let A be an S-act and $N < A$. If N is co-closed in A and N has a zero element with $|N| > 1$, then A is not hollow. Indeed, $\Theta \subset N$ is not co-small so that $N = N/\Theta \not\leqslant_s A/\Theta = A$.*

Theorem 2.6 (Characterization of co-closed subacts) L is a co-closed subact of an S-act A if and only if for any proper subact K of L, there is a subact N of A such that $L \cup N = A$ implies $K \cup N \neq A$.

Proof. Let L be a co-closed subact of A and $K < L$ such that for any $N \leqslant A$, if $L \cup N = A$, then $K \cup N = A$. We claim that $L/K \leqslant_s A/K$. Let $(L/K) \cup (N/K) = A/K$ where $K \leqslant N \leqslant A$. Then $(L \cup N)/K = A/K$ and so $L \cup N = A$. Using the assumption, $K \cup N = A$. Then $N = A$ and so $N/K = A/K$. Therefore, $K \subset L$ is co-small in A, which is a contradiction. Conversely, assume that L satisfies the mentioned condition. We show that L is co-closed. On the contrary, let $K < L$ be co-small in A and so $L/K \leqslant_s A/K$. Let $N \leqslant A$ and $L \cup N = A$. Then

$$A/K = (L \cup N)/K = (L \cup K \cup N)/K = (L/K) \cup ((K \cup N)/K).$$

Now from $L/K \leqslant_s A/K$ we conclude that $(K \cup N)/K = A/K$ and hence $K \cup N = A$, which contradicts the assumption. ■

In light of the above theorem, the following is immediate.

Corollary 2.7 Any direct summand of an S-act A is co-closed in A.

Recall from [5] that the *radical* of an S-act A, denoted as $\mathrm{Rad}(A)$, is the intersection of all maximal subacts of A, and if A contains no maximal subact, we put $\mathrm{Rad}(A) = A$. By [5, Proposition 4.6], $\mathrm{Rad}(A) = \bigcup\{B \mid B \leqslant_s A\}$.

Theorem 2.8 Let A be an S-act and $K \leqslant L \leqslant A$. Then the following assertions hold:
(i) If L is co-closed in A, then L/K is co-closed in A/K.
(ii) If L is co-closed in A, then $K \leqslant_s A$ implies $K \leqslant_s L$ and so $\mathrm{Rad}(L) = L \cap \mathrm{Rad}(A)$.
(iii) If L is hollow, then either L is co-closed in A or $L \leqslant_s A$.
(iv) If K is co-closed in A, then K is co-closed in L.

Proof. (i) Suppose there exists a proper subact N of L such that $N/K \subset L/K$ is co-small in A/K. Then

$$L/N \cong \frac{L/K}{N/K} \leqslant_s \frac{A/K}{N/K} \cong A/N,$$

and so $L/N \leqslant_s A/N$, showing that $N \subset L$ is co-small in A, which contradicts the assumption.

(ii) Let $K \leqslant_s A$ and $K \cup K' = L$ for some $K' \leqslant L$. Choose $K' \leqslant L' \leqslant A$ such that $A/K' = (L/K') \cup (L'/K')$. Then

$$A = L \cup L' = K \cup K' \cup L' = K \cup L'.$$

Since $K \leqslant_s A$, $L' = A$ and so $L'/K' = A/K'$, which shows that $L/K' \leqslant_s A/K'$. Since L is co-closed in A, $L = K'$ so that $K \leqslant_s L$. For the second assertion, first note that $\mathrm{Rad}(A) = \bigcup\{N \mid N \leqslant_s A\}$ and $\mathrm{Rad}(L) = \bigcup\{N \mid N \leqslant_s L\}$. We show that $\mathrm{Rad}(L) = L \cap \mathrm{Rad}(A)$. Take any $x \in \mathrm{Rad}(L)$. Thus $x \in N$ for some $N \leqslant_s L$. Using [5, Lemma 2.3(ii)], $N \leqslant_s A$ whence $x \in \mathrm{Rad}(A)$. Therefore, $x \in L \cap \mathrm{Rad}(A)$. For the reverse inclusion, let $x \in L \cap \mathrm{Rad}(A)$. Then $x \in L \cap B$ for some $B \leqslant_s A$. Then, $L \cap B \leqslant B \leqslant_s A$ whence $L \cap B \leqslant_s A$ by [5, Lemma 2.3(i)], so that $x \in L \cap B \leqslant_s L$ by the previous assertion of this part, which means that $x \in \mathrm{Rad}(L)$.

(iii) Assume that L is not co-closed in A. Then there exists $K < L$ such that $L/K \leqslant_s A/K$. Since L is hollow, $K \leqslant_s L$ and so $K \leqslant_s A$ by [5, Lemma 2.3(ii)]. Then from [5, Lemma 2.3(i)] one concludes that $L \leqslant_s A$.

(iv) Let there exist $X \subseteq K$ such that $K/X \leqslant_s L/X$. Then $K/X \subseteq L/X \subseteq A/X$ and by [5, Lemma 2.3(ii)], $K/X \leqslant_s A/X$. But being K co-closed in A implies $K = X$, which gives that K is co-closed in L. ■

Ultimately, we investigate some connections between lifting acts and supplemented acts studied extensively in [8]. Let us first recall some notions.

Let B be a subact of an S-act A. A subact C of A is said to be a *supplement* of B in A if C is minimal with respect to $A = B \cup C$, that is, $A = B \cup C$ and if $A = B \cup D$ for some subact D of C, then $D = C$. An S-act A is called *supplemented* if the supplement of any proper (non-empty) subact B of A is proper in A, that is, $B^s_A < A$, where $B^s = (A \setminus B)S$ is the unique supplement of B in A (see [8, Theorem 2.3]).

Theorem 2.9 Any lifting S-act is supplemented.

Proof. Let A be a lifting S-act. Take any proper subact B of A. Using the assumption, there exists a proper subact C of B and $A = C \dot{\cup} D$ for some $D < A$. Then, by [8, Lemma 2.4], we get $B^s_A \subseteq C^s_A \subseteq D \subset A$, which means that A is supplemented. ■

The converse of the above theorem is not generally true. For instance, the S-act $A = \{a, b\}$ over a monoid S with trivial action is supplemented by [8, Remark 3.2(iii)] whereas it is not clearly lifting.

Proposition 2.10 Let A be a lifting S-act and $B, B^s_A < A$. Then B^s_A is a direct summand of A.

Proof. By the assumption and Theorem 2.3, $B^s_A = L \dot{\cup} T$ where L is a direct summand of A and $T \leqslant_s A$. So, $A = B \cup B^s_A = B \cup L \dot{\cup} T = B \cup L$, where the last equality follows from $T \leqslant_s A$. Then minimality of B^s_A implies $B^s_A = L$ and hence B^s_A is a direct summand of A. ■

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