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On the frames in Hilbert C^*-modules

M. Rossafi[a,*], M. Ghiati[b], M. Mouniane[b]

[a] *LaSMA Laboratory, Department of Mathematics, Faculty of Sciences Dhar El Mahraz, Sidi Mohamed Ben Abdellah University, Fez, Morocco*
[b] *Laboratory of Analysis, Geometry and Applications (LAGA), Department of Mathematics, Ibn Tofail University, Kenitra, Morocco.*

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Abstract. Frame theory has been rapidly generalized and various generalizations have been developed. In this paper, we present a brief survey of the frames in Hilbert C^*-modules including frames, $*$-frames, g-frames, $*$-g-frames, $*$-K-g-frame, operator frame and $*$-K-operator frame in Hilbert C^*-modules. Various proofs are given for some results. We will also provide some new results. Moreover, non-trivial examples are presented.

Keywords: Frame, $*$-frame, g-frame, $*$-g-frame, $*$-K-g-frame, Hilbert C^*-modules.

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1. Introduction

One of the important concepts in the study of vector spaces is the concept of a basis for the vector space, which allows every vector to be uniquely represented as a linear combination of the basis elements. However, the linear independence property for a basis is restrictive; sometimes it is impossible to find vectors that both fulfill the basis requirements and also satisfy external conditions demanded by applied problems. For such purposes, we need to look for more flexible types of spanning sets. Frames provide these alternatives. They not only have a great variety for use in applications but also have a rich theory from a pure analysis point of view.

A frame is a set of vectors in a Hilbert space that can be used to reconstruct each vector in the space from its inner products with the frame vectors. These inner products

*Corresponding author.
E-mail address: rossafimohamed@gmail.com, mohamed.rossafi@usmba.ac.ma (M. Rossafi)
mhamed.ghiati@uit.ac.ma (M. Ghiati); mohammed.mouniane@uit.ac.ma (M. Mouniane).

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are generally called the frame coefficients of the vector. But unlike an orthonormal basis, each vector may have infinitely many different representations in terms of its frame coefficients. In 1952, frames for Hilbert spaces were introduced by Duffin and Schaefer [9] to study some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [11] for signal processing. In fact, Gabor showed that any function $f \in L^2(\mathbb{R})$ can be reconstructed via a Gabor system $\{g(x-ka)e^{2\pi imbx} : k, m \in \mathbb{Z}\}$ where g is a continuous compact support function. These ideas did not generate much interest outside of nonharmonic Fourier series and signal processing until the landmark paper of Daubechies et al. [8], where they developed the class of tight frames for signal reconstruction and they showed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$ which are very similar to the expansions using orthonormal bases. After this innovative work, the theory of frames began to be widely studied. While orthonormal bases have been widely used for many applications, it is the redundancy that makes frames useful in applications. Formally, a frame in a separable Hilbert space $\mathcal{H}$ is a sequence $\{f_i\}_{i\in I}$ for which there exist positive constants $A, B > 0$ called frame bounds such that $A\|x\|^2 \leqslant \sum_{i\in I} |\langle x, f_i\rangle|^2 \leqslant B\|x\|^2$ for all $x \in \mathcal{H}$. It is remarkable that this inequalities imply the existence of a dual frame $\{\tilde{f_i}\}_{i\in I}$, such that the following reconstruction formula holds for every $x \in \mathcal{H}$: $x = \sum_{i\in I}\langle x, \tilde{f_i}\rangle f_i$. In particular, any orthonormal basis for $\mathcal{H}$ is a frame. However, in general, a frame need not be a basis and most useful frames are over-complete. The redundancy that frames carry is what makes them very useful in many applications.

Hilbert space frames have been traditionally used in signal processing because of their resilience to additive noise, resilience to quantization, the numerical stability of reconstruction, and their ability to capture important signal characteristics. Today, frame theory is an exciting, dynamic, and fast-paced subject with applications to a wide variety of areas in mathematics and engineering, including sampling theory, operator theory, harmonic analysis, nonlinear sparse approximation, pseudodifferential operators, wavelet theory, wireless communication, data transmission with erasures, filter banks, signal processing, image processing, geophysics, quantum computing, sensor networks, and more. The last decades have seen tremendous activity in the development of frame theory and many generalizations of frames have come into existence [18, 20]. Hilbert C^*-modules is a generalization of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers.

2. Preliminaries

2.1 $\mathcal{C}^*$-algebra

Definition 2.1 [5] If $\mathcal{A}$ is a Banach algebra, an involution is a map $a \to a^*$ of $\mathcal{A}$ into itself such that for all a and b in $\mathcal{A}$ and all scalars α the following conditions hold:

(1) $(a^*)^* = a$.
(2) $(ab)^* = b^*a^*$.
(3) $(\alpha a + b)^* = \bar{\alpha}a^* + b^*$.

Definition 2.2 [5] A $\mathcal{C}^*$-algebra $\mathcal{A}$ is a Banach algebra with involution such that $\|a^*a\| = \|a\|^2$ for every a in $\mathcal{A}$.

***Example* 2.3**

(1) The algebra of bounded operators on a Hilbert space $\mathcal{H}$, that is $B(\mathcal{H})$, is a $\mathcal{C}^*$-

algebra, where for each operator A, A^* is the adjoint of A.

(2) The algebra of continuous functions on a compact space X, that is $C(X)$, is an abelian $\mathcal{C}^*$-algebra, where $f^*(x) := \overline{f(x)}$.

(3) The algebra of continuous functions on a locally compact space X that vanish at infinity, that is $C_0(X)$, is an abelian $\mathcal{C}^*$-algebra, where $f^*(x) := \overline{f(x)}$.

Definition 2.4 [5] An element a in a $\mathcal{C}^*$-algebra $\mathcal{A}$ is positive if $a^* = a$ and $sp(a) \subset \mathbb{R}^+$. We write $a \geqslant 0$ if a is positive. The set of all positive elements in $\mathcal{A}$ will be denoted by $\mathcal{A}^+$.

Lemma 2.5 [25] Let $T \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$ be a bounded operator with closed range $\mathcal{R}(T)$. Then there exists a bounded operator $U^\dagger \in End^*_{\mathcal{A}}(\mathcal{K}, \mathcal{H})$ for which $TT^\dagger x = x$ for $x \in \mathcal{R}(T)$.

Lemma 2.6 [19] Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module. If $T \in End^*_{\mathcal{A}}(\mathcal{H})$, then $\langle Tx, Tx\rangle \leqslant \|T\|^2\langle x, x\rangle$ for all $x \in \mathcal{H}$.

Lemma 2.7 [3] Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$. Then the following statements are equivalent:

(i) T is surjective.

(ii) T^* is bounded below with respect to norm, i.e., there exists $m > 0$ such that $\|T^*x\| \geqslant m\|x\|$ for all $x \in \mathcal{K}$.

(iii) T^* is bounded below with respect to the inner product, i.e., there exists $m' > 0$ such that $\langle T^*x, T^*x\rangle \geqslant m'\langle x, x\rangle$ for all $x \in \mathcal{K}$.

Lemma 2.8 [2] Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$.

(i) If T is injective and T has closed range, then the adjointable map T^*T is invertible and $\|(T^*T)^{-1}\|^{-1} \leqslant T^*T \leqslant \|T\|^2$.

(ii) If T is surjective, then the adjointable map TT^* is invertible and $\|(TT^*)^{-1}\|^{-1} \leqslant TT^* \leqslant \|T\|^2$.

2.2 *Hilbert $\mathcal{C}^*$-modules*

Definition 2.9 [13] Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle ., .\rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ such that is sesquilinear, positive definite and respects the module action. In the other words,

(1) $\langle x, x\rangle \geqslant 0$ for all $x \in \mathcal{H}$ and $\langle x, x\rangle = 0$ if and only if $x = 0$;

(2) $\langle ax + y, z\rangle = a\langle x, z\rangle + \langle y, z\rangle$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$;

(3) $\langle x, y\rangle = \langle y, x\rangle^*$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $||x|| = ||\langle x, x\rangle||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x\rangle^{\frac{1}{2}}$ for $x \in \mathcal{H}$.

***Example* 2.10**

(1) Let $\mathcal{H}$ be a Hilbert space, then $B(\mathcal{H})$ is a Hilbert C^*-module with the inner product $\langle T, S\rangle = TS^*$ for all $T, S \in B(\mathcal{H})$.

(2) Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and let $B(\mathcal{H}, \mathcal{K})$ be the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$. Then $B(\mathcal{H}, \mathcal{K})$ is a Hilbert $B(\mathcal{K})$-module with the inner product $\langle T, S\rangle = TS^*$ for all $T, S \in B(\mathcal{H}, \mathcal{K})$.

(3) Let X be a locally compact Hausdorff space and $\mathcal{H}$ a Hilbert space, the Banach space $C_0(X,\mathcal{H})$ of all continuous $\mathcal{H}$-valued functions vanishing at infinity is a Hilbert C^*-module over the C^*-algebra $C_0(X)$ with inner product $\langle f,g\rangle(x) := \langle f(x),g(x)\rangle$ and module operation $(\phi f)(x) = \phi(x)f(x)$ for all $\phi \in C_0(X)$ and $f \in C_0(X,\mathcal{H})$.

(4) If $\{\mathcal{H}_k\}_{k\in\mathbb{N}}$ is a countable set of Hilbert $\mathcal{A}$-modules, then one can define their direct sum $\oplus_{k\in\mathbb{N}}\mathcal{H}_k$. On the $\mathcal{A}$-module $\oplus_{k\in\mathbb{N}}\mathcal{H}_k$ of all sequences $x = (x_k)_{k\in\mathbb{N}} : x_k \in \mathcal{H}_k$, such that the series $\sum_{k\in\mathbb{N}}\langle x_k, x_k\rangle_{\mathcal{A}}$ is norm-convergent in the $\mathcal{C}^*$-algebra $\mathcal{A}$, we define the inner product by $\langle x, y\rangle := \sum_{k\in\mathbb{N}}\langle x_k, y_k\rangle_{\mathcal{A}}$ for $x, y \in \oplus_{k\in\mathbb{N}}\mathcal{H}_k$. Then $\oplus_{k\in\mathbb{N}}\mathcal{H}_k$ is a Hilbert $\mathcal{A}$-module. The direct sum of a countable number of copies of a Hilbert $\mathcal{C}^*$-module $\mathcal{H}$ is denoted by $l^2(\mathcal{H})$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules, A map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^*y\rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$. We also reserve the notation $End^*_{\mathcal{A}}(\mathcal{H},\mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End^*_{\mathcal{A}}(\mathcal{H},\mathcal{H})$ is abbreviated to $End^*_{\mathcal{A}}(\mathcal{H})$. Throughout the paper, we consider a unital C^*-algebra.

3. Frames in Hilbert $\mathcal{A}$-modules

Definition 3.1 [10]. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module. A family $\{x_i\}_{i\in I}$ of elements of $\mathcal{H}$ is a frame for $\mathcal{H}$, if there exist two positive constants A and B such that for all $x \in \mathcal{H}$,

$$A\langle x, x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle x, x_i\rangle_{\mathcal{A}}\langle x_i, x\rangle_{\mathcal{A}} \leqslant B\langle x, x\rangle_{\mathcal{A}}. \tag{1}$$

The numbers A and B are called lower and upper bounds of the frame, respectively. If $A = B = \lambda$, the frame is λ-tight. If $A = B = 1$, it is called a normalized tight frame or a Parseval frame. If the sum in the middle of (1) is convergent in norm, the frame is called standard.

***Example* 3.2** For $a \in \mathbb{R}$, define the translation operator $T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $T_af(x) = f(x-a)$. For $b \in \mathbb{R}$, define the modulation operator $E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $E_bf(x) = e^{2\pi ibx}f(x)$ A frame for $L^2(\mathbb{R})$ of the form

$$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$$

is called a Gabor frame.

Let $\{x_i\}_{i\in I}$ be a frame of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital $\mathcal{C}^*$-algebra $\mathcal{A}$. The operator $T : \mathcal{H} \to l^2(\mathcal{A})$ defined by $Tx = \{\langle x, x_i\rangle\}_{i\in I}$ is called the analysis operator. The adjoint operator $T^* : l^2(\mathcal{A}) \to \mathcal{H}$ is given by $T^*\{c_i\}_{i\in I} = \sum_{i\in I} c_ix_i$. T^* is called pre-frame operator or the synthesis operator. By composing T and T^*, we obtain the frame operator $S : \mathcal{H} \to \mathcal{H}$ given by $Sx = T^*Tx = \sum_{i\in I}\langle x, x_i\rangle x_i$.

Proposition 3.3 The frame operator S is positive, selfadjoint, bounded and invertible.

Proof. For all $x \in \mathcal{H}$, we have

$$\langle Sx, x\rangle_{\mathcal{A}} = \langle T^*Tx, x\rangle_{\mathcal{A}} = \langle\sum_{i\in I}\langle x, x_i\rangle x_i, x\rangle_{\mathcal{A}} = \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle_{\mathcal{A}}.$$

and $0 \leqslant A\langle x,x\rangle_{\mathcal{A}} \leqslant \langle Sx;x\rangle \leqslant B\langle x,x\rangle_{\mathcal{A}}$. Then S is a positive and sefladjoint operator. Now, we have $A\langle x,x\rangle_{\mathcal{A}} \leqslant \langle Sx,x\rangle_{\mathcal{A}} \leqslant B\langle x,x\rangle_{\mathcal{A}}$ for $x \in \mathcal{H}$. So, $A.I_{\mathcal{H}} \leqslant S_C \leqslant B.I_{\mathcal{H}}$. Then S_C is a bounded operator. Moreover, $0 \leqslant I_{\mathcal{H}} - B^{-1}S_C \leqslant \frac{B-A}{B}.I_{\mathcal{H}}$. Consequently,

$$\|I_{\mathcal{H}} - B^{-1}S_C\| = \sup_{x\in\mathcal{H},\|x\|=1} \|\langle (I_{\mathcal{H}} - B^{-1}S_C)x, x\rangle_{\mathcal{A}}\| \leqslant \frac{B-A}{B} < 1.$$

The shows that S is invertible. ■

The frame operator is positive, invertible, and is the unique operator in $End^*_{\mathcal{A}}(\mathcal{H})$ such that the reconstruction formula

$$x = \sum_{i\in I}\langle x, S^{-1}x_i\rangle x_i = \sum_{i\in I}\langle x, x_i\rangle S^{-1}x_i$$

holds for all $x \in \mathcal{H}$. The sequences $\{S^{-1}x_i\}_{i\in I}$ and $\{S^{-\frac{1}{2}}x_i\}_{i\in I}$ are frames for $\mathcal{H}$. The frame $\{S^{-1}x_i\}_{i\in I}$ is said to be the canonical dual frame of $\{x_i\}_{i\in I}$ and the frame $\{S^{-\frac{1}{2}}x_i\}_{i\in I}$ is said to be the canonical Parseval frame of $\{x_i\}_{i\in I}$. The following theorem gives a characterization of standard frame.

Theorem 3.4 Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module over a unital $\mathcal{C}^*$-algebra $\mathcal{A}$, and $\{x_i\}_i \subset \mathcal{H}$ a sequence such that $\sum_{i\in I}\langle x, x_i\rangle_{\mathcal{A}}\langle x_i, x\rangle_{\mathcal{A}}$ converges in norm for every $x \in \mathcal{H}$. Then $\{x_i\}_i$ is a frame of $\mathcal{H}$ with bounds A and B if and only if

$$A\|x\|^2 \leqslant \left\| \sum_{i\in I}\langle x, x_i\rangle_{\mathcal{A}}\langle x_i, x\rangle_{\mathcal{A}} \right\| \leqslant B\|x\|^2 \tag{2}$$

for all $x \in \mathcal{H}$.

Proof. Suppose that $\{x_i : i \in I\}$ is a frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with bounds A and B. We have $A\langle x,x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle x, x_i\rangle_{\mathcal{A}}\langle x_i, x\rangle_{\mathcal{A}} \leqslant B\langle x,x\rangle_{\mathcal{A}}$. Since $\langle x,x\rangle \geqslant 0$, we get

$$A\|\langle x,x\rangle_{\mathcal{A}}\| \leqslant \|\sum_{i\in I}\langle x, x_i\rangle_{\mathcal{A}}\langle x_i, x\rangle_{\mathcal{A}}\| \leqslant B\|\langle x,x\rangle_{\mathcal{A}}\|.$$

Also, we have $A\|x\|^2 \leqslant \|\sum_{i\in I}\langle x, x_i\rangle_{\mathcal{A}}\langle x_i, x\rangle_{\mathcal{A}}\| \leqslant B\|x\|^2$. Now, suppose that (2) holds. We know that the frame operator S is positive, self-adjoint and inversible and

$$\langle S^{1/2}x, S^{1/2}x\rangle = \langle Sx, x\rangle = \langle \sum_{i\in I}\langle x, x_i\rangle_{\mathcal{A}} x_i, x\rangle = \sum_{i\in I}\langle x, x_i\rangle_{\mathcal{A}}\langle x_i, x\rangle_{\mathcal{A}}.$$

Hence, $\sqrt{A}\|x\| \leqslant \|S^{1/2}x\| \leqslant \sqrt{B}\|x\|$ and $A_1\langle x,x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle x, x_i\rangle_{\mathcal{A}}\langle x, x_i\rangle_{\mathcal{A}} \leqslant B_1\langle x,x\rangle_{\mathcal{A}}$, which implies that $\{x_i : i \in I\}$ is a frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$. ■

Theorem 3.5 Let $\{x_i : i \in I\}$ be a frame for $\mathcal{H}$ with lower and upper bounds A and B, respectively. Then the frame transform $T : H \to l^2(\{\mathcal{V}_i\})$ defined by $Tx = \{\langle x, x_i\rangle : i \in I\}$ is injective and adjointable, and has a closed range with $\|T\| \leqslant \|B\|^{\frac{1}{2}}$. The adjoint operator T^* given by $T^*x = \sum_{i\in I} c_i x_i$, where $x = \{x_i\}_{i\in I}$, is surjective.

Proof. Let $x \in H$. By the definition of frame for $\mathcal{H}$, we have

$$A\langle x,x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle x,x_i\rangle_{\mathcal{A}}\langle x_i,x\rangle_{\mathcal{A}} \leqslant B\langle x,x\rangle_{\mathcal{A}}. \tag{3}$$

Thus, $A\langle x,x\rangle_{\mathcal{A}} \leqslant \langle\sum_{i\in I}\langle x,x_i\rangle_{\mathcal{A}}x_i,x\rangle_{\mathcal{A}} \leqslant B\langle x,x\rangle_{\mathcal{A}}$ and $A\langle x,x\rangle \leqslant \langle T^*Tx,x\rangle \leqslant B\langle x,x\rangle$, which implies that

$$A\langle x,x\rangle \leqslant \langle Tx,Tx\rangle \leqslant B\langle x,x\rangle. \tag{4}$$

If $Tx = 0$ then $\langle x,x\rangle = 0$ and so $x = 0$, i.e., T is injective. We now show that the range of T is closed. Let $\{Tx_n\}_{n\in\mathbb{N}}$ be a sequence in the range of T such that $\lim_{n\to\infty} Tx_n = y$. By (4), for $n,m \in \mathbb{N}$, we have

$$\|A\langle x_n - x_m, x_n - x_m\rangle\| \leqslant \|\langle T(x_n - x_m), T(x_n - x_m)\rangle\| = \|T(x_n - x_m)\|^2.$$

Since $\{Tx_n\}_{n\in\mathbb{N}}$ is Cauchy sequence in H, $\|A\langle x_n - x_m, x_n - x_m\rangle\| \to 0$ as $n,m \to \infty$. Note that for $n,m \in \mathbb{N}$,

$$\|\langle x_n - x_m, x_n - x_m\rangle\| = \|A^{-1}A\langle x_n - x_m, x_n - x_m\rangle\| \leqslant \|A^{-1}\|\|A\langle x_n - x_m, x_n - x_m\rangle\|.$$

Therefore, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy and hence there exists $x \in H$ such that $x_n \to x$ as $n \to \infty$. Again, by (4), we have $\|T(x_n - x)\|^2 \leqslant \|B\|\|\langle x_n - x, x_n - x\rangle\|$. Thus, $\|Tx_n - Tx\| \to 0$ as $n \to \infty$ implies that $Tx = y$. It concludes that the range of T is closed. For all $x \in H$ and $y = \{y_i\} \in l^2(\{\mathcal{V}_i\})$, we have

$$\langle Tx, y\rangle = \langle\{\langle x,x_i\rangle\}_{i\in I}x, y_i\rangle = \left\langle x, \sum_{i\in I}\langle x,x_i\rangle y_i\right\rangle.$$

Then T is adjointable and $T^*y = \sum_{i\in I}\langle x,x_i\rangle y_i$. By (4), $\|Tx\|^2 \leqslant \|B\|\|x\|^2$ and so $\|T\| \leqslant \|B\|^{\frac{1}{2}}$ and $\|Tx\| \geqslant \|A^{-1}\|^{-1}\|x\|$ for all $x \in \mathcal{H}$ and so, T^* is surjective. This completes the proof. ■

Theorem 3.6 Let $\{x_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with frame transform T. Then $\{x_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds $||(T^*T)^{-1}||^{-1}$ and $||T||^2$, respectively.

Proof. By Theorem 3.5, T is injective and has a closed range and by Lemma 2.8,

$$||(T^*T)^{-1}||^{-1}\langle x,x\rangle \leqslant \langle T^*Tx,x\rangle \leqslant ||T||^2\langle x,x\rangle, \qquad \forall x \in U.$$

So,

$$||(T^*T)^{-1}||^{-1}\langle x,x\rangle \leqslant \langle\sum_{i\in I}\langle x,x_i\rangle_{\mathcal{A}}x_i,x\rangle_{\mathcal{A}} \leqslant ||T||^2\langle x,x\rangle, \qquad \forall x \in U.$$

Hence, $\{x_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds $||(T^*T)^{-1}||^{-1}$ and $||T||^2$, respectively. ■

4. *g*-Frames

Definition 4.1 [15]. We call a sequence $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, V_i) : i \in I\}$ a g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with respect to $\{V_i : i \in I\}$ if there exist two positive constants C and D such that for all $x \in \mathcal{H}$,

$$C\langle x, x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \leqslant D\langle x, x\rangle_{\mathcal{A}}. \tag{5}$$

The numbers C and D are called lower and upper bounds of the g-frame, respectively. If $C = D = \lambda$, the g-frame is λ-tight. If $C = D = 1$, it is called a g-Parseval frame. If the sum in the middle of (5) is convergent in norm, the g-frame is called standard.

***Example* 4.2** Let $\mathbb{C}^2$ be the Hilbert $\mathbb{C}^2$-module with $\mathbb{C}^2$-inner product $\langle (x_1, x_2), (y_1, y_2)\rangle = (x_1\bar{y_1}, x_2\bar{y_2})$ and $\mathcal{A}$ be the totality of all diagonal operators $diag\{a, b\}$ on $\mathbb{C}^2$, sending $(z_1, z_2)^t$ to $(az_1, bz_2)^t$. Fix $\{a_i\}_i$ and $\{b_i\}_i$ in l^2. Define $\Lambda_i : \mathbb{C}^2 \to \mathbb{C}^2, (z_1, z_2) \to (a_i z_1, b_i z_2)$. Then $\{\Lambda_i\}_i$ is a g-frame for $\mathbb{C}^2$ with bounds $\min\{\sum_i |a_i|^2, \sum_i |b_i|^2\}$ and $\max\{\sum_i |a_i|^2, \sum_i |b_i|^2\}$, respectively.

Like frames, we define the frame transform T, the synthesis operator T^* and the g-frame operator S as follows: $T : \mathcal{H} \to \oplus_{i\in I}V_i$, $Tx = \{\Lambda_i x\}_{i\in I}$, $T^* : \oplus_{i\in I}V_i \to \mathcal{H}$, $T^*y = \sum_{i\in I}\Lambda_i^* y_i$ for all $y = \{y_i\}_{i\in I}$ in $\oplus_{i\in I}V_i$, and $S = T^*T : \mathcal{H} \to \mathcal{H}$ is given by $Sx = \sum_{i\in I}\Lambda_i^*\Lambda_i x$ for each $x \in \mathcal{H}$. The g-frame operator is positive, invertible, and the follow reconstruction formula $x = \sum_{i\in I}\Lambda_i^*\Lambda_i S^{-1}x = \sum_{i\in I}S^{-1}\Lambda_i^*\Lambda_i x$ holds for all $x \in \mathcal{H}$.

The following theorem gives a characterization of standard g-frame.

Theorem 4.3 Let $\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, V_i)$ for any $i \in I$ and $\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}}$ converge in norm for $x \in \mathcal{H}$. Then $\{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ if and only if there exist constants A, $B > 0$ such that

$$A\|x\|^2 \leqslant \left\|\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}}\right\| \leqslant B\|x\|^2 \tag{6}$$

for all $x \in \mathcal{H}$.

Proof. Suppose that $\{\Lambda_i : i \in I\}$ is a g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then $A\langle x, x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \leqslant B\langle x, x\rangle_{\mathcal{A}}$. Since $\langle x, x\rangle \geqslant 0$, we have

$$A\|\langle x, x\rangle_{\mathcal{A}}\| \leqslant \|\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}}\| \leqslant B\|\langle f, f\rangle_{\mathcal{A}}\|.$$

Thus, $A\|x\|^2 \leqslant \|\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}}\| \leqslant B\|x\|^2$. Now, suppose that (6) holds, we know that the $g-$ frame operator S_Λ is positive self-adjoint and inversible

$$\langle S^{1/2}x, S^{1/2}x\rangle = \langle Sx, x\rangle = \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}}.$$

Hence, $\sqrt{A}\|x\| \leqslant \|S^{1/2}x\| \leqslant \sqrt{B}\|x\|$ and $A_1\langle x, x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \leqslant B_1\langle x, x\rangle_{\mathcal{A}}$, which implies that $\{\Lambda_i : i \in I\}$ is a g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. ∎

Theorem 4.4 If $\Lambda := \{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H},\mathcal{H}_i)\}_{i\in I}$ is a g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ with bounds A and B, then the g-frame operator $S\colon \mathcal{H}\to\mathcal{H}$ defined by $Sf = \sum\limits_{i\in I}\Lambda_i^*\Lambda_i x$ is a bounded invertible operator. Also, if $\widetilde{\Lambda}_i = \Lambda_i S^{-1}$, then $\{\widetilde{\Lambda}_i \in End^*_{\mathcal{A}}(\mathcal{H},\mathcal{H}_i)\}_{i\in I}$ is a g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ with bounds B^{-1} and A^{-1} and it satisfies $x = \sum\limits_{i\in I}\Lambda_i^*\widetilde{\Lambda}_i x = \sum\limits_{i\in I}\widetilde{\Lambda}_i^*\Lambda_i x$ for $x\in\mathcal{H}$.

Proof. For $x\in\mathcal{H}$, we have

$$\begin{aligned}
\left\|\sum_{i\in I_1}\Lambda_i^*\Lambda_i x\right\| &= \sup_{y\in\mathcal{H},\|y\|=1}\left\|\left\langle\sum_{i\in I_1}\Lambda_i^*\Lambda_i x, y\right\rangle_{\mathcal{A}}\right\| \\
&= \sup_{y\in\mathcal{H},\|y\|=1}\left\|\sum_{i\in I_1}\langle\Lambda_i x,\Lambda_i y\rangle_{\mathcal{A}}\right\| \\
&\leqslant \sup_{x\in\mathcal{H},\|y\|=1}\left\|\sum_{i\in I_1}\langle\Lambda_i x,\Lambda_i x\rangle_{\mathcal{A}}\right\|^{1/2}\left\|\sum_{i\in I_1}\langle\Lambda_i x,\Lambda_i y\rangle_{\mathcal{A}}\right\|^{1/2} \\
&\leqslant \sqrt{B}\left\|\sum_{i\in I_1}\langle\Lambda_i y,\Lambda_i y\rangle_{\mathcal{A}}\right\|^{1/2}.
\end{aligned}$$

Hence, the series in $\sum_{i\in I_1}\Lambda_i^*\Lambda_i$ are convergent. Therefore, Sx is well-defined for any $x\in\mathcal{H}$. On the other hand, it is easy to check for any $x,g\in\mathcal{H}$ that

$$\langle Sx,g\rangle_{\mathcal{A}} = \sum_{i\in I_1}\langle\Lambda_i^*\Lambda_i x,g\rangle_{\mathcal{A}} = \sum_{i\in I_1}\langle x,\Lambda_i^*\Lambda_i g\rangle_{\mathcal{A}} = \langle f,Sg\rangle_{\mathcal{A}}.$$

Hence, S is a self-adjoint operator. Therefore,

$$\|S\| = \sup_{x\in\mathcal{H},\|x\|=1}\|\langle Sx,x\rangle_{\mathcal{A}}\| = \sup_{x\in\mathcal{H},\|x\|=1}\|\langle\sum_{i\in I}\Lambda_i^*\Lambda_i x,x\rangle_{\mathcal{A}}\| = \sup_{x\in\mathcal{H},\|x\|=1}\|\sum_{i\in I}\langle\Lambda_i x,\Lambda_i x\rangle_{\mathcal{A}}\| \leqslant B,$$

which shows that S is a bounded operator. Since there is $\langle x,x\rangle_{\mathcal{A}}\geqslant 0$, then for all $x\in\mathcal{H}$, $A\|\langle x,x\rangle_{\mathcal{A}}\|\leqslant\|\langle Sx,x\rangle_{\mathcal{A}}\|\leqslant B\|\langle x,x\rangle_{\mathcal{A}}\|$, which implies that $A\|x\|^2\leqslant\|\langle Sx,x\rangle_{\mathcal{A}}\|\leqslant B\|x\|^2$. Then $A\|x\|^2\leqslant\|\langle Sx,x\rangle_{\mathcal{A}}\|\leqslant\|Sx\|\|x\|$. It follows that $\|Sx\|\geqslant A\|x\|$. Thus, S is injective and $S\mathcal{H}$ is closed in $\mathcal{H}$. Let $g\in\mathcal{H}$ be such that $\langle Sx,g\rangle_{\mathcal{A}}=0$ for any $x\in\mathcal{H}$. Then, we have $\langle x,Sg\rangle_{\mathcal{A}}=0$ for $x\in\mathcal{H}$. This implies that $Sg=0$ and therefore $g=0$. Hence, $S\mathcal{H}=\mathcal{H}$. Consequently, S is invertible and $\|S^{-1}\|\leqslant\dfrac{1}{A}$. For any $x\in\mathcal{H}$, we have

$$x = SS^{-1}x = S^{-1}Sx = \sum_{i\in I}\Lambda_i^*\Lambda_i S^{-1}x = \sum_{i\in I}S^{-1}\Lambda_i^*\Lambda_i x.$$

Let $\widetilde{\Lambda}_i=\Lambda_i S^{-1}$. Then the above equalities become $x=\sum\limits_{i\in I}\Lambda_i^*\widetilde{\Lambda}_i x=\sum\limits_{i\in I}\widetilde{\Lambda}_i^*\Lambda_i x$. Now, we prove that $\{\widetilde{\Lambda}_i\in End^*_{\mathcal{A}}(\mathcal{H},\mathcal{H}_i)\}_{i\in I}$ is also a g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. In fact, for any $x \in \mathcal{H}$, we have

$$\begin{aligned}
\left\|\sum_{i\in I} \widetilde{\Lambda}_i x\right\|^2 &= \left\|\sum_{i\in I} \left\langle \Lambda_i S^{-1}x, \Lambda_i S^{-1}x\right\rangle_{\mathcal{A}}\right\| \\
&= \left\|\sum_{i\in I} \langle \Lambda_i^* \Lambda_i S^{-1}x, S^{-1}x\rangle_{\mathcal{A}}\right\| \\
&= \left\|\langle S^{-1}Sx, S^{-1}x\rangle_{\mathcal{A}}\right\| \\
&= \left\|\langle \Lambda_i^* \Lambda_i S^{-1}x, S^{-1}x\rangle_{\mathcal{A}}\right\| \\
&= \|\langle S^{-1}Sx, S^{-1}x\rangle_{\mathcal{A}}\| \\
&= \left\|\langle f, S^{-1}x\rangle_{\mathcal{A}}\right\| \leqslant \frac{1}{A}\|\langle x, x\rangle_{\mathcal{A}}\|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|x\|^2 = \|\langle x, x\rangle_{\mathcal{A}}\| &= \|\langle \sum_{i\in I} \widetilde{\Lambda}_i^{\,*} \Lambda_i x, x\rangle_{\mathcal{A}}\| \\
&= \|\sum_{i\in I} \langle \Lambda_i x, \widetilde{\Lambda}_i x\rangle_{\mathcal{A}}\| \\
&\leqslant \|\sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}}\|^{1/2} \|\sum_{i\in I} \langle \widetilde{\Lambda}_i x, \widetilde{\Lambda}_i x\rangle_{\mathcal{A}}\|^{1/2} \\
&\leqslant \sqrt{B}\|x\| \|\sum_{i\in I} \langle \widetilde{\Lambda}_i x, \widetilde{\Lambda}_i x\rangle_{\mathcal{A}}\|^{1/2}.
\end{aligned}$$

So, $\|\sum_{i\in I} \langle \widetilde{\Lambda}_i x, \widetilde{\Lambda}_i x\rangle_{\mathcal{A}}\| \geqslant \frac{1}{B}\|\langle x, x\rangle_{\mathcal{A}}\|$. Hence, $\{\widetilde{\Lambda}_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ is a g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with frame bounds $\frac{1}{A}$ and $\frac{1}{B}$. Let $\widetilde{S}$ be the $g-$frame operator associated with $\Lambda := \{\Lambda_i \in\}_{i\in I}$. Then

$$S\widetilde{S}x = \sum_{i\in I} S\widetilde{\Lambda_i^*}\widetilde{\Lambda}_i x = \sum_{i\in I} SS^{-1}\Lambda_i^*\Lambda_i S^{-1}x = \sum_{i\in I} \Lambda_i^*\Lambda_i S^{-1}x = SS^{-1}\Lambda_i^* x = x.$$

Hence, $\widetilde{S} = S^{-1}$ and $\widetilde{\Lambda}_i\widetilde{S^{-1}}S = \Lambda_i$. ■

Theorem 4.5 Let $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ be g-Bessel sequences for Hilbert C^*-modules U_1 and U_2 with g-Bessel bounds B_1 and B_2, respectively. Then $\{\Lambda_w^*\Gamma_w\}_{w\in\Omega}$ is a-g-Bessel sequence for U_2 with respect to U_1.

Proof. For each $x \in U_2$, we have

$$\begin{aligned}
\sum_{i\in I} \langle \Lambda_i^*\Gamma_i x, \Lambda_i^*\Gamma_i x\rangle_{\mathcal{A}} &\leqslant \sum_{i\in I} ||\Lambda_i^*||^2 \langle \Gamma_i x, \Gamma_i x\rangle_{\mathcal{A}} \leqslant ||B_1||^2 \sum_{i\in I} \langle \Gamma_i x, \Gamma_i x\rangle_{\mathcal{A}} \\
&\leqslant ||B_1||^2 B_2 \langle x, x\rangle_{\mathcal{A}} \leqslant ||B_1|| B_2 \langle x, x\rangle_{\mathcal{A}}.
\end{aligned}$$

Hence, $\{\Lambda_i^*\Gamma_i\}_{i\in I}$ is a g-Bessel sequence for U_2 with respect to U_1. ■

Theorem 4.6 A sequence $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{V}_i)\}_{i\in I}$ is a g-frame in Hilbert $\mathcal{A}$ module $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ if and only if

$$Q : \{g_i\}_{i\in I} \to \sum_{i\in I} \Lambda_i^* g \tag{7}$$

is a well defined bounded linear operator from $l^2\left(\{\mathcal{H}_i\}_{i\in I}\right)$ onto $\mathcal{H}$, where the g-frame bounds are $\|Q^+\|^{-2}$ and $\|Q\|^2$, and Q^+ is the pseudo-inverse of Q.

Proof. (1) $\Rightarrow$ If $\Lambda := \{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ is a g-frame in Hilbert $\mathcal{A}$ module $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ with bounds A and B, then for any finite subset $I_1 \subset I$, we have

$$\begin{aligned}
\left\|\sum_{i\in I_1} \Lambda_i^* g_j\right\| &= \sup_{f\in\mathcal{H}, \|f\|=1} \left\|\left\langle \sum_{i\in I_1} \Lambda_i^* g_i, f\right\rangle\right\| \\
&= \sup_{f\in\mathcal{H}, \|f\|=1} \left\|\sum_{i\in I_1} \langle g_i, \Lambda_i f\rangle\right\| \\
&\leqslant \sup_{f\in\mathcal{H}, \|f\|=1} \|\sum_{i\in I_1}\langle g_i, g_i\rangle\|^{1/2} \|\sum_{i\in I_1}\langle \Lambda_i f, \Lambda_i f\rangle\|^{1/2} \\
&\leqslant \sqrt{B} \|\sum_{i\in I_1}\langle g_i, g_i\rangle\|^{1/2}.
\end{aligned}$$

Hence, the series $\sum_{i\in i} \Lambda_i^* g_i$ converges in $\mathcal{H}$ and the operator defined by (7) is well defined from $l^2\left(\{\mathcal{V}_i\}_{i\in I}\right)$ into $\mathcal{H}$ with $\|Q\| \leqslant \sqrt{B}$. For every $f \in \mathcal{H}$, there exists a $g \in \mathcal{H}$ such that $f = Sg = \sum_{i\in I} \Lambda_i^*\Lambda_i g$. Since $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, then $\{\Lambda_i g\}_{i\in I} \in l^2\left(\{\mathcal{V}_i\}_{i\in I}\right)$ and $Q\left(\{\Lambda_i g\}_{i\in I}\right) = \sum_{i\in I} \Lambda_i^*\Lambda_i g = f$. This implies that the operator Q is onto.

(2) $\Leftarrow$ If Q is a well defined bounded linear operator from $l^2\left(\{\mathcal{H}_i\}_{i\in I}\right)$ onto $\mathcal{H}$, then for any $f \in \mathcal{H}$ and any finite subset $I_1 \subset I$, we have

$$\begin{aligned}
\left\|\sum_{i\in I_1} \Lambda_i f\right\|^2 &= \left\|\sum_{i\in I_1} \langle \Lambda_i f, \Lambda_i f\rangle\right\| = \left\|\sum_{i\in I_1} \langle f, \Lambda_i^*\Lambda_i f\rangle\right\| = \left\|\langle f, \sum_{i\in I_1} \Lambda_i^*\Lambda_i f\rangle\right\| \\
&\leqslant \|\langle f, f\rangle\|^{1/2} \|\langle \sum_{i\in I_1} \Lambda_i^*\Lambda_i f, \sum_{i\in I_1} \Lambda_i^*\Lambda_i f\rangle\|^{1/2} \\
&\leqslant \|f\| \left\|\sum_{i\in I_1} \Lambda_i^*\Lambda_i f\right\| = \|f\| \left\|Q\left(\{\Lambda_i f\}_{i\in I_1}\right)\right\|.
\end{aligned}$$

It follows that $\left\|\sum_{i\in I_1} \Lambda_i f\right\|^2 \leqslant \|Q\|^2\|f\|^2$ for all $f \in \mathcal{H}$ and any finite subset $I_1 \subset I$.

Hence, we obtain $\left\|\sum_{i\in I}\Lambda_i f\right\|^2 \leqslant \|Q\|^2\|f\|^2$ for all $f \in \mathcal{H}$. On the other hand, since $Q\left(\ell^2\left(\{\mathcal{H}_i\}_{i\in I}\right)\right) = \mathcal{H}$, there exists a unique bounded operator $Q^+ : \mathcal{H} \to l^2\left(\{\mathcal{H}_i\}_{i\in I}\right)$ satisfying $QQ^+f = f, f \in Q\left(l^2\left(\{\mathcal{V}_i\}_{i\in I}\right)\right) = \mathcal{H}$. Let $Q^+f = \{a_i\}_{i\in I}$. Then we have

$$\begin{aligned}&\left\|\sum_{i\in I} a_i\right\|^2 = \|Q^+f\|^2 \leqslant \|Q^+\|^2\|f\|^2, \quad f \in \mathcal{H},\\ &f = QQ^+f = \sum_{i\in I}\Lambda_i^* a_i, \quad f \in \mathcal{H}.\end{aligned}$$

Hence, we obtain

$$\|f\|^4 = \|\langle f, f\rangle\|^2 = \left\|\left\langle\sum_{i\in I}\Lambda_i^* a_i, f\right\rangle\right\|^2 = \left\|\sum_{i\in I}\langle a_i, \Lambda_i f\rangle\right\|^2 \leqslant \|Q^+\|^2\|f\|^2\left\|\sum_{i\in I}\Lambda_i f\right\|^2.$$

This implies that $\frac{1}{\|Q^+\|^2}\|f\|^2 \leqslant \left\|\sum_{i\in I}\Lambda_i f\right\|^2$ for all $f \in \mathcal{H}$. ■

Theorem 4.7 Let a sequence $\Lambda := \{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{V}_i)\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_i\}_{i\in I}$ with bounds A and B. If P is the orthogonal projection from $l^2\left(\{\mathcal{V}_i\}_{i\in I}\right)$ onto R_{Q^*}, then P is defined by $P\left(\{g_i\}_{i\in I}\right) := \{\Lambda_i\sum_{k\in I}\widetilde{\Lambda}_k^* g_k\}_{i\in I}$ for any $\{g_i\}_{i\in I}$ belongs to $l^2\left(\{\mathcal{V}_i\}_{i\in I}\right)$.

Proof. Let $\tilde{T} : l^2\left(\{\mathcal{H}_i\}_{i\in I}\right) \to R_{Q^*}$ be the operator defined by $\widetilde{T}\left(\{g_i\}_{i\in I}\right) := \{\Lambda_i\sum_{k\in I}\widetilde{\Lambda}_k^* g_k\}_{i\in I}$. Firstly, we prove that $\tilde{T}$ is a bounded linear operator. For all $\{g_i\}_{i\in I}, \{h_i\}_{i\in I} \in l^2\left(\{\mathcal{H}_i\}_{i\in I}\right)$, we consider that

$$\begin{aligned}\tilde{T}\left(\{g_i\}_{i\in I} + \{h_i\}_{i\in I}\right) &= \widetilde{T}\left(\{g_i + h_i\}_{i\in I}\right)\\ &= \left\{\Lambda_i\sum_{k\in I}\widetilde{\Lambda}_k^*\left(g_k + h_k\right)\right\}_{i\in I}\\ &= \left\{\Lambda_i\left(\sum_{k\in I}\widetilde{\Lambda}_k^* g_k + \sum_{k\in I}\widetilde{\Lambda}_k^* h_k\right)\right\}_{i\in I}\\ &= \left\{\Lambda_i\sum_{k\in I}\widetilde{\Lambda}_k^* g_k\right\}_{i\in I} + \left\{\Lambda_i\sum_{k\in I}\widetilde{\Lambda}_k^* h_k\right\}_{i\in I}\\ &= \tilde{T}\left(\{g_i\}_{i\in I}\right) + \tilde{T}\left(\{h_i\}_{i\in I}\right)\end{aligned}$$

and

$$\left\|\tilde{T}\left(\{g_i\}_{i\in I}\right)\right\|^2=\left\|\left\{\Lambda_i\sum_{k\in I}\widetilde{\Lambda}_k^*g_k\right\}_{i\in I}\right\|^2=\left\|\left\{\Lambda_i\sum_{k\in I}S^{-1}\Lambda_k^*g_k\right\}_{i\in I}\right\|^2$$
$$=\sum_{i\in I}\left\|\Lambda_i\sum_{k\in I}S^{-1}\Lambda_k^*g_k\right\|^2\leqslant B\left\|\sum_{k\in I}S^{-1}\Lambda_k^*g_k\right\|^2\leqslant\frac{B}{A}\left\|\{g_k\}_{k\in I}\right\|^2.$$

Hence, $\tilde{T}$ is a bounded linear operator. Secondly, let $g=\sum_{i\in I}\widetilde{\Lambda}_i^*g_i$. Since

$$\tilde{T}^2\left(\{g_i\}_{i\in I}\right)=\widetilde{T}\left(\{\Lambda_i g\}_{i\in I}\right)=\left\{\Lambda_I\sum_{k\in I}\tilde{\Lambda}_k^*\Lambda_k g\right\}_{i\in I}$$
$$=\left\{\Lambda_i\sum_{k\in I}S^{-1}\Lambda_k^*\Lambda_k g\right\}_{i\in I}=\left\{\Lambda_i S^{-1}\sum_{k\in I}\Lambda_k^*\Lambda_k g\right\}_{i\in I}$$
$$=\{\Lambda_i g\}_{i\in I}=\widetilde{T}\left(\{g_i\}_{i\in I}\right),$$

we obtain $\tilde{T}^2=\tilde{T}$. Hence, we have that $\tilde{T}$ is a projection from $l^2\left(\{\mathcal{V}_i\}_{i\in I}\right)$ onto R_{Q^2}. Finally, for all $\{g_i\}_{i\in I},\{f_i\}_{i\in I}\in l^2\left(\{\mathcal{V}_i\}_{i\in I}\right)$, we obtain

$$\left\langle\tilde{T}\left(\{g_i\}_{i\in I}\right),\{f_i\}_{i\in I}\right\rangle_{\mathcal{A}}=\left\langle\left\{\Lambda_i\sum_{k\in I}\widetilde{\Lambda}_k^*g_k\right\}_{i\in I},\{f_i\}_{i\in I}\right\rangle_{\mathcal{A}}$$
$$=\sum_{i\in I}\left\langle\Lambda_i\sum_{k\in I}\widetilde{\Lambda}_k^*g_k,f_i\right\rangle_{\mathcal{A}}$$
$$=\sum_{i\in I}\left\langle\sum_{k\in I}\widetilde{\Lambda}_k^*g_k,\Lambda_i^*f_i\right\rangle_{\mathcal{A}}$$
$$=\left\langle\sum_{k\in I}\widetilde{\Lambda}_k^*g_k,\sum_{i\in I}\Lambda_i^*f_i\right\rangle_{\mathcal{A}}=\left\langle\sum_{k\in I}S^{-1}\Lambda_k^*g_k,\sum_{i\in I}\Lambda_i^*f_i\right\rangle_{\mathcal{A}}$$
$$=\left\langle S^{-1}\sum_{k\in I}\Lambda_k^*g_k,\sum_{i\in I}\Lambda_i^*f_i\right\rangle_{\mathcal{A}}=\left\langle\sum_{k\in I}\Lambda_k^*g_k,\sum_{i\in I}S^{-1}\Lambda_i^*f_i\right\rangle_{\mathcal{A}}$$
$$=\left\langle\sum_{k\in I}\Lambda_k^*g_k,\sum_{i\in I}\widetilde{\Lambda}_i^*f_i\right\rangle_{\mathcal{A}}=\sum_{k\in I}\left\langle g_k,\Lambda_k\sum_{i\in I}\widetilde{\Lambda}_i^*f_i\right\rangle_{\mathcal{A}}$$
$$=\left\langle\{g_k\}_{k\in I},\left\{\Lambda_k\sum_{i\in I}\widetilde{\Lambda}_i^*f_i\right\}_{k\in I}\right\rangle_{\mathcal{A}}$$
$$=\left\langle\{g_i\}_{i\in I},\tilde{T}\left(\{f_i\}_{i\in I}\right)\right\rangle_{\mathcal{A}}.$$

It shows that $\widetilde{T}^* = \widetilde{T}$. Hence, $\tilde{T}$ is an orthogonal projection from $l^2\left(\{\mathcal{V}_i\}_{i\in I}\right)$ onto R_{Q^*}. Since orthogonal projection is unique, we have $P = \widetilde{T}$. ■

5. ∗-Frames

Definition 5.1 [2] Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module over a unital C^*-algebra. A family $\{x_i\}_{i\in I}$ of elements of $\mathcal{H}$ is a ∗-frame for $\mathcal{H}$ if there exist strictly nonzero elements A and B in $\mathcal{A}$ such that for all $x \in \mathcal{H}$,

$$A\langle x, x\rangle_{\mathcal{A}} A^* \leqslant \sum_{i\in I} \langle x, x_i\rangle_{\mathcal{A}} \langle x_i, x\rangle_{\mathcal{A}} \leqslant B\langle x, x\rangle_{\mathcal{A}} B^*. \tag{8}$$

The elements A and B are called lower and upper bounds of the ∗-frame, respectively. If $A = B = \lambda_1$, the ∗-frame is λ_1-tight. If $A = B = 1$, it is called a normalized tight ∗-frame or a Parseval ∗-frame. If the sum in the middle of (8) is convergent in norm, the ∗-frame is called standard.

***Example* 5.2** Let $\mathcal{A}$ be the $\mathcal{C}^*$-algebra of the set of all diagonal matrices in $M_{2,2}(\mathbb{C})$ and suppose $\mathcal{A}$ is the Hilbert $\mathcal{A}$-module over itself. Consider $A_i = \begin{bmatrix} \frac{1}{2^i} & 0 \\ 0 & \frac{1}{3^i} \end{bmatrix}$ for all $i \in \mathbb{N}$. For $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathcal{A}$, we have

$$\sum_{i\in\mathbb{N}} \langle A, A_i\rangle\langle A_i, A\rangle = \begin{bmatrix} \frac{|a|^2}{3} & 0 \\ 0 & \frac{|b|^2}{8} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{bmatrix} \langle A, A\rangle \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{bmatrix}.$$

Then $\{A_i\}_{i\in\mathbb{N}}$ is $\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{bmatrix}$-tight ∗-frame for Hilbert $\mathcal{A}$-module $\mathcal{A}$.

Remark 1

(1) The set of all frames in Hilbert $\mathcal{A}$-modules can be considered as a subset of ∗-frames.
(2) We see that ∗-frames can be studied as frames with different bounds.

Now we define the ∗-frame operator and compare its properties with ordinary case.

Definition 5.3 Let $\{x_i\}_{i\in I}$ be a ∗-frame for $\mathcal{H}$ with pre-∗-frame operator T and lower and upper ∗-frame bounds A and B, respectively. The ∗-frame operator $S : \mathcal{H} \to \mathcal{H}$ is defined by $Sx = T^*Tx = \sum_{i\in I} \langle x, x_i\rangle_{\mathcal{A}} x_i$.

The ∗-frame operator has some similar properties with frame operator in ordinary frames, but the other properties are different. The main cause of differences is $\mathcal{A}$-valued bounds. However, the reconstruction formula is given from the ∗-frame operator.

Theorem 5.4 Let $\{x_i\}_{i\in I}$ be a ∗-frame for $\mathcal{H}$ with ∗-frame operator S and lower and upper ∗-frame bounds A and B, respectively. Then S is positive, invertible and adjointable. Also, the following inequality $\|A^{-1}\|^{-2} \leqslant \|S\| \leqslant \|B\|^2$ holds, and the reconstruction formula $x = \sum_{i\in I} \langle x, S^{-1}x_i\rangle_{\mathcal{A}} x_i$ holds for all $x \in \mathcal{H}$.

In the following corollary, we see that ∗-frames can be studied as frames with different bounds.

Corollary 5.5 Let $\{x_i\}_{i\in I}$ be a $*$-frame for $\mathcal{H}$ with pre-$*$-frame operator T and lower and upper $*$-frame bounds A and B, respectively. Then $\{x_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds $\|(T^*T)^{-1}\|^{-1}$ and $\|T\|^2$, respectively.

5.1 *$*$-g-Frames*

The following definition was introduced independently by Alijani [1] and Bounader [4], which is a generalization of g-frames.

Definition 5.6 [1, 4] We call a sequence $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, V_i) : i \in I\}$ a $*$-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra with respect to $\{V_i : i \in I\}$ if there exist strictly nonzero elements A adn B in $\mathcal{A}$ such that for all $x \in \mathcal{H}$,

$$A\langle x, x\rangle_{\mathcal{A}} A^* \leqslant \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \leqslant B\langle x, x\rangle_{\mathcal{A}} B^*. \tag{9}$$

The elements A and B are called lower and upper bounds of the $*$-g-frame, respectively. If $A = B = \lambda_1$, the $*$-g-frame is λ_1-tight. If $A = B = 1$, it is called an $*$-g-Parseval frame. If the sum in the middle of (9) is convergent in norm, the $*$-g-frame is called standard.

***Example* 5.7** Let $\{x_i\}_{i\in\mathbb{I}}$ be a $*$-frame for $\mathcal{H}$ with bounds A and B, respectively. For each $i \in \mathbb{I}$, we define $\Lambda_i : \mathcal{H} \to \mathcal{A}$ by $\Lambda_i x = \langle x, x_i\rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$. Λ_i is adjointable and $\Lambda_i^* a = a x_i$ for each $a \in \mathcal{A}$, and we have $A\langle x, x\rangle_{\mathcal{A}} A^* \leqslant \sum_{i\in I} \langle x, x_i\rangle_{\mathcal{A}} \langle x_i, x\rangle_{\mathcal{A}} \leqslant B\langle x, x\rangle_{\mathcal{A}} B^*$ for all $x \in \mathcal{H}$. Then $A\langle x, x\rangle_{\mathcal{A}} A^* \leqslant \sum_{i\in I} \langle T_i x, T_i x\rangle \leqslant B\langle x, x\rangle_{\mathcal{A}} B^*$ for all $x \in \mathcal{H}$. So, $\{\Lambda_i\}_{i\in\mathbb{I}}$ is a $*$-g-frame with bounds A and B, respectively, in $\mathcal{H}$ with respect to $\mathcal{A}$.

Remark 2

(1) The set of all g-frames in Hilbert $\mathcal{A}$-modules can be considered as a subset of $*$-g-frames.
(2) We see that $*$-g-frames can be studied as g-frames with different bounds.

Now we define the $*$-g-frame operator.

Definition 5.8 Let $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, V_i) : i \in I\}$ be a $*$-g-frame for $\mathcal{H}$ with lower and upper $*$-g-frame bounds A and B, respectively. The $*$-g-frame operator $S : \mathcal{H} \to \mathcal{H}$ is defined by $Sx = \sum_{i\in I} \Lambda_i^* \Lambda_i x$.

The $*$-g-frame operator has some similar properties with g-frame operator, but the other properties are different. The main cause of differences is $\mathcal{A}$-valued bounds. However, the reconstruction formula is given from the $*$-g-frame operator.

Theorem 5.9 Let $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, V_i) : i \in I\}$ be a $*$-g-frame for $\mathcal{H}$ with $*$-g-frame operator S and lower and upper $*$-g-frame bounds A and B, respectively. Then S is positive, invertible and adjointable. Also, the inequality $\|A^{-1}\|^{-2} \leqslant \|S\| \leqslant \|B\|^2$ holds, and the reconstruction formula $x = \sum_{i\in I} \Lambda_i^* \Lambda_i S^{-1} x = \sum_{i\in I} S^{-1} \Lambda_i^* \Lambda_i x$ holds for all $x \in \mathcal{H}$.

In the following corollary we see that $*$-g-frames can be studied as g-frames with different bounds.

Corollary 5.10 Let $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, V_i) : i \in I\}$ be a $*$-g-frame for $\mathcal{H}$ with pre-$*$-g-frame operator T and lower and upper $*$-g-frame bounds A and B, respectively. Then $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, V_i) : i \in I\}$ is a g-frame for $\mathcal{H}$ with lower and upper g-frame bounds $\|(T^*T)^{-1}\|^{-1}$ and $\|T\|^2$, respectively.

Theorem 5.11 Let $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, V_i) : i \in I\}$. If the operator $\theta : \oplus_{i\in I} V_i \to \mathcal{H}$ defined by $\theta(\{x_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^* x_i$ is surjective, then $\{\Lambda_i\}_{i\in I}$ is a $*$-g-frame for $\mathcal{H}$.

Proof. For each $x \in \mathcal{H}$,

$$\begin{aligned}\left\|\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle\right\| &= \left\|\sum_{i\in I}\langle x, \Lambda_i^*\Lambda_i x\rangle\right\| = \left\|\langle x, \sum_{i\in I}\Lambda_i^*\Lambda_i x\rangle\right\| \\ &\leqslant \|x\| \left\|\sum_{i\in I}\Lambda_i^*\Lambda_i x\right\| \leqslant \|x\| \ \|\theta(\{\Lambda_i x\}_{i\in I})\| \\ &\leqslant \|x\| \ \|\theta\| \ \|\{\Lambda_i x\}_{i\in I}\| \leqslant \|x\| \ \|\theta\| \left\|\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle\right\|^{\frac{1}{2}}.\end{aligned}$$

Thus, $\left\|\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle\right\|^{\frac{1}{2}} \leqslant \|\theta\| \ \|x\|$. So,

$$\left\|\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle\right\| \leqslant \|\theta\|^2 \ \|x\|^2, \qquad \forall x \in \mathcal{H}. \tag{10}$$

Since θ is surjective, by Lemma 2.7 there exists $\nu > 0$ such that $||\theta^* x|| \geqslant \nu ||x||$ for all $x \in \mathcal{H}$. Therefore, θ^* is injective. Hence $\theta^* : \mathcal{H} \to \mathcal{R}(\theta^*)$ is invertible, and for each $x \in \mathcal{H}$, $(\theta^*_{/\mathcal{R}(\theta^*)})^{-1}\theta^* x = x$. So, for each $x \in \mathcal{H}$, $\|x\| = \|(\theta^*_{/\mathcal{R}(\theta^*)})^{-1}\theta^* x\| \leqslant \|(\theta^*_{/\mathcal{R}(\theta^*)})^{-1}\| \ \|\theta^* x\|$. Hence,

$$\|(\theta^*_{/\mathcal{R}(\theta^*)})^{-1}\|^{-2} \ \|x\|^2 \leqslant \left\|\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle\right\|. \tag{11}$$

From (10) and (11), $\{\Lambda_i\}_{i\in I}$ is a $*$-g-frame for $\mathcal{H}$. ∎

Theorem 5.12 Let $\{\Lambda_i\}_{i\in I}$ be a $*$-g-frame for $\mathcal{H}$. If $\{\Gamma_i\}_{i\in I}$ is a $*$-g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i : i \in I\}$ and the operator $F : \mathcal{H} \to \mathcal{H}$ defined by $Fx = \sum_{i\in I} \Gamma_i^* \Lambda_i x$ is surjective, then $\{\Gamma_i\}_{i\in I}$ is a $*$-g-frame for $\mathcal{H}$.

Proof. Since $\{\Lambda_i\}_{i\in I}$ is a $*$-g-frame for $\mathcal{H}$, we have a $*$-g-frame transform $T : \mathcal{H} \to \oplus_{i\in I} V_i$ defined by $Tx = \{\Lambda_i x\}_{i\in I}$. Now, the operator $K : \oplus_{i\in I} V_w \to \mathcal{H}$ defined by $K(\{x_i\}_{i\in I}) = \sum_{i\in I} \Gamma_i^* x_i$ is well-defined. Since

$$\begin{aligned}\left\|\sum_{i\in I}\Gamma_i^* x_i\right\| &= \sup_{\|y\|=1}\left\|\langle\sum_{i\in I}\Gamma_i^* x_i, y\rangle\right\| = \sup_{\|y\|=1}\left\|\sum_{i\in I}\langle x_i, \Gamma_i y\rangle\right\| \\ &\leqslant \sup_{\|y\|=1}\left\|\sum_{i\in I}\langle x_i, x_i\rangle\right\|^{\frac{1}{2}}\left\|\sum_{i\in I}\langle \Gamma_i y, \Gamma_i y\rangle\right\|^{\frac{1}{2}} \\ &\leqslant \sup_{\|y\|=1}\|\{x_i\}_{i\in I}\|\|\langle y, y\rangle\|^{\frac{1}{2}} = \|\{x_i\}_{i\in I}\|,\end{aligned}$$

For each $x \in \mathcal{H}$, we have $Fx = \sum_{i\in I} \Gamma_i^* \Lambda_i x = KTx$; that is, $F = KT$. Since F is surjective, for each $x \in \mathcal{H}$ there exists $y \in \mathcal{H}$ such that $Fy = x$, which implies $x = Fy =$

KTy and $Ty \in \oplus_{i\in I} V_i$ and so K is surjective. From Theorem 5.11, we conclude that $\{\Gamma_i\}_{i\in I}$ is a $*$-g-frame for $\mathcal{H}$. ■

Now, we study $*$-g-frames in two Hilbert C^*-modules with different C^*-algebras.

Theorem 5.13 Let $(\mathcal{H}, \mathcal{A}, \langle .,. \rangle_{\mathcal{A}})$ and $(\mathcal{H}, \mathcal{B}, \langle .,. \rangle_{\mathcal{B}})$ be two Hilbert C^*-modules, $\phi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism and θ be an adjointable map on $\mathcal{H}$ such that $\langle \theta x, \theta y \rangle_{\mathcal{B}} = \phi(\langle x, y \rangle_{\mathcal{A}})$ for all $x, y \in \mathcal{H}$. Also, suppose that $\{\Lambda_i\}_{i\in I}$ is a $*$-g-frame for $(\mathcal{H}, \mathcal{A}, \langle .,. \rangle_{\mathcal{A}})$ with $*$-g-frame operator $S_{\mathcal{A}}$ and lower and upper bounds A and B, respectively. If θ is surjective and $\theta \Lambda_i = \Lambda_i \theta$ for all $i \in I$, then $\{\Lambda_i\}_{i\in I}$ is a $*$-g-frame for $(\mathcal{H}, \mathcal{B}, \langle .,. \rangle_{\mathcal{B}})$ with $*$-g-frame operator $S_{\mathcal{B}}$ and lower and upper bounds $\phi(A)$ and $\phi(B)$, respectively, and $\langle S_{\mathcal{B}} \theta x, \theta y \rangle_{\mathcal{B}} = \phi(\langle S_{\mathcal{A}} x, y \rangle_{\mathcal{A}})$.

Proof. Let $y \in \mathcal{H}$. Since θ is surjective, there exists $x \in \mathcal{H}$ such that $\theta x = y$, and we have $A\langle x, x \rangle_{\mathcal{A}} A^* \leqslant \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}} \leqslant B\langle x, x \rangle_{\mathcal{A}} B^*$. Thus,

$$\phi(A\langle x, x \rangle_{\mathcal{A}} A^*) \leqslant \phi\big(\sum_{i\in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}}\big) \leqslant \phi(B\langle x, x \rangle_{\mathcal{A}} B^*).$$

By definition of $*$-homomorphism, we have

$$\phi(A)\phi(\langle x, x \rangle_{\mathcal{A}})\phi(A^*) \leqslant \sum_{i\in I} \phi\big(\langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}}\big) \leqslant \phi(B)\phi(\langle x, x \rangle_{\mathcal{A}})\phi(B^*).$$

By the relation betwen θ and ϕ, we get

$$\phi(A)\langle y, y \rangle_{\mathcal{B}} \phi(A)^* \leqslant \sum_{i\in I} \langle \Lambda_i y, \Lambda_i y \rangle_{\mathcal{B}} \leqslant \phi(B)\langle y, y \rangle_{\mathcal{B}} \phi(B)^*.$$

On the other hand, we have

$$\begin{aligned}\phi(\langle S_{\mathcal{A}} x, y \rangle_{\mathcal{A}}) &= \phi(\langle \sum_{i\in I} \Lambda_i^* \Lambda_i x, y \rangle_{\mathcal{A}}) = \sum_{i\in I} \phi(\langle \Lambda_i x, \Lambda_i y \rangle_{\mathcal{A}}) \\ &= \sum_{i\in I} \langle \Lambda_i \theta x, \Lambda_i \theta y \rangle_{\mathcal{B}} = \langle \sum_{i\in I} \Lambda_i^* \Lambda_i \theta x, \theta y \rangle_{\mathcal{B}} = \langle S_{\mathcal{B}} \theta x, \theta y \rangle_{\mathcal{B}}.\end{aligned}$$

This completes the proof. ■

Theorem 5.14 Let $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, V_i) : i \in I\}$ be a $*$-g-frame for $\mathcal{H}$ with lower and upper bounds A and B, respectively. Let $\theta \in End_{\mathcal{A}}^*(\mathcal{H})$ be injective and have a closed range. Then $\{\theta \Lambda_i\}_{i\in I}$ is a $*$-g-frame for $\mathcal{H}$.

Proof. We have $A\langle x, x \rangle A^* \leqslant \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x \rangle \leqslant B\langle x, x \rangle B^*$ for all $x \in \mathcal{H}$. Then

$$\sum_{i\in I} \langle \theta \Lambda_i x, \theta \Lambda_i x \rangle \leqslant ||\theta||^2 B\langle x, x \rangle B^* \leqslant (||\theta|| B)\langle x, x \rangle (||\theta|| B)^*. \tag{12}$$

By Lemma 2.8, we have $||(\theta^* \theta)^{-1}||^{-1} \langle \Lambda_i x, \Lambda_i x \rangle \leqslant \langle \theta \Lambda_i x, \theta \Lambda_i x \rangle$ for each $x \in \mathcal{H}$ and $||\theta^{-1}||^{-2} \leqslant ||(\theta^* \theta)^{-1}||^{-1}$. Thus,

$$||\theta^{-1}||^{-1} A\langle x, x \rangle (||\theta^{-1}||^{-1} A)^* \leqslant \sum_{i\in I} \langle \theta \Lambda_i x, \theta \Lambda_i x \rangle. \tag{13}$$

From (12) and (13), we have

$$||\theta^{-1}||^{-1}A\langle x,x\rangle(||\theta^{-1}||^{-1}A)^* \leqslant \sum_{i\in I}\langle\theta\Lambda_i x,\theta\Lambda_i x\rangle \leqslant ||\theta||^2 B\langle x,x\rangle B^* \leqslant (||\theta||B)\langle x,x\rangle(||\theta||B)^*.$$

for each $x \in \mathcal{H}$. Hence, $\{\theta\Lambda_i\}_{i\in I}$ is a $*$-g-frame for $\mathcal{H}$. ■

6. *K*-Frames

Definition 6.1 [17] Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$. A family $\{x_i\}_{i\in I}$ of elements of $\mathcal{H}$ is a K-frame for $\mathcal{H}$, if there exist two positive constants A and B such that for all $x \in \mathcal{H}$,

$$A\langle K^*x, K^*x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle x, x_i\rangle_{\mathcal{A}}\langle x_i, x\rangle_{\mathcal{A}} \leqslant B\langle x, x\rangle_{\mathcal{A}}. \tag{14}$$

The numbers A and B are called lower and upper bounds of the K-frame, respectively.

The following theorem give a condition for getting a frame from a K-frame.

Theorem 6.2 Let $\{x_i\}_{i\in I}$ be a K-frame for $\mathcal{H}$ with bounds $A, B > 0$. If the operator K is surjective, then $\{x_i\}_{i\in I}$ is a frame for $\mathcal{H}$.

Proposition 6.3 A Bessel sequence $\{x_i\}_{i\in I}$ of $\mathcal{H}$ is a K-frame with bounds $A, B > 0$ if and only if $S \geqslant AKK^*$, where S is the frame operator for $\{x_i\}_{i\in I}$.

7. *K*-g-Frames

Definition 7.1 [24] Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, V_i)$ for all $i \in I$, then $\{\Lambda_i\}_{i\in I}$ is said to be a K-g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ if there exist two constants $C, D > 0$ such that $C\langle K^*x, K^*x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle\Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \leqslant D\langle x, x\rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$. The numbers C and D are called K-g-frame bounds. Particularly, if $C\langle K^*x, K^*x\rangle = \sum_{i\in I}\langle\Lambda_i x, \Lambda_i x\rangle$ for all $x \in \mathcal{H}$, the K-g-frame is C-tight.

***Example* 7.2** Let l^∞ be the set of all bounded complex-valued sequences. For any $u = \{u_j\}_{j\in\mathbb{N}}, v = \{v_j\}_{j\in\mathbb{N}} \in l^\infty$, we define $uv = \{u_j v_j\}_{j\in\mathbb{N}}, u^* = \{\bar{u_j}\}_{j\in\mathbb{N}}, \|u\| = \sup_{j\in\mathbb{N}}|u_j|$. Then $\mathcal{A} = \{l^\infty, \|.\|\}$ is a $\mathbb{C}^*$-algebra. Let $\mathcal{H} = C_0$ be the set of all sequences converging to zero. For any $u, v \in \mathcal{H}$, we define $\langle u, v\rangle = uv^* = \{u_j\bar{u_j}\}_{j\in\mathbb{N}}$. Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module. Now, let $\{e_j\}_{j\in\mathbb{N}}$ be the standard orthonormal basis of $\mathcal{H}$. For each $j \in \mathbb{N}$ define the adjointable operator $\Lambda_j : \mathcal{H} \to \overline{span}\{e_j\}$ by $\Lambda_j x = \langle x, e_j\rangle e_j$, then we have $\sum_{j\in\mathbb{N}}\langle\Lambda_j x, \Lambda_j x\rangle = \langle x, x\rangle$ for every $x \in \mathcal{H}$. Fix $N \in \mathbb{N}^*$ and define

$$K : \mathcal{H} \to \mathcal{H}, \quad Ke_j = \begin{cases} je_j & \text{if } j \leqslant N, \\ 0 & \text{if } j > N. \end{cases}$$

It is easy to check that K is adjointable and satisfies

$$K^*e_j = \begin{cases} je_j & \text{if } j \leqslant N, \\ 0 & \text{if } j > N. \end{cases}$$

For any $x \in \mathcal{H}$, we have $\dfrac{1}{N^2}\langle K^*x, K^*x\rangle \leqslant \sum_{j\in\mathbb{N}}\langle \Lambda_j x, \Lambda_j x\rangle = \langle x, x\rangle$. This shows that $\{\Lambda_j\}_{j\in\mathbb{N}}$ is a K-g-frame with bounds $\dfrac{1}{N^2}$ and 1.

Remark 3 *If $K \in End^*_{\mathcal{A}}(\mathcal{H})$ is a surjective operator, then every K-g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ is a g-frame.*

Theorem 7.3 Let $\{\Lambda_i\}_{i\in I}$ be a $K - g$ frame in Hilbert $\mathcal{A}$ module $\mathcal{H}$ and $T \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $\mathcal{R}(T) \subset \mathcal{R}(K)$. Then $\{\Lambda_i\}_{i\in I}$ is a $T - g$-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$.

Proof. Suppose that C is a lower frame bound of $\{\Lambda_i\}_{i\in I}$. There exists $\alpha > 0$ such that $TT^* \leqslant \alpha^2 KK^*$. Now, for each $x \in \mathcal{H}$, we have $\langle TT^*x, x\rangle_{\mathcal{A}} \leqslant \alpha^2\langle KK^*x, x\rangle_{\mathcal{A}}$. So,

$$\frac{C}{\alpha^2}\langle T^*x, T^*x\rangle_{\mathcal{A}} \leqslant C\langle K^*x, K^*x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \leqslant D\langle x, x\rangle_{\mathcal{A}}.$$

■

Theorem 7.4 Let $\{\Lambda_i\}_{i\in I}$ be a $K - g$- frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$. Assume that K has a closed range and $T \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $\mathcal{R}(T^*) \subset \mathcal{R}(K)$. Then $\{\Lambda_i T^*\}_{i\in I}$ is a $K - g$- frame for $\mathcal{R}(T)$ if and only if there exists $\delta > 0$ such that for each $x \in \mathcal{R}(T)$, $\|T^*x\| \geqslant \delta\|K^*x\|$.

Proof. Suppose that $\{\Lambda_i T^*\}_{i\in I}$ is a K-g-frame in Hilbert $\mathcal{A}$ module $\mathcal{H}$ with a lower frame bound $E > 0$. If F is an upper frame bound of $\{\Lambda_i\}_{i\in I}$, then we have $E\langle K^*x, K^*x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle \Lambda_i T^*x, \Lambda_i T^*x\rangle_{\mathcal{A}}$ for each $x \in \mathcal{R}(T)$. Thus,

$$E\langle K^*x, K^*x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle \Lambda_i T^*x, \Lambda_i T^*x\rangle_{\mathcal{A}} \leqslant F\langle T^*x, T^*x\rangle_{\mathcal{A}}$$

and

$$E\|\langle K^*x, K^*x\rangle_{\mathcal{A}}\| \leqslant \|\sum_{i\in I}\langle \Lambda_i T^*x, \Lambda_i T^*x\rangle_{\mathcal{A}}\| \leqslant F\|\langle T^*x, T^*x\rangle_{\mathcal{A}}\|.$$

Hence, $E\|K^*x\|^2 \leqslant F\|T^*x\|^2$ and $\sqrt{\frac{E}{F}}\|K^*x\| \leqslant \|T^*x\|$ for the opposite implication. For each $x \in H$, we have $\|T^*x\| = \|(K^\dagger)^*K^*T^*x\| \leqslant \|(K^\dagger)\|\|K^*U^*x\|$. Therefore, if E is a lower frame bound of $\{\Lambda_i\}_{i\in I}$, we have

$$E\delta^2\|K^\dagger\|^{-2}\langle K^*x, K^*x\rangle \leqslant E\|K^\dagger\|^{-2}\langle T^*x, T^*x\rangle \leqslant E\|K^*T^*x\|^2 \leqslant \sum_{i\in I}\langle \Lambda_i T^*x, \Lambda_i T^*x\rangle_{\mathcal{A}}.$$

For the upper bound, it is clear that $\sum_{i\in I}\langle \Lambda_i T^*x, \Lambda_i T^*x\rangle_{\mathcal{A}} \leqslant F\langle T^*x, T^*x\rangle_{\mathcal{A}} \leqslant F\|T\|^2\langle x, x\rangle_{\mathcal{A}}$. So, $(\Lambda_i T^*)_{i\in I}$ is a $K-g$-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with frame bounds $E\delta^2\|K^\dagger\|^{-2}$ and $F\|T\|^2$. ■

Theorem 7.5 Let $\{\Lambda_i\}_{i\in I}$ be a $K-g$-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ and the operator K has a dense rang. Assume that $T \in End^*_{\mathcal{A}}(\mathcal{H})$ has a closed range and T and T^* commute. If $\{\Lambda_i T^*\}_{i\in I}$ and $\{\Lambda_i T\}_{i\in I}$ are $K-g$-frame in Hilbert $\mathcal{A}$- module $\mathcal{H}$, then T is invertible.

Proof. Suppose that $\{\Lambda_i T^*\}_{i\in I}$ is a $K - g$-frame in Hilbert $\mathcal{A}$ module $\mathcal{H}$ with a lower frame bound A_1 and B_1. Then $A_1\langle K^*x, K^*x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle \Lambda_i T^*x, \Lambda_i T^*x\rangle_{\mathcal{A}} \leqslant B_1\langle x, x\rangle_{\mathcal{A}}$ for each $x \in \mathcal{H}$. Hence,

$$\|A_1\langle K^*x, K^*x\rangle_{\mathcal{A}}\| \leqslant \|\sum_{i\in I}\langle \Lambda_i T^*x, \Lambda_i T^*x\rangle_{\mathcal{A}}\| \leqslant \|B_1\langle x, x\rangle_{\mathcal{A}}\| \tag{15}$$

and $A_1\|K^*x\|^2 \leqslant \|\sum_{i\in I}\langle \Lambda_i T^*x, \Lambda_i T^*x\rangle\| \leqslant B_1\|x\|^2$. Since K has a dense range, K^* is injective. Moreover, $\mathcal{R}(T) = (\ker T^*)^{\perp} = H$ and so T is surjective. Suppose that $\{\Lambda_i T^*\}_{i\in I}$ is a $K-g$-frame in Hilbert $\mathcal{A}$ module $\mathcal{H}$ with a lower frame bound A_2 and B_2. Then, for each $x \in \mathcal{H}$, $A_2\langle K^*x, K^*x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle \Lambda_i T^*x, \Lambda_i T^*x\rangle_{\mathcal{A}} \leqslant B_2\langle x, x\rangle_{\mathcal{A}}$, which implies that

$$\|A_2\langle K^*x, K^*x\rangle_{\mathcal{A}}\| \leqslant \|\sum_{i\in I}\langle \Lambda_i T^*x, \Lambda_i T^*x\rangle_{\mathcal{A}}\| \leqslant \|B_2\langle x, x\rangle_{\mathcal{A}}\|$$

and $A_2\|K^*x\|^2 \leqslant \|\sum_{i\in I}\langle \Lambda_i T^*x, \Lambda_i T^*x\rangle_{\mathcal{A}}\| \leqslant B_2\|x\|^2$ and since $\ker U \subseteq \ker K^*$, T is injective and hence, T is an invertible operator. ■

Theorem 7.6 Let $\{\Lambda_i\}_{i\in I}$ be a $K-g$-frame in Hilbert $\mathcal{A}$- module $\mathcal{H}$ and $T \in End^*_{\mathcal{A}}(\mathcal{H})$ be a co-isometry (i.e. $TT^* = Id_H$) such that $TK = KT$. Then $\{\Lambda_i T^*\}_{i\in I}$ is a $K-g$-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$.

Proof. Suppose $\{\Lambda_i\}_{i\in I}$ be a $K-g$- frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with a lower frame bound A_1 and B_1. For each $x \in \mathcal{H}$, we have $\sum_{i\in I}\langle \Lambda_i T^*x, \Lambda_i T^*x\rangle_{\mathcal{A}} \leqslant B_1\langle T^*x, T^*x\rangle_{\mathcal{A}}$. Hence, $\sum_{i\in I}\langle \Lambda_i T^*x, \Lambda_i T^*x\rangle_{\mathcal{A}} \leqslant B_1\|T^*\|^2\langle x, x\rangle_{\mathcal{A}}$. So, $\{\Lambda_i T^*\}_{i\in I}$ is a g-Bessel sequence. For the lower bound, we can write

$$\begin{aligned}
\sum_{i\in I}\langle \Lambda_i T^*x, \Lambda_i T^*x\rangle_{\mathcal{A}} \geqslant A_1\langle K^*T^*x, K^*T^*x\rangle_{\mathcal{A}} \\
&= A_1\langle (TK)^*x, (TK)^*x\rangle_{\mathcal{A}} \\
&= A_1\langle (KT)^*x, (KT)^*x\rangle_{\mathcal{A}} \\
&= A_1\langle T^*K^*x, T^*K^*x\rangle_{\mathcal{A}} \\
&= A_1\langle TT^*K^*x, K^*x\rangle_{\mathcal{A}} \\
&= A_1\langle K^*x, K^*x\rangle_{\mathcal{A}}.
\end{aligned}$$

■

Theorem 7.7 Let $\Lambda := \{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{V}_i)\}_{i\in I}$ and $\ominus := \{\ominus_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{V}_i)\}_{i\in I}$ be tow $K-g$- Bessel sequences in Hilbert $\mathcal{A}$- module $\mathcal{H}$ with bounds B_Λ and $B_\ominus$ respectively. Suppose that T_Λ and $T_\ominus$ are their synthesis operators such that $T_{\ominus,}T^*_\Lambda = K^*$. Then Λ and $\ominus$ are K and $K^* - g$-frames, respectively.

Proof. For each $x \in \mathcal{H}$, we have

$$\begin{aligned}
\|K^*x\|^2 = \|\langle K^*x, K^*x\rangle_{\mathcal{A}}\| &= \|\langle T_\ominus T^*_\Lambda x, K^*x\rangle_{\mathcal{A}}\| \leqslant \|T^*_\Lambda x\|\|T^*_{\ominus,}K^*x\| \\
&\leqslant (\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}})^{1/2}B_\ominus\|\langle K^*x, K^*x\rangle_{\mathcal{A}}\|.
\end{aligned}$$

So, $\|\langle K^*x, K^*x\rangle_{\mathcal{A}}\| \leqslant \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}}B_\ominus$. ■

8. $*$-K-Frames

Definition 8.1 [6] Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$. A family $\{x_i\}_{i\in I}$ of elements of $\mathcal{H}$ is a $*$-K-frame for $\mathcal{H}$ if there exist strictly nonzero elements A and B in $\mathcal{A}$ such that for all $x \in \mathcal{H}$,

$$A\langle K^*x, K^*x\rangle_{\mathcal{A}} A^* \leqslant \sum_{i\in I}\langle x, x_i\rangle_{\mathcal{A}}\langle x_i, x\rangle_{\mathcal{A}} \leqslant B\langle x, x\rangle_{\mathcal{A}} B^*.$$

The elements A and B are called lower and upper bound of the $*$-K-frame, respectively.

Remark 4 *Every $*$-frame is a $*$-K-frame.*

9. $*$-K-g-Frames in Hilbert A-modules

Definition 9.1 [21] Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$. We call a sequence $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a $*$-K-g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ if there exist strictly nonzero elements A, B in $\mathcal{A}$ such that $A\langle K^*x, K^*x\rangle_{\mathcal{A}} A^* \leqslant \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \leqslant B\langle x, x\rangle_{\mathcal{A}} B^*$ for all $x \in \mathcal{H}$. The numbers A and B are called lower and upper bounds of the $*$-K-g-frame, respectively. If $A\langle K^*x, K^*x\rangle A^* = \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle$ for all $x \in \mathcal{H}$. The $*$-K-g-frame is A-tight.

Remark 5 *[21]*

1. *(1) Every $*$-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ is a $*$-K-g-frame, for any $K \in End^*_{\mathcal{A}}(\mathcal{H})$: $K \neq 0$.*
2. *(2) If $K \in End^*_{\mathcal{A}}(\mathcal{H})$ is a surjective operator, then every $*$-K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ is a $*$-g-frame.*

***Example* 9.2** [21] Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module. Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$: $K \neq 0$. Let $\mathcal{A}$ be a Hilbert $\mathcal{A}$-module over itself with the inner product $\langle a, b\rangle = ab^*$. Let $\{x_i\}_{i\in I}$ be a $*$-frame for $\mathcal{H}$ with bounds A and B, respectively. For each $i \in I$, we define $\Lambda_i : \mathcal{H} \to \mathcal{A}$ by $\Lambda_i x = \langle x, x_i\rangle$ for all $x \in \mathcal{H}$. Λ_i is adjointable and $\Lambda_i^* a = ax_i$ for each $a \in \mathcal{A}$, and we have $A\langle x, x\rangle A^* \leqslant \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leqslant B\langle x, x\rangle B^*$ for all $x \in \mathcal{H}$ or $\langle K^*x, K^*x\rangle \leqslant \|K\|^2\langle x, x\rangle$ for all $x \in \mathcal{H}$. Then

$$\|K\|^{-1}A\langle K^*x, K^*x\rangle(\|K\|^{-1}A)^* \leqslant \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \leqslant B\langle x, x\rangle B^*, \quad \forall x \in \mathcal{H}.$$

So $\{\Lambda_i\}_{i\in I}$ is a $*$-K-g-frame for $\mathcal{H}$ with bounds $\|K\|^{-1}A$ and B, respectively.

Let $\{\Lambda_i\}_{i\in I}$ be a $*$-K-g-frame in $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$. Define an operator $T : \mathcal{H} \to \oplus_{i\in I}\mathcal{H}_i$ by $Tx = \{\Lambda_i x\}_i, \forall x \in \mathcal{H}$, then T is called the analysis operator. So it's adjoint operator is $T^* : \oplus_{i\in I}\mathcal{H}_i \to \mathcal{H}$ given by $T^*(\{x_i\}_i) = \sum_{i\in I}\Lambda_i^* x_i$ for all $\{x_i\}_i \in \oplus_{i\in I}\mathcal{H}_i$. The operator T^* is called the synthesis operator. By composing T and T^*, the frame operator $S : \mathcal{H} \to \mathcal{H}$ is given by $Sx = T^*Tx = \sum_{i\in I}\Lambda_i^*\Lambda_i x$. Note that S need not be invertible in general. But under some condition S will be invertible.

Theorem 9.3 [21] Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ be a surjective operator. If $\{\Lambda_i\}_{i\in I}$ is a $*$-K-g-frame in $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$, then the frame operator S is positive, invertible and adjointable. Moreover, we have the reconstruction formula $x = \sum_{i\in I}\Lambda_i^*\Lambda_i S^{-1}x$ for all $x \in \mathcal{H}$.

Proof. Result of (2) in Remark *5* and Theorem 3.8 in [1]. ■

Let $K \in End_{\mathcal{A}}^{*}(\mathcal{H})$. Now, using a $*$-g-frame, we constructed a $*$-K-g-frame.

Theorem 9.4 [21] Let $K \in End_{\mathcal{A}}^{*}(\mathcal{H})$ and $\{\Lambda_i\}_{i\in I}$ be a $*$-g-frame in $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ with bounds A and B. Then $\{\Lambda_i K\}_{i\in I}$ is a $*$-K^*-g-frame in $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ with bounds A and $\|K\|B$. The frame operator of $\{\Lambda_i K\}_{i\in I}$ is $S' = K^*SK$, where S is the frame operator of $\{\Lambda_i\}_{i\in I}$.

Proof. Form $A\langle x, x\rangle_{\mathcal{A}} A^* \leqslant \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \leqslant B\langle x, x\rangle_{\mathcal{A}} B^*$ for all $x \in \mathcal{H}$, we get

$$A\langle Kx, Kx\rangle_{\mathcal{A}} A^* \leqslant \sum_{i\in I}\langle \Lambda_i Kx, \Lambda_i Kx\rangle_{\mathcal{A}} \leqslant B\langle Kx, Kx\rangle_{\mathcal{A}} B^* \leqslant \|K\|B\langle x, x\rangle_{\mathcal{A}}(\|K\|B)^*.$$

Then $\{\Lambda_i K\}_{i\in I}$ is a $*$-K^*-g-frame in $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ with bounds A and $\|K\|B$. By definition of S, we have $SKx = \sum_{i\in I}\Lambda_i^*\Lambda_i Kx$. Then

$$K^*SKx = K^*\sum_{i\in I}\Lambda_i^*\Lambda_i Kx = \sum_{i\in I}K^*\Lambda_i^*\Lambda_i Kx.$$

Hence, $S' = K^*SK$. ∎

Corollary 9.5 [21] Let $K \in End_{\mathcal{A}}^{*}(\mathcal{H})$ and $\{\Lambda_i\}_{i\in I}$ be a $*$-g-frame. Then $\{\Lambda_i S^{-1}K\}_{i\in I}$ is a $*$-K^*-g-frame, where S is the frame operator of $\{\Lambda_i\}_{i\in I}$.

Proof. Result of the Theorem 9.4 for the $*$-g-frame $\{\Lambda_i S^{-1}\}_{i\in I}$. ∎

Remark 6 *Note that*

- *If $A, B \in \mathbf{C}$ and $K = I$ in Definition 9.1 we find the definition of the g-frame.*
- *If $A, B \in \mathbf{C}$ in Definition 9.1 we find the definition of the K-g-frame.*
- *For $K = I$ in Definition 9.1 we find the definition of the $*$-g-frame.*

10. Operator frame

Definition 10.1 [22] A family of adjointable operators $\{T_i\}_{i\in\mathbb{J}}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said to be an operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ if there exists positive constants $A, B > 0$ such that

$$A\langle x, x\rangle \leqslant \sum_{i\in\mathbb{J}}\langle T_i x, T_i x\rangle \leqslant B\langle x, x\rangle, \quad \forall x \in \mathcal{H}. \tag{16}$$

The numbers A and B are called lower and upper bounds of the operator frame, respectively. If $A = B = \lambda$, the operator frame is λ-tight. If $A = B = 1$, it is called a normalized tight operator frame or a Parseval operator frame. If only upper inequality of (16) hold, then $\{T_i\}_{i\in\mathbb{J}}$ is called an operator Bessel sequence for $End_{\mathcal{A}}^{*}(\mathcal{H})$. If the sum in the middle of (16) is convergent in norm, the operator frame is called standard.

Throughout the paper, series like (16) are assumed to be convergent in the norm sense.

***Example* 10.2** [22] Let $\mathcal{A}$ be a Hilbert C^*-module over itself with the inner product $\langle a, b\rangle = ab^*$ and $\{x_i\}_{i\in I}$ be a frame for $\mathcal{A}$ with bounds A and B, respectively. For each $i \in I$, we define $T_i : \mathcal{A} \to \mathcal{A}$ by $T_i x = \langle x, x_i\rangle$ for all $x \in \mathcal{A}$. T_i is adjointable and $T_i^* a = ax_i$ for each $a \in \mathcal{A}$, and we have $A\langle x, x\rangle \leqslant \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leqslant B\langle x, x\rangle$ for all $x \in \mathcal{A}$. Thus,

$A\langle x,x\rangle \leqslant \sum_{i\in I}\langle T_ix,T_ix\rangle \leqslant B\langle x,x\rangle, \forall x\in\mathcal{A}$. So, $\{T_i\}_{i\in I}$ is an operator frame in $\mathcal{A}$ with bounds A and B, respectively.

***Example* 10.3** [22] Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and let $B(\mathcal{H},\mathcal{K})$ be the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$. $B(\mathcal{H},\mathcal{K})$ is a Hilbert $B(\mathcal{K})$-module with a $B(\mathcal{K})$-valued inner product $\langle S,T\rangle = ST^*$ for all $S,T\in B(\mathcal{H},\mathcal{K})$, and with a linear operation of $B(\mathcal{K})$ on $B(\mathcal{H},\mathcal{K})$ by composition of operators. Let $\mathbb{J}=\mathbb{N}$ and fix $(a_i)_{i\in\mathbb{N}}\in l^2(\mathbf{C})$. Define: $T_i(X)=a_iX$ for all $X\in B(\mathcal{H},\mathcal{K})$ and $i\in\mathbb{N}$. We have for all $X\in B(\mathcal{H},\mathcal{K})$ that $\sum_{i\in\mathbb{N}}\langle T_ix,T_ix\rangle = \sum_{i\in\mathbb{N}}|a_i|^2\langle X,X\rangle$, and $\{T_i\}_{i\in\mathbb{N}}$ is $\sum_{i\in\mathbb{N}}|a_i|^2$-tight operator frame.

***Example* 10.4** [22] Let $\{W_i\}_{i\in\mathbb{J}}$ be a frame of submodules with respect to $\{v_i\}_{i\in\mathbb{J}}$ for $\mathcal{H}$. Put $T_i=v_i\pi_{W_i}, \forall i\in\mathbb{J}$, then we get a sequence of operators $\{T_i\}_{i\in\mathbb{J}}$. Then there exist constants $A,B>0$ such that $A\langle x,x\rangle \leqslant \sum_{i\in\mathbb{J}} v_i^2\langle \pi_{W_i}x,\pi_{W_i}x\rangle \leqslant B\langle x,x\rangle$ for all $x\in\mathcal{H}$. So, we have $A\langle x,x\rangle \leqslant \sum_{i\in\mathbb{J}}\langle T_ix,T_ix\rangle \leqslant B\langle x,x\rangle$ for all $x\in\mathcal{H}$. Thus, the sequence $\{T_i\}_{i\in\mathbb{J}}$ becomes an operator frame for $\mathcal{H}$.

In this example, a frame of submodules can be viewed as a special case of operator frames.

Theorem 10.5 Let $\{T_i\}_{i\in I}$ be an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with bounds ν and δ. If $\{R_i\}_{i\in I}\subset End_{\mathcal{A}}^*(\mathcal{H})$ is an operator Bessel family with bound $\xi<\nu$, then $\{T_i\mp R_i\}_{i\in I}$ is an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$.

Proof. We just proof the case that $\{T_w+R_w\}_{w\in\Omega}$ is an operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$. On one hand, for each $x\in\mathcal{H}$, we have

$$\begin{aligned}
\|\{(T_i+R_i)f\}_{i\in I}\| &= \|\sum_{i\in I}\langle (T_i+R_i)f,(T_i+R_i)f\rangle_{\mathcal{A}}\|^{\frac{1}{2}}\\
&\leqslant \|\{T_if\}_{i\in I}\| + \|\{R_if\}_{i\in I}\|\\
&= \|\sum_{i\in I}\langle T_if,T_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}} + \|\sum_{i\in I}\langle R_if,R_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}}\\
&\leqslant \sqrt{\delta}\|f\| + \sqrt{\xi}\|f\|.
\end{aligned}$$

Hence,

$$\|\sum_{i\in I}\langle (T_i+R_i)f,(T_i+R_i)f\rangle_{\mathcal{A}}\|^{\frac{1}{2}} \leqslant (\sqrt{\delta}+\sqrt{\xi})\|f\|. \tag{17}$$

One the other hand, we have

$$\begin{aligned}
\|\{(T_i+R_i)f\}_{i\in I}\| &= \|\sum_{i\in I}\langle (T_i+R_i)f,(T_i+R_i)f\rangle_{\mathcal{A}}\|^{\frac{1}{2}}\\
&\geqslant \|\{T_if\}_{i\in I}\| - \|\{R_if\}_{i\in I}\|\\
&= \|\sum_{i\in I}\langle T_if,T_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}} - \|\sum_{i\in I}\langle R_if,R_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}}\\
&\geqslant \sqrt{\nu}\|f\| - \sqrt{\xi}\|f\|.
\end{aligned}$$

Then

$$\|\sum_{i\in I}\langle (T_i+R_i)f,(T_i+R_i)f\rangle_{\mathcal{A}}\|^{\frac{1}{2}} \geqslant (\sqrt{\nu}-\sqrt{\xi})\|f\|. \tag{18}$$

From (17) and (18), we get

$$(\sqrt{\nu}-\sqrt{\xi})^2\|f\|^2 \leqslant \|\sum_{i\in I}\langle (T_i+R_i)f,(T_i+R_i)f\rangle_{\mathcal{A}}\| \leqslant (\sqrt{\delta}+\sqrt{\xi})^2\|f\|^2.$$

Therefore, $\{T_i+R_i\}_{i\in I}$ is an operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$. ■

Theorem 10.6 Let $\{T_i\}_{i\in I}$ be an operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ with bounds ν and δ and let $\{R_i\}_{i\in I}\subset End^*_{\mathcal{A}}(\mathcal{H})$. The following statements are equivalent:
(i) $\{R_i\}_{i\in I}$ is an operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$.
(ii) There exists a constant $\xi>0$ such that for all $x\in\mathcal{H}$,

$$\|\sum_{i\in I}\langle (T_i-R_i)f,(T_i-R_i)f\rangle_{\mathcal{A}}\| \leqslant \xi.min(\|\sum_{i\in I}\langle T_if,T_if\rangle_{\mathcal{A}}\|,\|\sum_{i\in I}\langle R_if,R_if\rangle_{\mathcal{A}}\|). \tag{19}$$

Proof. Suppose that $\{R_i\}_{i\in I}$ is an operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ with bound η and ρ. Then for all $f\in\mathcal{H}$ we have

$$\begin{aligned}\|\{(T_i-R_i)f\}_{i\in I}\| &= \|\sum_{i\in I}\langle (T_i-R_i)f,(T_i-R_i)f\rangle_{\mathcal{A}}\|^{\frac{1}{2}} \leqslant \|\{T_if\}_{i\in I}\|+\|\{R_if\}_{i\in I}\|\\ &= \|\langle T_if,T_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}}+\|\sum_{i\in I}\langle R_if,R_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}} \leqslant \|\sum_{i\in I}\langle T_if,T_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}}+\sqrt{\rho}\|f\|\\ &\leqslant \|\sum_{i\in I}\langle T_if,T_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}}+\sqrt{\frac{\rho}{\nu}}\|\sum_{i\in I}\langle T_if,T_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}}\\ &= (1+\sqrt{\frac{\rho}{\nu}})\|\sum_{i\in I}\langle T_if,T_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}}.\end{aligned}$$

In the same way, we have

$$\|\sum_{i\in I}\langle (T_i-R_i)f,(T_i-R_i)f\rangle_{\mathcal{A}}\|^{\frac{1}{2}} \leqslant \left(1+\sqrt{\frac{\delta}{\eta}}\right)\|\sum_{i\in I}\langle R_if,R_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}}.$$

For (19), we take $\xi=min(1+\sqrt{\frac{\delta}{\eta}},1+\sqrt{\frac{\rho}{\nu}})$. Now, we assume that (19) holds. For each $f\in\mathcal{H}$, we have from (19) that $\|\sum_{i\in I}\langle (T_i-R_i)f,(T_i-R_i)f\rangle_{\mathcal{A}}\| \leqslant \xi\|\sum_{i\in I}\langle R_if,R_if\rangle_{\mathcal{A}}\|$. Then

$$\|\sum_{i\in I}\langle T_if,T_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}} \leqslant \sqrt{\xi}\|\sum_{i\in I}\langle R_if,R_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}}+\|\sum_{i\in I}\langle R_if,R_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}}.$$

Hence,

$$\sqrt{\nu}\|f\| \leqslant \|\sum_{i\in I}\langle T_if, T_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}} \leqslant (1+\sqrt{\xi})\|\sum_{i\in I}\langle R_if, R_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}}. \tag{20}$$

Also, we have

$$\begin{aligned}\|\{R_if\}_{i\in I}\| &= \|\sum_{i\in I}\langle R_if, R_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}} \\ &= \|\{(R_if - T_if) + T_if\}_{i\in I}\| \\ &\leqslant \|\{(T_i - R_i)f\}_{i\in I}\| + \|\{T_if\}_{i\in I}\| \\ &= \|\sum_{i\in I}\langle (T_i - R_i)f, (T_i - R_i)f\rangle_{\mathcal{A}}\|^{\frac{1}{2}} + \|\sum_{i\in I}\langle T_if, T_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}}.\end{aligned}$$

From (19), we have $\|\sum_{i\in I}\langle (T_i - R_i)f, (T_i - R_i)f\rangle_{\mathcal{A}}\| \leqslant \xi\|\sum_{i\in I}\langle T_if, T_if\rangle_{\mathcal{A}}\|$. Then $\|\sum_{i\in I}\langle R_if, R_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}} \leqslant (1+\sqrt{\xi})\|\sum_{i\in I}\langle T_if, T_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}}$. So,

$$\|\sum_{i\in I}\langle R_if, R_if\rangle_{\mathcal{A}}\|^{\frac{1}{2}} \leqslant (1+\sqrt{\xi})\sqrt{\delta}\|f\|. \tag{21}$$

From (20) and (21), we get $\frac{\nu}{(1+\sqrt{\xi})^2}\|f\|^2 \leqslant \|\sum_{i\in I}\langle R_if, R_if\rangle_{\mathcal{A}}\| \leqslant \delta(1+\sqrt{\xi})^2\|f\|^2$. Therefore, $\{R_i\}_{i\in I}$ is an operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$. ■

Theorem 10.7 Let $\{T_{k,i}\}_{i\in I} \subset End_{\mathcal{A}}^{*}(\mathcal{H})$ be an operator frames with bounds A_k and B_k for $k = 1, 2, ..., n$ and $\{R_{k,i}\}_{i\in I} \subset End_{\mathcal{A}}^{*}(\mathcal{H})$ and $L : l^2(\mathcal{H}) \longrightarrow l^2(\mathcal{H})$ be a bounded linear operator such that $L(\{\sum_{k=1}^{n} R_{k,i}x\}_{i\in I}) = \{T_{p,i}x\}_{i\in I}$ for some $p \in \{1, 2, .., n\}$. If there exists a constant $\lambda > 0$ such that

$$\|\sum_{i\in I}\langle (T_{k,i} - R_{k,i})x, (T_{k,i} - R_{k,i})x\rangle_{\mathcal{A}}\| \leqslant \lambda\|\sum_{i\in I}\langle T_{k,i}x, T_{k,i}x\rangle_{\mathcal{A}}\|$$

for each $x \in \mathcal{H}$ and $k = 1, .., n$, then $\{\sum_{k=1}^{n} R_{k,i}\}_{i\in I}$ is an operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$.

Proof. For all $x \in \mathcal{H}$, we have

$$\begin{aligned}\|\{\sum_{k=1}^{n} R_{k,i}x\}_{i\in I}\| &\leqslant \sum_{k=1}^{n}\|\{R_{k,i}x\}_{i\in I}\| \leqslant \sum_{k=1}^{n}(\|\{T_{k,i} - R_{k,i}x\}_{i\in I}\| + \|\{T_{k,i}x\}_{i\in I}\|) \\ &\leqslant (1+\sqrt{\lambda})\|\sum_{k=1}^{n}\|\{T_{k,i}x\}_{i\in I}\| \leqslant (1+\sqrt{\lambda})(\sum_{k=1}^{n}\sqrt{B_k})\|\langle x, x\rangle_{\mathcal{A}}\|^{\frac{1}{2}}.\end{aligned}$$

Since we have $\|L(\{\sum_{k=1}^{n} R_{k,i}x\}_{i\in I})\| = \|\{T_{p,i}x\}_{i\in I}\|$ for any $x \in \mathcal{H}$, then

$$\sqrt{A_p}\|\langle x, x\rangle_{\mathcal{A}}\|^{\frac{1}{2}} \leqslant \|\{T_{p,i}x\}_{i\in I}\| = \|L(\{\sum_{k=1}^{n} R_{k,i}x\}_{i\in I})\| \leqslant \|L\|\|\{\sum_{k=1}^{n} R_{k,i}x\}_{i\in I}\|.$$

Hence, $\frac{\sqrt{A_p}}{\|L\|}\|\langle x,x\rangle_{\mathcal{A}}\|^{\frac{1}{2}} \leqslant \|\{\sum_{k=1}^{n} R_{k,i}x\}_{i\in I}\|$ for all $x\in\mathcal{H}$. Therefore,

$$\frac{\sqrt{A_p}}{\|L\|}\|\langle x,x\rangle_{\mathcal{A}}\|^{\frac{1}{2}} \leqslant \|\{\sum_{k=1}^{n} R_{k,i}\}_{i\in I}\| \leqslant (1+\sqrt{\lambda})(\sum_{k=1}^{n}\sqrt{B_k})\|\langle x,x\rangle_{\mathcal{A}}\|^{\frac{1}{2}}.$$

This gives that $\{\sum_{k=1}^{n} R_{k,i}\}_{i\in I}$ is an operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$. ■

11. *K*-operator frame

Definition 11.1 Let $K \in End_{\mathcal{A}}^{*}(\mathcal{H})$. A family of adjointable operators $\{T_i\}_{i\in\mathbb{J}}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said to be a K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ if there exists positive constants $A, B > 0$ such that

$$A\langle K^*x, K^*x\rangle \leqslant \sum_{i\in\mathbb{J}}\langle T_ix, T_ix\rangle \leqslant B\langle x,x\rangle, \quad \forall x\in\mathcal{H}. \tag{22}$$

The numbers A and B are called lower and upper bound of the K-operator frame, respectively. If $A\langle K^*x, K^*x\rangle = \sum_{i\in\mathbb{J}}\langle T_ix, T_ix\rangle$, the K-operator frame is A-tight. If $A = 1$, it is called a normalized tight K-operator frame or a Parseval K-operator frame.

***Example* 11.2** Let l^∞ be the set of all bounded complex-valued sequences. For any $u = \{u_j\}_{j\in\mathbb{N}}, v = \{v_j\}_{j\in\mathbb{N}} \in l^\infty$, we define $uv = \{u_jv_j\}_{j\in\mathbb{N}}$, $u^* = \{\bar{u_j}\}_{j\in\mathbb{N}}$ and $\|u\| = \sup_{j\in\mathbb{N}}|u_j|$. Then $\mathcal{A} = \{l^\infty, \|.\|\}$ is a $\mathbb{C}^*$-algebra. Let $\mathcal{H} = C_0$ be the set of all sequences converging to zero. For any $u, v \in \mathcal{H}$, we define $\langle u, v\rangle = uv^* = \{u_j\bar{u_j}\}_{j\in\mathbb{N}}$. Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module. Now, let $\{e_j\}_{j\in\mathbb{N}}$ be the standard orthonormal basis of $\mathcal{H}$. For each $j\in\mathbb{N}$, define the adjointable operator $T_j : \mathcal{H}\to\mathcal{H}$ by $T_jx = \langle x, e_j\rangle e_j$. Then for every $x\in\mathcal{H}$, we have $\sum_{j\in\mathbb{N}}\langle T_jx, T_jx\rangle = \langle x,x\rangle$. Fix $N\in\mathbb{N}^*$ and define

$$K:\mathcal{H}\to\mathcal{H}, \;\; Ke_j = \begin{cases} je_j & \text{if } j\leqslant N,\\ 0 & \text{if } j > N.\end{cases}$$

It is easy to check that K is adjointable and satisfies

$$K^*e_j = \begin{cases} je_j & \text{if } j\leqslant N,\\ 0 & \text{if } j > N.\end{cases}$$

For any $x\in\mathcal{H}$, we have $\dfrac{1}{N^2}\langle K^*x, K^*x\rangle \leqslant \sum_{j\in\mathbb{N}}\langle T_jx, T_jx\rangle = \langle x,x\rangle$. This shows that $\{T_j\}_{j\in\mathbb{N}}$ is a K-operator frame with bounds $\dfrac{1}{N^2}, 1$.

One may ask for the class of operators K which can guarantee the existence of K-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$. The following remark and proposition answer this query.

***Remark* 7** *Every operator frame is a K-operator frame for any $K\in End_{\mathcal{A}}^{*}(\mathcal{H})$ where $K\neq 0$. Indeed, for any $K\in End_{\mathcal{A}}^{*}(\mathcal{H})$, the inequality $\langle K^*x, K^*x\rangle \leqslant \|K\|^2\langle x,x\rangle$ for all*

$x \in \mathcal{H}$ holds. Now, if $\{T_i\}_{i\in\mathbb{J}}$ is an operator frame with bounds A and B, then

$$A\|K\|^{-2}\langle K^*x, K^*x\rangle \leqslant A\langle x, x\rangle \leqslant \sum_{i\in\mathbb{J}}\langle T_ix, T_ix\rangle \leqslant B\langle x, x\rangle, \quad \forall x \in \mathcal{H}.$$

Therefore, $\{T_i\}_{i\in\mathbb{J}}$ is a K-operator frame with bounds $A\|K\|^{-2}$ and B.

12. K-operator frame in tensor products of Hilbert C^*-modules

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and take $\mathcal{A}\otimes\mathcal{B}$ as the completion of $\mathcal{A}\otimes_{alg}\mathcal{B}$ with the spatial norm. $\mathcal{A}\otimes\mathcal{B}$ is the spatial tensor product of $\mathcal{A}$ and $\mathcal{B}$. Also suppose that $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module and $\mathcal{K}$ is a Hilbert $\mathcal{B}$-module. We want to define $\mathcal{H}\otimes\mathcal{K}$ as a Hilbert $(\mathcal{A}\otimes\mathcal{B})$-module. Start by forming the algebraic tensor product $\mathcal{H}\otimes_{alg}\mathcal{K}$ of the vector spaces $\mathcal{H}$, $\mathcal{K}$ (over $\mathbb{C}$). This is a left module over $(\mathcal{A}\otimes_{alg}\mathcal{B})$ (the module action being given by $(a\otimes b)(x\otimes y) = ax\otimes by$, where $a\in\mathcal{A}$, $b\in\mathcal{B}$, $x\in\mathcal{H}$ and $y\in\mathcal{K}$). For $x_1, x_2\in\mathcal{H}$ and $y_1, y_2\in\mathcal{K}$, we define

$$\langle x_1\otimes y_1, x_2\otimes y_2\rangle_{\mathcal{A}\otimes\mathcal{B}} = \langle x_1, x_2\rangle_{\mathcal{A}}\otimes\langle y_1, y_2\rangle_{\mathcal{B}}.$$

We also know that for $z=\sum_{i=1}^n x_i\otimes y_i\in\mathcal{H}\otimes_{alg}\mathcal{K}$, $\langle z,z\rangle_{\mathcal{A}\otimes\mathcal{B}} = \sum_{i,j}\langle x_i, x_j\rangle_{\mathcal{A}}\otimes\langle y_i, y_j\rangle_{\mathcal{B}} \geqslant 0$ and $\langle z,z\rangle_{\mathcal{A}\otimes\mathcal{B}} = 0$ iff $z=0$. This extends by linearity to an $(\mathcal{A}\otimes_{alg}\mathcal{B})$-valued sesquilinear form on $\mathcal{H}\otimes_{alg}\mathcal{K}$, which makes $\mathcal{H}\otimes_{alg}\mathcal{K}$ into a semi-inner-product module over the pre-C^*-algebra $(\mathcal{A}\otimes_{alg}\mathcal{B})$. The semi-inner-product on $\mathcal{H}\otimes_{alg}\mathcal{K}$ is actually an inner product, see [16]. Then $\mathcal{H}\otimes_{alg}\mathcal{K}$ is an inner-product module over the pre-C^*-algebra $(\mathcal{A}\otimes_{alg}\mathcal{B})$, and we can perform the double completion discussed in chapter 1 of [16] to conclude that the completion $\mathcal{H}\otimes\mathcal{K}$ of $\mathcal{H}\otimes_{alg}\mathcal{K}$ is a Hilbert $(\mathcal{A}\otimes\mathcal{B})$-module. We call $\mathcal{H}\otimes\mathcal{K}$ the exterior tensor product of $\mathcal{H}$ and $\mathcal{K}$. With $\mathcal{H}$,$\mathcal{K}$ as above, we wish to investigate the adjointable operators on $\mathcal{H}\otimes\mathcal{K}$. Suppose that $S\in End^*_{\mathcal{A}}(\mathcal{H})$ and $T\in End^*_{\mathcal{B}}(\mathcal{K})$. We define a linear operator $S\otimes T$ on $\mathcal{H}\otimes\mathcal{K}$ by $S\otimes T(x\otimes y) = Sx\otimes Ty (x\in\mathcal{H}, y\in\mathcal{K})$. It is a routine verification that is $S^*\otimes T^*$ is the adjoint of $S\otimes T$, so in fact $S\otimes T\in End^*_{\mathcal{A}\otimes\mathcal{B}}(\mathcal{H}\otimes\mathcal{K})$. For more details see [7, 16]. We note that if $a\in\mathcal{A}^+$ and $b\in\mathcal{B}^+$, then $a\otimes b\in(\mathcal{A}\otimes\mathcal{B})^+$. Plainly if a , b are Hermitian elements of $\mathcal{A}$ and $a\geqslant b$, then for every positive element x of $\mathcal{B}$, we have $a\otimes x\geqslant b\otimes x$.

Theorem 12.1 Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert C^*-modules over unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\{\Lambda_i\}_{i\in I}\subset End^*_{\mathcal{A}}(\mathcal{H})$ be a K_1-operator frame for $\mathcal{H}$ and $\{\Gamma_j\}_{j\in J}\subset End^*_{\mathcal{B}}(\mathcal{K})$ be a K_2-operator frame for $\mathcal{K}$ with frame operators S_Λ and S_Γ and operator frame bounds (A,B) and (C,D) respectively. Then $\{\Lambda_i\otimes\Gamma_j\}_{i\in I, j\in J}$ is a $K_1\otimes K_2$-operator frame for Hibert $\mathcal{A}\otimes\mathcal{B}$-module $\mathcal{H}\otimes\mathcal{K}$ with frame operator $S_\Lambda\otimes S_\Gamma$ and lower and upper operator frame bounds AC and BD, respectively.

Proof. By the definition of K_1-operator frame $\{\Lambda_i\}_{i\in I}$ and K_2-operator frame $\{\Gamma_j\}_{j\in J}$, we have

$$A\langle K_1^*x, K_1^*x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle\Lambda_ix, \Lambda_ix\rangle_{\mathcal{A}} \leqslant B\langle x, x\rangle_{\mathcal{A}}, \quad \forall x\in\mathcal{H},$$

$$C\langle K_2^*y, K_2^*y\rangle_{\mathcal{B}} \leqslant \sum_{j\in J}\langle\Gamma_jy, \Gamma_jy\rangle_{\mathcal{B}} \leqslant D\langle y, y\rangle_{\mathcal{B}}, \quad \forall y\in\mathcal{K}.$$

Therefore,

$$\begin{aligned}(A\langle K_1^*x, K_1^*x\rangle_{\mathcal{A}}) \otimes (C\langle K_2^*y, K_2^*y\rangle_{\mathcal{B}}) &\leqslant \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \otimes \sum_{j\in J}\langle \Gamma_j y, \Gamma_j y\rangle_{\mathcal{B}} \\ &\leqslant (B\langle x, x\rangle_{\mathcal{A}}) \otimes (D\langle y, y\rangle_{\mathcal{B}}), \forall x \in \mathcal{H}, \forall y \in \mathcal{K}.\end{aligned}$$

Then

$$\begin{aligned}AC(\langle K_1^*x, K_1^*x\rangle_{\mathcal{A}} \otimes \langle K_2^*y, K_2^*y\rangle_{\mathcal{B}}) &\leqslant \sum_{i\in I, j\in J}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \otimes \langle \Gamma_j y, \Gamma_j y\rangle_{\mathcal{B}} \\ &\leqslant BD(\langle x, x\rangle_{\mathcal{A}} \otimes \langle y, y\rangle_{\mathcal{B}}), \forall x \in \mathcal{H}, \forall y \in \mathcal{K}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}AC\langle K_1^*x \otimes K_2^*y, K_1^*x \otimes K_2^*y\rangle_{\mathcal{A}\otimes\mathcal{B}} &\leqslant \sum_{i\in I, j\in J}\langle \Lambda_i x \otimes \Gamma_j y, \Lambda_i x \otimes \Gamma_j y\rangle_{\mathcal{A}\otimes\mathcal{B}} \\ &\leqslant BD\langle x\otimes y, x\otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}}, \forall x \in \mathcal{H}, \forall y \in \mathcal{K}.\end{aligned}$$

Then, for all $x \otimes y$ in $\mathcal{H} \otimes \mathcal{K}$, we have

$$\begin{aligned}&AC\langle (K_1 \otimes K_2)^*(x\otimes y), (K_1 \otimes K_2)^*(x\otimes y)\rangle_{\mathcal{A}\otimes\mathcal{B}} \\ &\qquad \leqslant \sum_{i\in I, j\in J}\langle (\Lambda_i \otimes \Gamma_j)(x\otimes y), (\Lambda_i \otimes \Gamma_j)(x\otimes y)\rangle_{\mathcal{A}\otimes\mathcal{B}} \leqslant BD\langle x\otimes y, x\otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}}.\end{aligned}$$

The last inequality is satisfied for every finite sum of elements in $\mathcal{H} \otimes_{alg} \mathcal{K}$ and then it's satisfied for all $z \in \mathcal{H} \otimes \mathcal{K}$. It shows that $\{\Lambda_i \otimes \Gamma_j\}_{i\in I, j\in J}$ is a $K_1 \otimes K_2$-operator frame for Hilbert $\mathcal{A}\otimes\mathcal{B}$-module $\mathcal{H}\otimes\mathcal{K}$ with lower and upper operator frame bounds AC and BD, respectively. By the definition of frame operator S_Λ and S_Γ, we have $S_\Lambda x = \sum_{i\in I}\Lambda_i^*\Lambda_i x$ for all $x \in \mathcal{H}$ and $S_\Gamma y = \sum_{j\in J}\Gamma_j^*\Gamma_j y$ for all $y \in \mathcal{K}$. Therefore, we have

$$\begin{aligned}(S_\Lambda \otimes S_\Gamma)(x\otimes y) &= S_\Lambda x \otimes S_\Gamma y \\ &= \sum_{i\in I}\Lambda_i^*\Lambda_i x \otimes \sum_{j\in J}\Gamma_j^*\Gamma_j y \\ &= \sum_{i\in I, j\in J}\Lambda_i^*\Lambda_i x \otimes \Gamma_j^*\Gamma_j y \\ &= \sum_{i\in I, j\in J}(\Lambda_i^* \otimes \Gamma_j^*)(\Lambda_i x \otimes \Gamma_j y) \\ &= \sum_{i\in I, j\in J}(\Lambda_i^* \otimes \Gamma_j^*)(\Lambda_i \otimes \Gamma_j)(x\otimes y) \\ &= \sum_{i\in I, j\in J}(\Lambda_i \otimes \Gamma_j)^*)(\Lambda_i \otimes \Gamma_j)(x\otimes y).\end{aligned}$$

Now, by the uniqueness of frame operator, the last expression is equal to $S_{\Lambda\otimes\Gamma}(x\otimes y)$. Consequently, we have $(S_\Lambda \otimes S_\Gamma)(x\otimes y) = S_{\Lambda\otimes\Gamma}(x\otimes y)$. The last equality is satisfied for every finite sum of elements in $\mathcal{H} \otimes_{alg} \mathcal{K}$ and then it's satisfied for all $z \in \mathcal{H} \otimes \mathcal{K}$. It shows that $(S_\Lambda \otimes S_\Gamma)(z) = S_{\Lambda\otimes\Gamma}(z)$. So, $S_{\Lambda\otimes\Gamma} = S_\Lambda \otimes S_\Gamma$. ∎

Theorem 12.2 Assume $Q \in End^*_{\mathcal{A}}(\mathcal{H})$ is invertible and $\{\Lambda_i\}_{i\in I} \subset End^*_{\mathcal{A}\otimes\mathcal{B}}(\mathcal{H} \otimes \mathcal{K})$ is a K-operator frame for $\mathcal{H} \otimes \mathcal{K}$ with lower and upper operator frame bounds A and B, respectively and frame operator S. If K commute with $Q \otimes I$, then $\{\Lambda_i(Q^* \otimes I)\}_{i\in I}$ is a K-operator frame for $\mathcal{H} \otimes \mathcal{K}$ with lower and upper operator frame bounds $\|Q^{*-1}\|^{-2}A$ and $\|Q\|^2B$, respectively and frame operator $(Q \otimes I)S(Q^* \otimes I)$.

Proof. Since $Q \in End^*_{\mathcal{A}}(\mathcal{H})$, $Q\otimes I \in End^*_{\mathcal{A}\otimes\mathcal{B}}(\mathcal{H} \otimes \mathcal{K})$ with inverse $Q^{-1}\otimes I$. It is obvious that the adjoint of $Q \otimes I$ is $Q^* \otimes I$. An easy calculation shows that for every elementary tensor $x \otimes y$,

$$\|(Q \otimes I)(x \otimes y)\|^2 = \|Q(x) \otimes y\|^2 = \|Q(x)\|^2\|y\|^2 \leqslant \|Q\|^2\|x\|^2\|y\|^2 = \|Q\|^2\|x \otimes y\|^2.$$

So $Q \otimes I$ is bounded, and therefore it can be extended to $\mathcal{H} \otimes \mathcal{K}$. Similarly, for $Q^* \otimes I$, $Q \otimes I$ is $\mathcal{A} \otimes \mathcal{B}$-linear, adjointable with adjoint $Q^* \otimes I$. Hence, for every $z \in \mathcal{H} \otimes \mathcal{K}$, we have $\|Q^{*-1}\|^{-1}.|z| \leqslant |(Q^* \otimes I)z| \leqslant \|Q\|.|z|$. By the definition of K-operator frames, we have $A\langle K^*z, K^*z\rangle_{\mathcal{A}\otimes\mathcal{B}} \leqslant \sum_{i\in I}\langle\Lambda_i z, \Lambda_i z\rangle_{\mathcal{A}\otimes\mathcal{B}} \leqslant B\langle z, z\rangle_{\mathcal{A}\otimes\mathcal{B}}$. Then

$$\begin{aligned} A\langle K^*(Q^* \otimes I)z, K^*(Q^* \otimes I)z\rangle_{\mathcal{A}\otimes\mathcal{B}} &\leqslant \sum_{i\in I}\langle\Lambda_i(Q^* \otimes I)z, \Lambda_i(Q^* \otimes I)z\rangle_{\mathcal{A}\otimes\mathcal{B}} \\ &\leqslant B\langle(Q^* \otimes I)z, (Q^* \otimes I)z\rangle_{\mathcal{A}\otimes\mathcal{B}} \\ &\leqslant \|Q\|^2B\langle z, z\rangle_{\mathcal{A}\otimes\mathcal{B}} \end{aligned}$$

or

$$\begin{aligned} A\langle K^*(Q^* \otimes I)z, K^*(Q^* \otimes I)z\rangle_{\mathcal{A}\otimes\mathcal{B}} &= A\langle(Q^* \otimes I)K^*z, (Q^* \otimes I)K^*z\rangle_{\mathcal{A}\otimes\mathcal{B}} \\ &\geqslant \|Q^{*-1}\|^{-2}A\langle K^*z, K^*z\rangle_{\mathcal{A}\otimes\mathcal{B}}. \end{aligned}$$

So, we have

$$\|Q^{*-1}\|^{-2}A\langle K^*z, K^*z\rangle_{\mathcal{A}\otimes\mathcal{B}} \leqslant \sum_{i\in I}\langle\Lambda_i(Q^* \otimes I)z, \Lambda_i(Q^* \otimes I)z\rangle_{\mathcal{A}\otimes\mathcal{B}} \leqslant \|Q\|^2B\langle z, z\rangle_{\mathcal{A}\otimes\mathcal{B}}.$$

Now,

$$\begin{aligned} (Q \otimes I)S(Q^* \otimes I) &= (Q \otimes I)(\sum_{i\in I}\Lambda_i^*\Lambda_i)(Q^* \otimes I) \\ &= \sum_{i\in I}(Q \otimes I)\Lambda_i^*\Lambda_i(Q^* \otimes I) \\ &= \sum_{i\in I}(\Lambda_i(Q^* \otimes I))^*\Lambda_i(Q^* \otimes I), \end{aligned}$$

which completes the proof. ∎

Theorem 12.3 Assume that $Q \in End^*_{\mathcal{B}}(\mathcal{K})$ is invertible and $\{\Lambda_i\}_{i\in I} \subset End^*_{\mathcal{A}\otimes\mathcal{B}}(\mathcal{H} \otimes \mathcal{K})$ is a K-operator frame for $\mathcal{H} \otimes \mathcal{K}$ with lower and upper operator frame bounds A and B respectively and frame operator S. If K commute with $I \otimes Q$, then $\{\Lambda_i(I \otimes Q^*)\}_{i\in I}$ is a K-operator frame for $\mathcal{H} \otimes \mathcal{K}$ with lower and upper operator frame bounds $\|Q^{*-1}\|^{-2}A$ and $\|Q\|^2 B$ respectively and frame operator $(I \otimes Q)S(I \otimes Q^*)$.

Proof. Similar to the proof of the previous theorem. ■

13. K-operator frame in Hilbert C^*-modules with different C^*-algebras

Studying operator frame in Hilbert C^*-modules with different C^*-algebras is interesting and important. In the following theorem we study this situation.

Theorem 13.1 Let $(\mathcal{H}, \mathcal{A}, \langle ., .\rangle_{\mathcal{A}})$ and $(\mathcal{H}, \mathcal{B}, \langle ., .\rangle_{\mathcal{B}})$ be two Hilbert $\mathcal{C}^*$-modules and let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a $*$-homomorphism and θ be a map on $\mathcal{H}$ such that $\langle \theta x, \theta y\rangle_{\mathcal{B}} = \varphi(\langle x, y\rangle_{\mathcal{A}})$ for all $x, y \in \mathcal{H}$. Also, suppose that $\{\Lambda_i\}_{i\in I} \subset End^*_{\mathcal{A}}(\mathcal{H})$ is a K-operator frame for $(\mathcal{H}, \mathcal{A}, \langle ., .\rangle_{\mathcal{A}})$ with frame operator $S_{\mathcal{A}}$ and lower and upper operator frame bounds A and B, respectively. If θ is surjective, $\theta K^* = K^*\theta$, $\theta\Lambda_i = \Lambda_i\theta$ and $\theta\Lambda_i^* = \Lambda_i^*\theta$ for each i in I, then $\{\Lambda_i\}_{i\in I}$ is a K-operator frame for $(\mathcal{H}, \mathcal{B}, \langle ., .\rangle_{\mathcal{B}})$ with frame operator $S_{\mathcal{B}}$ and lower and upper operator frame bounds A and B, respectively and $\langle S_{\mathcal{B}}\theta x, \theta y\rangle_{\mathcal{B}} = \varphi(\langle S_{\mathcal{A}}x, y\rangle_{\mathcal{A}})$.

Proof. Let $y \in \mathcal{H}$ then there exists $x \in \mathcal{H}$ such that $\theta x = y$ (θ is surjective). By the definition of K-operator frames, we have $A\langle K^*x, K^*x\rangle_{\mathcal{A}} \leqslant \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \leqslant B\langle x, x\rangle_{\mathcal{A}}$ and $\varphi(A\langle K^*x, K^*x\rangle_{\mathcal{A}}) \leqslant \varphi(\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}}) \leqslant \varphi(B\langle x, x\rangle_{\mathcal{A}})$. By the definition of $*$-homomorphism, we have $A\varphi(\langle K^*x, K^*x\rangle_{\mathcal{A}}) \leqslant \sum_{i\in I}\varphi(\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}}) \leqslant B\varphi(\langle x, x\rangle_{\mathcal{A}})$. By the relation betwen θ and φ, we get $A\langle \theta K^*x, \theta K^*x\rangle_{\mathcal{B}} \leqslant \sum_{i\in I}\langle \theta\Lambda_i x, \theta\Lambda_i x\rangle_{\mathcal{B}} \leqslant B\langle \theta x, \theta x\rangle_{\mathcal{B}}$. By the relation betwen θ, K^* and Λ_i, we have

$$A\langle K^*\theta x, K^*\theta x\rangle_{\mathcal{B}} \leqslant \sum_{i\in I}\langle \Lambda_i\theta x, \Lambda_i\theta x\rangle_{\mathcal{B}} \leqslant B\langle \theta x, \theta x\rangle_{\mathcal{B}}.$$

Then $A\langle K^*y, K^*y\rangle_{\mathcal{B}} \leqslant \sum_{i\in I}\langle \Lambda_i y, \Lambda_i y\rangle_{\mathcal{B}} \leqslant B\langle y, y\rangle_{\mathcal{B}}$ for all $y \in \mathcal{H}$. On the other hand,

$$\begin{aligned}\varphi(\langle S_{\mathcal{A}}x, y\rangle_{\mathcal{A}}) &= \varphi(\langle\sum_{i\in I}\Lambda_i^*\Lambda_i x, y\rangle_{\mathcal{A}}) = \sum_{i\in I}\varphi(\langle \Lambda_i x, \Lambda_i y\rangle_{\mathcal{A}}) = \sum_{i\in I}\langle \theta\Lambda_i x, \theta\Lambda_i y\rangle_{\mathcal{B}}\\ &= \sum_{i\in I}\langle \Lambda_i\theta x, \Lambda_i\theta y\rangle_{\mathcal{B}} = \langle\sum_{i\in I}\Lambda_i^*\Lambda_i\theta x, \theta y\rangle_{\mathcal{B}} = \langle S_{\mathcal{B}}\theta x, \theta y\rangle_{\mathcal{B}}.\end{aligned}$$

This completes the proof. ■

14. Duals of K-operator frame

In the following we define the Dual K-operator frame and we give some properties.

Definition 14.1 Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}), i \in I\}$ be a K-operator frame for the Hilbert $\mathcal{A}$-module H. An operator Bessel sequences $\{\Gamma_i \in End^*_{\mathcal{A}}(\mathcal{H}), i \in I\}$ is called a K-duals operator frame for $\{\Lambda_i\}_{i\in I}$ if $Kf = \sum_{i\in I}\Lambda_i^*\Gamma_i f$ for all $f \in H$.

***Example* 14.2** Let $K \in End_{\mathcal{A}}{}^*(\mathcal{H})$ be a surjective operator and $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}), i \in I\}$ be a K-operator frame for $\mathcal{H}$ with K-frame operator S, then S is invertible. For all $f \in \mathcal{H}$, we have $Sf = \sum_{i\in I}\Lambda_i^*\Lambda_i f$. So $Kf = \sum_{i\in I}\Lambda_i^*\Lambda_i S^{-1}Kf$. Then the sequence $\{\Lambda_i S^{-1}K \in End^*_{\mathcal{A}}(\mathcal{H}), i \in I\}$ is a dual K-operator frame of $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}), i \in I\}$

Theorem 14.3 Let $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_j\}_{j\in J}$ are K-operator frame and L-operator frame respectively in $\mathcal{H}$ and $\mathcal{K}$ with duals $\{\tilde{\Lambda_i}\}_{i\in I}$ and $\{\tilde{\Gamma_j}\}_{j\in J}$ respectively, then $\{\tilde{\Lambda_i}\otimes\tilde{\Gamma_j}\}_{i,j\in I,J}$ is a dual of $\{\Lambda_i\otimes\Gamma_j\}_{i,j\in I,J}$.

Proof. By definition, for all $x\in\mathcal{H}$ and $y\in\mathcal{K}$, we have $\sum_{i\in I}\Lambda_i^*\tilde{\Lambda}_i x = Kx$ and $\sum_{j\in J}\Gamma_j^*\tilde{\Gamma}_j y = Ly$. Then

$$(K\otimes L)(x\otimes y) = Kx\otimes Ly = \sum_{i\in I}\Lambda_i^*\tilde{\Lambda}_i x\otimes\sum_{j\in J}\Gamma_j^*\tilde{\Gamma}_j y$$

and

$$\begin{aligned}\sum_{i\in I}\Lambda_i^*\tilde{\Lambda}_i x\otimes\sum_{j\in J}\Gamma_j^*\tilde{\Gamma}_j y &= \sum_{i,j\in I,J}\Lambda_i^*\tilde{\Lambda}_i x\otimes\Gamma_j^*\tilde{\Gamma}_j y\\ &= \sum_{i,j\in I,J}(\Lambda_i^*\otimes\Gamma_j^*).(\tilde{\Lambda}_i x\otimes\tilde{\Gamma}_j y)\\ &= \sum_{i,j\in I,J}(\Lambda_i\otimes\Gamma_j)^*.(\tilde{\Lambda_i}\otimes\tilde{\Gamma_j}).(x\otimes y)\end{aligned}$$

Thus, $\{\tilde{\Lambda_i}\otimes\tilde{\Gamma_j}\}_{i,j\in I,J}$ is a dual of $\{\Lambda_i\otimes\Gamma_j\}_{i,j\in I,J}$. ∎

Corollary 14.4 Let $\{\Lambda_{i,j}\}_{0\leqslant i\leqslant n;j\in J}$ be a family of K_i-operator frames such that $0\leqslant i\leqslant n$ and $\{\tilde{\Lambda}_{i,j}\}_{0\leqslant i\leqslant n;j\in J}$ their dual. Then $\{\tilde{\Lambda}_{0,j}\otimes\tilde{\Lambda}_{1,j}\otimes ......\otimes\tilde{\Lambda}_{n,j}\}_{j\in J}$ is a dual of $\{\Lambda_{0,j}\otimes\Lambda_{1,j}\otimes ......\otimes\Lambda_{n,j}\}_{j\in J}$.

15. $*$-operator frame

Definition 15.1 [12] A family of adjointable operators $\{T_i\}_{i\in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said to be a $*$-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$, if there exists two strictly nonzero elements A and B in $\mathcal{A}$ such that

$$A\langle x,x\rangle A^*\leqslant\sum_{i\in I}\langle T_i x,T_i x\rangle\leqslant B\langle x,x\rangle B^*,\quad\forall x\in\mathcal{H}. \tag{23}$$

The elements A and B are called lower and upper bounds of the $*$-operator frame, respectively. If $A=B=\lambda$, the $*$-operator frame is λ-tight. If $A=B=1_{\mathcal{A}}$, it is called a normalized tight $*$-operator frame or a Parseval $*$-operator frame. If only upper inequality of (23) hold, then $\{T_i\}_{i\in i}$ is called an $*$-operator Bessel sequence for $End_{\mathcal{A}}^*(\mathcal{H})$.

We mentioned that the set of all of operator frames for $End_{\mathcal{A}}^*(\mathcal{H})$ can be considered as a subset of $*$-operator frame. To illustrate this, let $\{T_j\}_{i\in I}$ be an operator frame for Hilbert $\mathcal{A}$-module $\mathcal{H}$ with operator frame real bounds A and B. Note that for $x\in\mathcal{H}$,

$$(\sqrt{A})1_{\mathcal{A}}\langle x,x\rangle_{\mathcal{A}}(\sqrt{A})1_{\mathcal{A}}\leqslant\sum_{i\in I}\langle T_i x,T_i x\rangle\leqslant(\sqrt{B})1_{\mathcal{A}}\langle x,x\rangle_{\mathcal{A}}(\sqrt{B})1_{\mathcal{A}}.$$

Therefore, every operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with real bounds A and B is a $*$-operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with $\mathcal{A}$-valued $*$-operator frame bounds $(\sqrt{A})1_{\mathcal{A}}$ and $(\sqrt{B})1_{\mathcal{B}}$.

***Example* 15.2** [12] Let $\mathcal{A}$ be a Hilbert C^*-module over itself with the inner product $\langle a, b\rangle = ab^*$ and $\{x_i\}_{i\in I}$ be a $*$-frame for $\mathcal{A}$ with bounds A and B, respectively. For each $i \in I$, we define $T_i : \mathcal{A} \to \mathcal{A}$ by $T_i x = \langle x, x_i\rangle$ for all $x \in \mathcal{A}$. T_i is adjointable and $T_i^* a = ax_i$ for each $a \in \mathcal{A}$, and we have $A\langle x, x\rangle A^* \leqslant \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leqslant B\langle x, x\rangle B^*$ for all $x \in \mathcal{A}$. Then $A\langle x, x\rangle A^* \leqslant \sum_{i\in I}\langle T_i x, T_i x\rangle \leqslant B\langle x, x\rangle B^*$ for all $x \in \mathcal{A}$. So $\{T_i\}_{i\in I}$ is a $*$-operator frame in $\mathcal{A}$ with bounds A and B, respectively.

***Example* 15.3** [12] Let $\{W_i\}_{i\in\mathbb{J}}$ be a $*$-frame of submodules with respect to $\{v_i\}_{i\in\mathbb{J}}$ for $\mathcal{H}$. Put $T_i = v_i\pi_{W_i}$ for all $i \in \mathbb{J}$, then we get a sequence of operators $\{T_i\}_{i\in\mathbb{J}}$. Then there exist $A, B \in \mathcal{A}$ such that $A\langle x, x\rangle A^* \leqslant \sum_{i\in\mathbb{J}} v_i^2\langle \pi_{W_i} x, \pi_{W_i} x\rangle \leqslant B\langle x, x\rangle B^*$ for all $x \in \mathcal{H}$. So, we have $A\langle x, x\rangle A^* \leqslant \sum_{i\in\mathbb{J}}\langle T_i x, T_i x\rangle \leqslant B\langle x, x\rangle B^*$ for all $x \in \mathcal{H}$. Thus, the sequence $\{T_i\}_{i\in\mathbb{J}}$ becomes a $*$-operator frame for $\mathcal{H}$.

With this example a $*$-frame of submodules can be viewed as a special case of $*$-operator frames.

***Remark* 8** *The examples* 3.3 *and* 3.4 *in [1] are examples of* $*$*-operator frame.*

16. $*$-K-operator frame

Now, we define the $*$-K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$.

Definition 16.1 [23] Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$. A family of adjointable operators $\{T_i\}_{i\in I}$, on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said a $*$-K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ if there exists two nonzero elements A and B in $\mathcal{A}$ such that

$$A\langle K^*x, K^*x\rangle A^* \leqslant \sum_{i\in I}\langle T_i x, T_i x\rangle \leqslant B\langle x, x\rangle B^*, \forall x \in \mathcal{H}. \tag{24}$$

The elements A and B are called lower and upper bounds of the $*$-K-operator frame, respectively. If $A\langle K^*x, K^*x\rangle_{\mathcal{A}} A^* = \sum_{i\in\mathbb{J}}\langle T_i x, T_i x\rangle_{\mathcal{A}}$, the $*$-K-operator frame is an A-tight. If $A = 1$, it is called a normalized tight $*$-K-operator frame or a Parseval $*$-K-operator frame.

***Example* 16.2** [23] Let l^∞ be the set of all bounded complex-valued sequences. For any $u = \{u_j\}_{j\in\mathbb{N}}, v = \{v_j\}_{j\in\mathbb{N}} \in l^\infty$, we define $uv = \{u_j v_j\}_{j\in\mathbb{N}}$, $u^* = \{\bar{u_j}\}_{j\in\mathbb{N}}$ and $\|u\| = \sup_{j\in\mathbb{N}} |u_j|$. Then $\mathcal{A} = \{l^\infty, \|.\|\}$ is a $\mathbb{C}^*$-algebra. Let $\mathcal{H} = C_0$ be the set of all null sequences. For any $u, v \in \mathcal{H}$, we define $\langle u, v\rangle = uv^* = \{u_j\bar{u_j}\}_{j\in\mathbb{N}}$. Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module. Define $f_j = \{f_i^j\}_{i\in\mathbb{N}^*}$ by $f_i^j = \frac{1}{2} + \frac{1}{i}$ if $i = j$ and $f_i^j = 0$ if $i \neq j$ for all $j \in \mathbb{N}^*$. Now, define the adjointable operator $T_j : \mathcal{H} \to \mathcal{H}$ by $T_j\{(x_i)_i\} = (x_i f_i^j)_i$. Then, for every $x \in \mathcal{H}$, we have

$$\sum_{j\in\mathbb{N}}\langle T_j x, T_j x\rangle = \{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathbb{N}^*}\langle x, x\rangle\{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathbb{N}^*}.$$

So, $\{T_j\}_j$ is a $\{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathbb{N}^*}$-tight $*$-operator frame. Let $K : \mathcal{H} \to \mathcal{H}$ defined by $Kx = \{\frac{x_i}{i}\}_{i\in\mathbb{N}^*}$. Then, for every $x \in \mathcal{H}$, we have

$$\langle K^*x, K^*x\rangle_{\mathcal{A}} \leqslant \sum_{j\in\mathbb{N}}\langle T_j x, T_j x\rangle = \{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathbb{N}^*}\langle x, x\rangle\{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathbb{N}^*}.$$

This shows that $\{T_j\}_{j\in\mathbb{N}}$ is a $*$-K-operator frame with bounds $1, \{\frac{1}{2}+\frac{1}{i}\}_{i\in\mathbb{N}^*}$.

Remark 9

(1) Every $*$-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ is an $*$-K-operator frame for any $K \in End^*_{\mathcal{A}}(\mathcal{H})$ where $K \neq 0$.

(2) If $K \in End^*_{\mathcal{A}}(\mathcal{H})$ is a surjective operator, then every $*$-K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ is an $*$-operator frame.

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