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Characterization of matrices using m-projectors and singular value decomposition in Minkowski space

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Abstract. In this paper we characterize different classes of matrices in Minkowski space $\mathcal{M}$ by generalizing the singular value decomposition in terms of m-projectors. Furthermore, we establish results on the relation between the range spaces and rank of the range disjoint matrices by utilizing the singular value decomposition obtained in terms of m-projectors. Since there is no result on the formulation of Minkowski inverse of the sum of two matrices, we have established an expression for the Minkowski inverse of the sum of a range disjoint matrix with its Minkowski adjoint, which will ease to formulate an expression for the Minkowski inverse of the sum of two matrices in general case.

Keywords: Singular value decomposition, range symmetric, Minkowski inverse, m-projectors, range disjoint, full range.

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1. Introduction and Preliminaries

Let us denote by $M_{(m,n)}(\mathbb{C})$ the set of $m \times n$ matrices and when $m = n$ we write $M_n(\mathbb{C})$ for $M_{(n,n)}(\mathbb{C})$. The symbols A^*, $A^\sim$, $A^\oplus$, $A^\dagger$, $R(A)$, and $N(A)$ denote the conjugate transpose, Minkowski adjoint, Minkowski inverse, Moore-Penrose inverse, range space and null space of a matrix A respectively. I_n denote the identity matrix of order $n \times n$. $\bar{A}$ denotes the matrix $I - A$; I is the identity matrix of suitable order, and $\{1^\sim\} = \{M : AMA = A\}$ denotes the set of Minkowski $\{1\}$-inverses. Furthermore, we use the following notation to denote different classes of matrices:

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(i) $\mathbb{C}_n^{mp}$ the set of all m-projections. i.e. $\mathbb{C}_n^{mp} = \{P : P^2 = P = P^{\sim}\}$.
(ii) $\mathbb{C}_n^{PIM} = \{H \in M_n(\mathbb{C}) : H^{\oplus} = H^{\sim}\}$ denotes the set of partial isometries in $\mathcal{M}$
(iii) $\mathbb{C}_n^{GN} = \{H \in M_n(\mathbb{C}) : HH^{\sim} = H^{\sim}H\}$ denotes the set of G-normal matrices.
(iv) $\mathbb{C}_n^{MIA} = \{H \in M_n(\mathbb{C}) : H^{\oplus}H^{\sim} = H^{\sim}H^{\oplus}\}$ denotes the set for which the Minkowski adjoint commutes with its Minkowski inverse.
(v) $\mathbb{C}_n^{Gmp} = \{H \in M_n(\mathbb{C}) : H^2 = H^{\sim}\}$ denotes the set of generalized m-projectors in Minkowski space.
(vi) $\mathbb{C}_n^{HGmp} = \{H \in M_n(\mathbb{C}) : H^2 = H^{\oplus}\}$ denotes the set of hypergeneralized m-projectors in Minkowski space.
(vii) $\Gamma = \{H \in M_n(\mathbb{C}) : HH^{\oplus} = H^{\oplus}H\}$ denotes the set of range symmetric matrices.

Also, we use the convention $P_w = WW^{\oplus}$ and $\tilde{P}_w = I_k - WW^{\oplus}$, where I_k is the identity matrix of suitable order.

Indefinite inner product is a scalar product defined by

$$[u, v] = \langle u, Mv \rangle = u^*Mv, \tag{1}$$

where $\langle\ ,\ \rangle$ denotes the conventional Hilbert space inner product and M is a hermitian matrix. This hermitian matrix M is referred to as metric matrix. Minkowski space $\mathcal{M}$ is an indefinite inner product space in which the metric matrix is denoted by G and is defined as

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix} \text{ satisfying } G^2 = I_n \text{ and } G^* = G.$$

G is called the Minkowski metric matrix. In case $u = (u_0, u_1, ..., u_{n-1}) \in \mathbb{C}^n$, G is called the Minkowski metric tensor and is defined as $Gu = (u_0, -u_1, ..., -u_{n-1})$. For detailed study of indefinite linear algebra refer to [5].

The singular value decomposition theorem holds a central place in the literature of the matrix theory (see [20]) and has enormous applications in the pure as well as applied mathematics (for instance, see [7, 10–12, 14, 18, 21, 22]). In particular, SVD plays a vital role in the computation of generalized inverses of a matrix e.g. see [4, 6, 8, 19, 24]. The SVD theorem is stated differently depending upon the field under consideration. Mostly the following statement from [4] is used:

Theorem 1.1 For any matrix $A \in M_{(m,n)}(\mathbb{C})$ with singular values $\sigma_1, \sigma_2 ..., \sigma_r \in \sigma(A)$ their exist two unitary matrices U_m and V_n such that the $m \times n$ matrix

$$\Sigma = U^*AV = \left[\begin{array}{ccc|c} \sigma_1 & & & O \\ & \ddots & & \\ & & \sigma_r & \\ \hline & O & & O \end{array}\right]$$

is diagonal.

Therefore, for any complex $m \times n$ matrix, we can find the unitary matrices U and V such that $A = U\Sigma V^*$.

The singular value decomposition of matrices obtained in the Euclidean space does not hold in the Minkowski space. This problem was first faced by Xing [23] while studying the polarization of light in the Minkowski space. Later on Renardy in [15] studied the singular value decomposition in the Minkowski space and established the conditions under which the singular value decomposition holds in the Minkowski space. Renardy established the Singular value decomposition in Minkowski space in the following form:

Theorem 1.2 The Matrix M can be written in the form $M = QDR$ with Q, R orthogonal and D diagonal, if and only if the following conditions hold:

(i) The eigenvalues of $M^{\sim}M$ are real and nonnegative.
(ii) $M^{\sim}M$ diagonalizable.
(iii) The null space of $M^{\sim}M$ is same as null space of M.

Every matrix in Euclidean space satisfy the conditions mentioned in the above theorem when Minkowski adjoint is replaced by conjugate transpose. If the condition (i) does not hold and the remaining two conditions hold we can still have a SVD in Euclidean space. However, this is not the case in Minkowski space. Kilicman and Zhour in [1] showed by counter examples that even if one condition is violated the result does not hold in the Minkowski space. In this paper we consider the set of all matrices in Minkowski space for which the SVD theorem holds. Let $H_{m\times n} \in \mathcal{M}$ has SVD given by $H = V\Sigma U^*$. Taking Minkowski adjoint on both sides, we get $H^{\sim} = RDS^{\sim}$, where $R = G_1U,\ V = S$ are unitary and $D = \Sigma G_2$ is diagonal matrix. G_1, G_2 are Minkowski metric matrices of suitable order. Thus, corresponding to every matrix $H \in \mathcal{M}$ having a SVD, there corresponds a matrix $W = H^{\sim} = RDS^{\sim}$. Furthermore, if we assume that $UG_1 = G_1U$ and $VG_2 = G_2V$, then U and V are G-unitary i.e. $UU^{\sim} = U^{\sim}U = I$ and $VV^{\sim} = V^{\sim}V = I$.

Consider the matrix

$$H = R\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} S^{\sim},$$

where R and S are G-unitary and D is diagonal sub block of rank r. Let

$$S^{\sim}R = \begin{pmatrix} J & K \\ L & M \end{pmatrix}.$$

Then, it can be easily verified that $S^{\sim}R$ is G-unitary. Post multiplying the above equality by $R^{\sim}$, we get

$$S^{\sim} = \begin{pmatrix} J & K \\ L & M \end{pmatrix} R^{\sim}.$$

Using this representation of $S^{\sim}$, we have

$$H = R\begin{pmatrix} DJ & DK \\ 0 & 0 \end{pmatrix} R^{\sim}. \tag{2}$$

Since $S^{\sim}R$ is G-unitary, then $(S^{\sim}R)(S^{\sim}R)^{\sim} = I$ gives $JJ^{\sim} - KG_1K^{\sim} = I$, where G_1 denotes the Minkowski metric matrix of order $n - r$.

The representation (2) is inspired by the representation obtained by Hartwig and Spindlebock [9]. This representation is very handy in characterizing different classes of matrices in terms of the generalized inverses. Furthermore, this representation helps in establishing the relation between different classes of matrices and particularly the projections as will be seen in the forthcoming sections.

2. Characterization of matrices using Hartwig Spindlebock Decomposition in Minkowski space.

In this section we use the representation (2) to obtain a characterization of different classes of matrices related to the projections. The following two results are extension of Lemma 1 and Theorem 1 from [2] to the Minkowski space.

Lemma 2.1 Let $H \in \mathcal{M}$ has the representation (2). Then

(i) $H \in \mathbb{C}_n^{mp}$ if and only if $D = I,\ J = I$ and $K = 0$.
(ii) $H \in \mathbb{C}_n^{PIM}$ if and only if $D = I_r$.
(iii) $H \in \mathbb{C}_n^{GN}$ if and only if $D^2J = JD^2$.
(iv) $H \in \mathbb{C}_n^{MIA}$ if and only if $D^2J^\sim = J^\sim D^2$.
(v) $H \in \Gamma$ if and only if J is G-unitary and $K = 0$.
(vi) $H \in \mathbb{C}_n^{Gmp}$ if and only if $J^3 = I$ and $K = 0$.
(vii) $H \in \mathbb{C}_n^{HGmp}$ if and only if $(JD)^3 = (DJ)^3 = I_r$ and $K = 0$.
(viii) H is nilpotent of index 2 if and only if $J = 0$.

Proof. (i) From the definition of the m-projectors, we have $H = H^2 = H^\sim$. Using the first half of the equality i.e., $H = H^2$, on using the representation (2) of H, gives $DJ = (DJ)^2$. From $H = H^\sim$, we have $K = 0$ as D is nonsingular. Therefore, from the fact that $JJ^\sim - KG_1K^\sim = I$, we have $JJ^\sim = I$ and hence $DJ = (DJ)^2$ gives $DJ = I$, which implies $D = I$ and $J = I$.

(ii) $H \in \mathbb{C}_n^{PIM}$ implies $H^\oplus = H^\sim$. Direct verification shows that the Minkowski inverse of H of the form (2) is

$$H^\oplus = U \begin{pmatrix} J^\sim D^{-1} & 0 \\ -G_1K^\sim D^{-1} & 0 \end{pmatrix} U^\sim. \tag{3}$$

Also doing some simple algebra, we get

$$H^\sim = U \begin{pmatrix} J^\sim D & 0 \\ -G_1K^\sim D & 0 \end{pmatrix} U^\sim. \tag{4}$$

Therefore, the equation $H^\sim = H^\oplus$, on using the representations (3) and (4) gives $D = D^{-1}$ implies $D = I$.

(iii) Since $H \in \mathbb{C}_n^{GN}$, then $HH^\sim = H^\sim H$. Using the representation (2) of H, we have

$$HH^\sim = U \begin{pmatrix} D^2 & 0 \\ 0 & 0 \end{pmatrix} U^\sim \quad \text{and} \quad H^\sim H = U \begin{pmatrix} J^\sim D^2 J & J^\sim D^2 K \\ -G_1K^\sim D^2 J & -G_1K^\sim D^2 K \end{pmatrix} U^\sim.$$

Therefore, the matrix equality $HH^\sim = H^\sim H$ yields $K = 0$ and $JD^2 = D^2J$.

(iv) For $H \in \mathbb{C}_n^{MIA}$, we have $H^\sim H^\oplus = H^\oplus H^\sim$. On using the representations (2), (3) and (4), we get

$$H^\sim H^\oplus = U \begin{pmatrix} J^\sim DJ^\sim D^{-1} & 0 \\ -G_1K^\sim DJ^\sim D^{-1} & 0 \end{pmatrix} U^\sim \quad \text{and} \quad H^\oplus H^\sim = U \begin{pmatrix} J^\sim D^{-1}J^\sim D & 0 \\ -G_1K^\sim D^{-1}J^\sim D & 0 \end{pmatrix} U^\sim.$$

Therefore, the matrix equality $H^\sim H^\oplus = H^\oplus H^\sim$ gives $J^\sim DJ^\sim D^{-1} = J^\sim D^{-1}J^\sim D$ and $-G_1K^\sim DJ^\sim D^{-1} = -G_1K^\sim D^{-1}J^\sim D$. Post ultiplying the first equality by J and second equality by K and then subtracting and using the fact that $JJ^{\sim} - KG_1K^{\sim} = I$, we get $D^2J^{\sim} = J^{\sim}D^2$.

(v) For $H \in \Gamma$, using the representations (2) and (3), the matrix equality $HH^{\oplus} = H^{\oplus}H$ implies

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} JJ^{\sim} & J^{\sim}K \\ -G_1K^{\sim}J & -G_1K^{\sim}K \end{pmatrix},$$

which gives $JJ^{\sim} = I_r$ i.e., J is G-unitary and $K = 0$.

(vi) $H \in \mathbb{C}_n^{Gmp}$ implies $H^2 = H^{\sim}$. This matrix equality, on using the representations (2) and (4), gives $(DJ)^2 = J^{\sim}D$, i.e., $DJDJ = J^{\sim}D$ and $K = 0$, which implies J is G-unitary. Now, $DJDJ = J^{\sim}D$ gives $D = JDJDJ$. But $DJDJ = J^{\sim}D \Rightarrow DJD = J^{\sim}DJ^{\sim}$. Substituting this in the last equality, we get $D = I_r$. Using $D = I_r$ in $DJDJ = J^{\sim}D$, the statement follows.

(vii) $H \in \mathbb{C}_n^{HGmp}$ implies $H^2 = H^{\oplus}$. Utilizing the representations (2) and (3), the matrix equality $H^2 = H^{\oplus}$ i.e.,

$$\begin{pmatrix} (DJ)^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} J^{\sim}D^{-1} & 0 \\ -G_1K^{\sim}D^{-1} & 0 \end{pmatrix},$$

implies $(DJ)^2 = J^{\sim}D^{-1}$ and $K = 0$, which further on using the implication that J is G-unitary, as a consequence of $K = 0$, gives $(JD)^3 = (DJ)^3 = I_r$.

(viii) Follows at once by direct verification. ■

Theorem 2.2 Let $H \in M_n(\mathbb{C})$ has the representation given by (2). Then $H \in \mathbb{C}_n^{HGmp}$ if and only if H^3 is the m-projector onto the $R(A)$.

Proof. For H given by (2), we have

$$H^3 = U \begin{pmatrix} (DJ)^3 & (DJ)^2DK \\ 0 & 0 \end{pmatrix} U^{\sim}.$$

Also, it can be easily verified that the m-projector onto the $R(H)$ is given by

$$P_{R(H)} = U \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} U^{\sim}.$$

This m-projector is equivalent to H^3 if and only if $(DJ)^3 = I_r$ and $K = 0$ which is the necessary and sufficient condition for H to be a hypergeneralized m-projector given by Lemma 2.1 statement (vii). ■

3. Null intersection or range disjoint matrices in Minkowski space

A matrix A is said to EP if and only if $R(A) = R(A^*)$. In view of this concept of range space equality Baksalary and Trenkler in [3] introduced the concept of disjoint range matrices and spanning range matrices. In this section we extend these concepts to the Minkowski space. Since a matrix is range symmetric if and only if $R(A) = R(A^{\sim})$. This equality of ranges has another equivalent condition that is the equality of m-projectors onto $R(A)$ and $R(A^{\sim})$. On contrary to this concept of equality of range space, we consider the class of null intersection matrices or range disjoint matrices and the class of range spanning matrices. These classes can be alternatively defined in terms of null spaces instead of range spaces. Thus, we have the following definitions:

Definition 3.1 Let $H \in M_n(\mathbb{C})$. Then

(i) H is said to be range symmetric whenever $R(H) = R(H^\sim)$.
(ii) H is said to be range disjoint or null intersection whenever $R(H) \cap R(H^\sim) = \{0\}$.
(iii) H is said to be of full range or range spanning whenever $R(H)+R(H^\sim) = M_{(n,1)}(\mathbb{C})$.

The above definitions can alternatively be expressed in terms of respective null spaces. We have the following Lemma as an immediate consequence of these definitions.

Lemma 3.2 Let $H \in M_n(\mathbb{C})$. Then

(i) H is Range symmetric and range disjoint if and only if $H = 0$.
(ii) H is range symmetric and of full range or range spanning if and only if H is invertible.
(iii) H is range disjoint and of full range if and only if $R(H) \bigoplus R(H^\sim) = M_{(n,1)}(\mathbb{C})$.

Let $H \in M_n(\mathbb{C})$ has the representation (2) and let $L = HH^\oplus$ and $M = H^\oplus H$, then a simple verification shows that

$$L = U \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} U^\sim \text{ and } M = U \begin{pmatrix} JJ^\sim & J^\sim K \\ -G_1K^\sim J & -G_1K^\sim K \end{pmatrix} U^\sim. \tag{5}$$

In [16] Saleem and Krishnaswamy expressed the m-projectors alternatively in the form

$$L = U \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} U^\sim \text{ and } M = U \begin{pmatrix} W & X \\ -G_1X^\sim & Z \end{pmatrix} U^\sim. \tag{6}$$

Since the two representation express the same m-projectors, we have $W = JJ^\sim,\ X = J^\sim K$ and $Z = -G_1K^\sim K$. Thus,

$$H = 0 \Leftrightarrow L = M = 0 \text{ and } rk(H) = n \Leftrightarrow L = I_n = M. \tag{7}$$

Theorem 3.3 Let $H \in M_n(\mathbb{C})$ has the representation (2) and let $L = HH^\oplus$ and $M = H^\oplus H$ has the form (6). Then

(i) $P_{R(H)\cap R(H^\sim)} = U \begin{pmatrix} \tilde{P}_{I_r - J^\sim J} & 0 \\ 0 & 0 \end{pmatrix} U^\sim$, where $dim[R(H) \cap R(H^\sim)] = r - rk(K)$.
(ii) $P_{R(H)+R(H^\sim)} = U \begin{pmatrix} I_r & 0 \\ 0 & P_k \end{pmatrix} U^\sim$, where $dim[R(H) + R(H^\sim)] = r + rk(K)$.

Proof. From Lemma 3.4, statement (i) of [16], we have

$$P_{R(L)\cap R(M)} = U \begin{pmatrix} \tilde{P}_{\bar{w}} & 0 \\ 0 & 0 \end{pmatrix} U^\sim$$

with $\dim[R(L) \cap R(M)] = rk(W) - rk(X)$.

Furthermore, for M as given in (6), on account of equation (9) of [16], applying Corollary 19.1 from [13] to M, gives

$$rk(M) = rk(W) + rk(Z + G_1X^\sim W^\oplus X).$$

Utilizing the statement (iv) of Theorem 2.10 from [16], we get

$$rk(M) = rk(W) + rk(\tilde{P}_{\bar{z}}) = rk(W) + n - r + rk(\bar{Z}).$$

Also, from [16] statement (ii) of Lemma 3.1, we have $rk(\bar{Z}) = n - r + rk(X) + rk(Z)$. Therefore, $rk(M) = rk(W) - rk(X) + rk(Z)$, which further on using $rk(H) = r$ and the equality of the matrix blocks in the two representation (5) and (6) of M, implies

$$r = rk(J) - rk(J^{\sim}K) + rk(K), \tag{8}$$

where $W = J^{\sim}J,\ X = J^{\sim}K$ and $Z = -G_1K^{\sim}K$. This completes the proof of (i).

Again, from Lemma 3.3 [16], statement (i), we have

$$P_{R(L)+R(M)} = U \begin{pmatrix} I_r & 0 \\ 0 & P_z \end{pmatrix} U^{\sim},$$

where $\dim[R(L)+R(M)] = r + rk(Z)$, which is equivalent to the required projector under the assumptions of the theorem that $L = HH^{\oplus}$ and $M = H^{\oplus}H$. The remaining portion follows at once on using the fact that G_1 is nonsingular and $Z = -G_1K^{\sim}K$. ■

The following characterization of range disjoint and of full range class of matrices is obtained as an immediate consequence of Theorem 3.3.

Corollary 3.4 Let $H \in M_n(\mathbb{C})$ has the representation (2). Then

(i) H is range disjoint $\Leftrightarrow rk(K) = r$ i.e. K is of full row rank.
(ii) H is of full range $\Leftrightarrow rk(K) = n - r$ i.e. K is of full column rank.

Corollary 3.5 Let $H \in M_n(\mathbb{C})$ has the representation (2). Then, H is range symmetric if and only if $\dim[R(H) \cap R(H^{\sim})] = rk(H)$ or $\dim[R(H) + R(H^{\sim})] = rk(H)$.

Two more characterizations of the null intersection i.e., range disjoint and of full range class of matrices is given by the following two lemmas.

Lemma 3.6 Let $H \in M_n(\mathbb{C})$ has the representation (2). Then, H is range disjoint if and only if $R(J) \subseteq R(K)$.

Proof. For $rk(K) = r$, the result is obvious. Now, $R(J) \subseteq R(K) \Leftrightarrow R(J^{\sim}J) \subseteq R(J^{\sim}K)$ i.e., $R(J^{\sim}) \subseteq R(J^{\sim}K)$. But $R(J^{\sim}K) \subseteq R(J^{\sim})$ is always true, we have $R(J^{\sim}) = R(J^{\sim}K)$ and hence $rk(J^{\sim}) = rk(J^{\sim}K)$. Using this implication in (8) i.e., $r = rk(J) - rk(J^{\sim}K) + rk(K)$, we get $rk(K) = r$. Therefore, from Corollary 3.4, statement (i) the result follows. ■

Lemma 3.7 Let $H \in M_n(\mathbb{C})$ has the representation (2). Then the following statements are equivalent:

(i) H is range disjoint.
(ii) $I_r - JJ^{\sim}$ is invertible.
(iii) $rk(J^{\sim}K) = rk(J)$.

Proof. From the fact that $(I_r - JJ^{\sim}) = -KG_1K^{\sim}$, we have

$$rk(I_r - JJ^{\sim}) = rk(KG_1K^{\sim}) = rk(K) = r.$$

Therefore, on using the statement (i) of Corollary 3.4, the equivalence of the first two statements i.e., (i)⇔(ii) follows. Also, from equation (8), we get $rk(J^{\sim}K) = rk(K) \Leftrightarrow rk(K) = r$. Hence the equivalence (i)⇔(iii) follows and the proof is complete. ■

Theorem 3.8 Let $H \in M_n(\mathbb{C})$ has the representation (2) and let $L = HH^{\oplus}$ and $M = H^{\oplus}H$ has the form (6). Then the following statements are equivalent:

(i) H is range disjoint.
(ii) LM is range disjoint.
(iii) $R(H) \cap R(H^{\sim}) = \{0\}$

Proof. Form the assumptions $L = HH^{\oplus}$ and $M = H^{\oplus}H$, it can be easily verified that $R(L) = R(H)$ and $R(M) = R(H^{\sim})$. Hence the equivalence (i)⇔(iii) follows at once.

Also, L and M of the form given in equation (6), we have

$$LM + ML = U \begin{pmatrix} 2W & X \\ -G_1X^{\sim} & 0 \end{pmatrix} U^{\sim}.$$

Direct verification shows that the Minkowski inverse of $LM+ML$, on using the fact that $R(X) \subseteq R(W)$, is given by $(LM + ML)^{\oplus}$

$$= U \begin{pmatrix} \frac{1}{2}W^{\oplus} - \frac{1}{2}W^{\oplus}X(G_1X^{\sim}W^{\oplus}X)^{\oplus}G_1X^{\sim}W^{\oplus} & W^{\oplus}X(G_1X^{\sim}W^{\oplus}X)^{\oplus} \\ (G_1X^{\sim}W^{\oplus}X)^{\oplus}G_1X^{\sim}W^{\oplus} & 2(G_1X^{\sim}W^{\oplus}X)^{\oplus} \end{pmatrix} U^{\sim}$$

Furthermore, it can be easily verified that $LM(LM + ML)^{\oplus}ML = L(LM + ML)^{\oplus}M$ and as a consequence of this we have $R(LM) \cap R(ML) = R(L) \cap R(M)$ and hence the equivalence (ii)⇔(iii) follows. ■

Theorem 3.9 Let $L,\ M \in \mathbb{C}^{mp}$. Then the following conditions are equivalent:

(i) $R(L) \cap R(M) = \{0\}$.
(ii) $rk(L - M) = rk(L) + rk(M)$.
(iii) $R(L - M) = R(L) + R(M)$.

Proof. From [17] Theorem 3, the statement (i) is equivalent to $rk(W) = rk(X)$. For $L,\ M$ of the form (6), from equation (30) of Theorem 8 in [17], we have

$$P_{R(L-M)} = U \begin{pmatrix} P_{\bar{w}} & 0 \\ 0 & P_z \end{pmatrix} U^{\sim}.$$

Also, from Lemma 3.1, statement (i) in [16], we have $rk(\bar{W}) = r - rk(W) + rk(X)$. Thus, as a combined observation from last two equations we have

$$rk(L - M) = r - rk(W) + rk(X) + rk(Z) \tag{9}$$

Also, from the proof of Theorem 3.3, it can be seen that $rk(M) = rk(W) - rk(X) + rk(Z)$. Adding to this equation $rk(L)$, we get

$$rk(L) + rk(M) = r + rk(W) - rk(X) + rk(Z). \tag{10}$$

On comparing the equations (9) and (10), it can be easily observed that the statement (ii) is also equivalent to the condition $rk(W) = rk(X)$. Thus, we have established the equivalence (i)⇔ (ii)

Finally, for the equivalence of the statement (iii), on comparing the m-projectors

$$P_{R(L-M)} = U \begin{pmatrix} P_{\bar{w}} & 0 \\ 0 & P_z \end{pmatrix} U^{\sim} \text{ and } P_{R(L)+R(M)} = U \begin{pmatrix} I_r & 0 \\ 0 & P_z \end{pmatrix} U^{\sim},$$

it can be easily observed that the statement (iii) becomes equivalent to the first two statements if and only if $rk(\bar{W}) = r$, which when combined with the fact $rk(\bar{W}) = r - rk(W) + rk(X)$, is equivalent to $rk(W) = rk(X)$. Hence, the proof is complete. ■

The next result adds some more conditions equivalent to the statement (i) of the Theorem 3.9.

Theorem 3.10 Let $L,\ M \in \mathbb{C}^{mp}$. Then the following conditions are equivalent:

(i) $R(L) \cap R(M) = \{0\}$.
(ii) $rk(L\bar{M}) + rk(\bar{L}M) = rk(L) + rk(M)$.
(iii) $R(L\bar{M}) \oplus^{[\perp]} R(\bar{L}M) = R(L) + R(M)$.

Proof. For $L,\ M$ of the form (6), using equation (8) from [17], we have

$$P_{R(L\bar{M})} = U \begin{pmatrix} \bar{W} & -X \\ 0 & 0 \end{pmatrix} U^{\sim} U \begin{pmatrix} P_{\bar{w}} & 0 \\ G_1 X^{\sim} W^{\oplus} & 0 \end{pmatrix} U^{\sim} = U \begin{pmatrix} P_{\bar{w}} & 0 \\ 0 & 0 \end{pmatrix} U^{\sim}$$

Also, $\bar{L}M = U \begin{pmatrix} 0 & 0 \\ -G_1 X^{\sim} & Z \end{pmatrix} U^{\sim}$. The Minkowski inverse of $\bar{L}M$, as a result of direct verification, on using statements (iv) and (iii) of Theorem 2.9 and Theorem 2.10 respectively from [16], is given by $(\bar{L}M)^{\oplus} = U \begin{pmatrix} 0 & XZ^{\oplus} \\ 0 & P_z \end{pmatrix} U^{\sim}$. Therefore, the m-projector onto the range space of $\bar{L}M$ is given by

$$P_{R(L\bar{M})} = U \begin{pmatrix} 0 & 0 \\ -G_1 X^{\sim} & Z \end{pmatrix} U^{\sim} U \begin{pmatrix} 0 & XZ^{\oplus} \\ 0 & P_z \end{pmatrix} U^{\sim} = U \begin{pmatrix} 0 & 0 \\ 0 & P_z \end{pmatrix} U^{\sim}.$$

The above m-projector is obtained on account of statement (iii) of Theorem 2.10 from [16]. Since $P_{R(L\bar{M})} P_{R(L\bar{M})} = 0$, therefore $P_{R(L\bar{M})} + P_{R(L\bar{M})} = P_{R(L\bar{M}) \oplus^{\perp} R(L\bar{M})}$ and hence

$$P_{R(L\bar{M}) \oplus^{\perp} R(L\bar{M})} = U \begin{pmatrix} P_{\bar{w}} & 0 \\ 0 & P_z \end{pmatrix} U^{\sim} = P_{R(L-M)}.$$

Combining the above relation with the statement (iii) of the Theorem 3.9, the equivalence follows. ■

The following result is a collective consequence of the Theorems 3.8, 3.9 and 3.10.

Theorem 3.11 Let $H \in M_n(\mathbb{C})$ has the representation (2) and let $L = HH^{\oplus}$ and $M = H^{\oplus}H$ has the form (6). Then the following statements are equivalent:

(i) H is range disjoint.
(ii) $rk(L - M) = rk(L) + rk(M)$.
(iii) $R(L - M) = R(L) + R(M)$.
(iv) $rk(L\bar{M}) + rk(\bar{L}M) = rk(L) + rk(M)$.
(v) $R(L\bar{M}) \oplus^{[\perp]} R(\bar{L}M) = R(L) + R(M)$.
(vi) $R(H) \cap R(H^{\sim}) = \{0\}$.

Lemma 3.12 Let $L,\ M \in \mathbb{C}^{mp}$ has the form (6). Then

(i) $rk(M) = rk(W) - rk(X) + rk(Z)$.
(ii) $rk(LM) = rk(W)$.
(iii) $rk(L\bar{M}) = rk(\bar{W})$.
(iv) $rk(\bar{L}M) = rk(Z)$.
(v) $rk(I_n - LM) = n - rk(W) + rk(X)$.
(vi) $rk(L + M) = r + rk(Z)$.
(vii) $rk(L + M - LM) = r + rk(Z)$.

Proof. The statement (i) follows from the proof of the Theorem 3.3. statements (ii), (iii), (iv) and (vi) follow at once on using the respective representations of L and M and a similar argument as for the statement (i). For (v), we have

$$(I_n - LM) = U \begin{pmatrix} I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} U^{\sim} - U \begin{pmatrix} W & X \\ 0 & 0 \end{pmatrix} U^{\sim} = U \begin{pmatrix} \bar{W} & -X \\ 0 & I_{n-r} \end{pmatrix} U^{\sim}.$$

Therefore, using Corollary 19.1 from [13], we have

$$rk(I_n - LM) = rk(\bar{W}) - rk(I_{n-r} - 0).$$

Further, using the statement (i) of Lemma 3.1 from [16], we have $rk(I_n - LM) = r - rk(W) + rk(X) + (n - r) = n - rk(W) + rk(X)$. Hence the statement (v) follows. Using the representations of L, M and LM, we get

$$(L + M - LM) = U \begin{pmatrix} I_r & 0 \\ G_1X^{\sim} & Z \end{pmatrix} U^{\sim}.$$

Again using the Corollary 19.1 from [13], we get $rk(L + M - LM) = r + rk(Z)$ and the statement (vii) follows. ■

Theorem 3.13 Let $H \in M_n(\mathbb{C})$ has the representation (2) and let $L = HH^{\oplus}$ and $M = H^{\oplus}H$ has the form (6). Then the following statements are equivalent:

(i) H is range disjoint.
(ii) $(I_n - LM)$ is nonsingular.
(iii) $rk(L\bar{M}) = r$.
(iv) $rk(\bar{M}L) = r$.
(v) $rk(L + M) = 2r$.
(vi) $rk(L + M - LM) = 2r$.

Proof. The equivalences (i)⇔(ii), (i)⇔(iii), (i)⇔(v) and (i)⇔(vi) follow at once on using the statement (i) of Corollary 3.4 and Lemma 3.12. Using the fact that the relation $R(L) \cap R(M) = \{0\}$ remains unaltered on interchanging L and M along with the statement (iii) of the Theorem proves that (i)⇒(iv). In order to prove the reverse implication we observe from statement (v) of Lemma 3.12 that $rk(I_n - LM) = n - rk(W) + rk(X)$. Since $W = J^{\sim}J$, and $X = J^{\sim}K$. Therefore, $rk(I_n - LM) = n - rk(J) + rk(J^{\sim}K)$. Thus, $(I_n - LM)$ is nonsingular if and only if $rk(J) = rk(J^{\sim}K)$, which according to Lemma 3.7, statement (iii) is equivalent to the fact that H is range disjoint and the proof is complete. ■

Theorem 3.14 Let $H \in M_n(\mathbb{C})$ has the representation (2). Then the following statements are equivalent:

(i) H is range disjoint.
(ii) $H^{\sim}$ is range disjoint.
(iii) $H^{\oplus}$ is range disjoint.

Proof. The proof follows at once from the definition of the range disjoint class of matrices and the fact that $R(H^{\sim}) = R(H^{\oplus})$ ■

The above equivalences are also satisfied by the class of full range matrices. The forthcoming results are established regarding the sum of matrix with its Minkowski adjoint.

Lemma 3.15 Let $H \in M_n(\mathbb{C})$ has the representation (2). Then $rk(H + H^{\sim}) = 2rk(K) + rk[\tilde{P}_S(R + R^{\sim})\tilde{P}_S]$, where $R = DJ$, $S = DK$ and $\tilde{P}_S = I_r - SS^{\oplus}$.

Proof. For $H \in M_n(\mathbb{C})$ of the form (2), we have from (4)

$$H^{\sim} = U \begin{pmatrix} J^{\sim}D & 0 \\ -G_1K^{\sim}D & 0 \end{pmatrix} U^{\sim}$$

and,

$$H + H^{\sim} = U \begin{pmatrix} R + R^{\sim} & S \\ -G_1S^{\sim} & 0 \end{pmatrix} U^{\sim}. \tag{11}$$

Since elementary transformation does not alter the rank of a matrix, therefore, without loss of generality, applying (8.3) from [13] to $H + H^{\sim}$, using Theorem 3 from [25] and the fact that G_1 nonsingular, we get

$$\begin{aligned} rk(H + H^{\sim}) &= rk(S) + rk(S^{\sim}) + rk[\tilde{P}_{SG_1}(R + R^{\sim})\tilde{P}_S] \\ &= 2rk(K) + rk[\tilde{P}_{SG_1}(R + R^{\sim})\tilde{P}_S]. \end{aligned}$$

Hence the Lemma follows. ■

Corollary 3.16 Let $H \in M_n(\mathbb{C})$ has the representation (2). Then $R(H + H^{\sim}) = R(H) + R(H^{\sim})$ if and only if $rk[\tilde{P}_{SG_1}(R + R^{\sim})\tilde{P}_S] = r - rk(K)$.

Proof. The inclusion $R(H + H^{\sim}) \subseteq R(H) + R(H^{\sim})$ is obvious. Thus, the result holds at once if and only the dimensions of the spaces on both sides are equal. Using Theorem 3.3, statement (i) and Lemma 3.15, the result follows. ■

Utilizing the Lemma 3.15, various classes of matrices can be characterized as follows:

(i) Let $H \in M_n(\mathbb{C})$ be a range symmetric matrix, then from the statement (v) of the lemma 2.1, we have J is G-unitary and $K = 0$. Thus, $\tilde{P}_{SG_1} = \tilde{P}_S = I_r$ and as an immediate consequence of this we have, $rk(H + H^{\sim}) = rk(R + R^{\sim})$. Also, $R(H + H^{\sim}) = R(H) + R(H^{\sim}) \Leftrightarrow rk(R + R^{\sim}) = r$.
(ii) Let $H \in M_n(\mathbb{C})$ be range disjoint, then using the statement (i) of the Corollary 3.4 we get H is of full row rank. Therefore, $\tilde{P}_{SG_1} = \tilde{P}_S = 0$. Hence $rk(H + H^{\sim}) = 2r$ and $R(H + H^{\sim}) = R(H) + R(H^{\sim})$ holds always.
(iii) Let $H \in M_n(\mathbb{C})$ be an oblique projector. Then $R = I_r$ and we have $rk(H + H^{\sim}) = r + rk(K)$.
(iv) Let $H \in M_n(\mathbb{C})$ be nilpotent matrix such that $ind(H) = 2$. Then $J = 0 \Rightarrow R = 0$ and $rk(K) = r$.

(v) Let $H \in M_n(\mathbb{C})$ be such that $SS^{\sim} = I_r$ i.e. S is G-unitary. Then $S = DK$ implies $S^{\sim} = K^{\sim}D$. Hence $SS^{\sim} = DKK^{\sim}D = I_r \Rightarrow KK^{\sim} = D^{-2}$ and therefore, $rk(K) = r$ i.e., H is range disjoint.

(vi) Let $R + R^{\sim}$ be positive definite. Then

$$rk[\tilde{P}_{SG_1}(R + R^{\sim})\tilde{P}_S] = rk[\tilde{P}_{SG_1}(R + R^{\sim})] = rk(\tilde{P}_{SG_1}) = r - rk(K).$$

(vii) Let $R(R + R^{\sim}) \subseteq R(S)$. Then direct conformation shows that the Minkowski inverse of $H + H^{\sim}$ is given by

$$(H + H^{\sim})^{\oplus} = U \begin{pmatrix} 0 & -(S^{\sim})^{\oplus}G_1 \\ S^{\oplus} & S^{\oplus}(R + R^{\sim})(S^{\sim})^{\oplus}G_1 \end{pmatrix} U^{\sim}. \tag{12}$$

Therefore, on using the representation (11) and (12). It can be easily verified that the m-projector on the $R(H + H^{\sim})$ is given by

$$P_{R(H+H^{\sim})} = U \begin{pmatrix} P_S & 0 \\ 0 & P_{S^{\sim}} \end{pmatrix} U^{\sim}. \tag{13}$$

As a consequence of the representation (13), we have $rk(H + H^{\sim}) = rk(K)$. However, if we assume that $SS^{\sim} = I_r$, then the Minkowski inverse of $H + H^{\sim}$ reduces to

$$(H + H^{\sim})^{\oplus} = U \begin{pmatrix} 0 & -SG_1 \\ S^{\sim} & S^{\sim}(R + R^{\sim})(S^{\sim})G_1 \end{pmatrix} U^{\sim}. \tag{14}$$

Since no expression for the Minkowski inverse of the sum of the two matrices has been established, the expressions (12) and (14) eases to formulate the Minkowski inverse of the sum $H + H^{\sim}$ in general case.

Lemma 3.17 Let $H \in M_n(\mathbb{C})$ of the form (2) be range disjoint. Then

$$(H + H^{\sim})^{\oplus} = U \begin{pmatrix} 0 & -(S^{\sim})^{\oplus} \\ S^{\oplus} & S^{\oplus}(K^{\oplus}J)^{\sim} - K^{\oplus}J(S^{\sim})^{\oplus} \end{pmatrix} U^{\sim}, \tag{15}$$

where $S = DK$.

Proof. Utilizing the fact that when H is range disjoint, then according to the statement (i) of the Corollary 3.4, i.e., $rk(K) = r$, we have $S = DK \Rightarrow S^{\oplus} = K^{\oplus}D^{-1}$ and $KK^{\oplus} = I_r$. This further gives $SS^{\oplus} = I_r$ and $S^{\oplus}S = K^{\oplus}K$. Now, using these implications, we verify that (15) is the Minkowski inverse of $(H + H^{\sim})$. Let

$$X = U \begin{pmatrix} 0 & -(S^{\sim})^{\oplus}G_1 \\ S^{\oplus} & S^{\oplus}(K^{\oplus}J)^{\sim} - K^{\oplus}J(S^{\sim})^{\oplus} \end{pmatrix} U^{\sim}$$

Then,

$$\begin{aligned}(H + H^{\sim})X(H + H^{\sim}) &= U \begin{pmatrix} I_r & 0 \\ 0 & G_1P_{K^{\sim}}G_1 \end{pmatrix} \begin{pmatrix} R + R^{\sim} & S \\ -G_1S^{\sim} & 0 \end{pmatrix} U^{\sim} \\ &= U \begin{pmatrix} R + R^{\sim} & S \\ -G_1S^{\sim} & 0 \end{pmatrix} U^{\sim}.\end{aligned}$$

From the above proved condition of the Minkowski inverse of $(H + H^{\sim})$, it is clear that

$$P_{H+H^{\sim}} = U \begin{pmatrix} I_r & 0 \\ 0 & G_1 P_{K^{\sim}} G_1 \end{pmatrix} U^{\sim}. \tag{16}$$

Therefore, using this fact, we have

$$\begin{aligned} X(H+H^{\sim})X &= U \begin{pmatrix} 0 & -(S^{\sim})^{\oplus} G_1 \\ S^{\oplus} & S^{\oplus}(K^{\oplus}J)^{\sim} - K^{\oplus}J(S^{\sim})^{\oplus} \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & G_1 P_{K^{\sim}} G_1 \end{pmatrix} U^{\sim} \\ &= U \begin{pmatrix} 0 & -(S^{\sim})^{\oplus} P_{K^{\sim}} G_1 \\ S^{\oplus} & S^{\oplus}(K^{\oplus}J)^{\sim} - K^{\oplus}J(S^{\sim})^{\oplus} P_{K^{\sim}} G_1 \end{pmatrix} U^{\sim} \\ &= U \begin{pmatrix} 0 & -(S^{\sim})^{\oplus} G_1 \\ S^{\oplus} & -S^{\oplus}(K^{\oplus}J)^{\sim} - K^{\oplus}J(S^{\sim})^{\oplus} \end{pmatrix} U^{\sim}. \end{aligned}$$

This proves the second condition of the Minkowski inverse. Finally, the m-symmetric conditions follow at once. ■

If we assume that H is nilpotent such that $ind(H) = 2$, i.e., $H^2 = 0$, then $J = 0$, $H = U \begin{pmatrix} 0 & DK \\ 0 & 0 \end{pmatrix} U^{\sim}$ and $H^{\sim} = U \begin{pmatrix} 0 & 0 \\ -G_1 K^{\sim} D & 0 \end{pmatrix} U^{\sim}$. Therefore,

$$H + H^{\sim} = U \begin{pmatrix} 0 & DK \\ -G_1 K^{\sim} D & 0 \end{pmatrix} U^{\sim}.$$

Direct conformation shows that the Minkowski inverse of $H + H^{\sim}$ is given by

$$(H + H^{\sim})^{\oplus} = U \begin{pmatrix} 0 & -D^{-1}K \\ -G_1 K^{\sim} D^{-1} & 0 \end{pmatrix} U^{\sim}.$$

Also, premultiplying (15) by H, we have

$$\begin{aligned} H(H+H^{\sim})^{\oplus} &= U \begin{pmatrix} R & S \\ 0 & 0 \end{pmatrix} U^{\sim} U \begin{pmatrix} 0 & -(S^{\sim})^{\oplus} \\ S^{\oplus} & S^{\oplus}(K^{\oplus}J)^{\sim} - K^{\oplus}J(S^{\sim})^{\oplus} \end{pmatrix} U^{\sim} \\ &= U \begin{pmatrix} I_r & (K^{\oplus}J)^{\sim} G_1 \\ 0 & 0 \end{pmatrix} U^{\sim}. \end{aligned} \tag{17}$$

Clearly, (17) is an oblique projector. Using the results from [17] we can determine the onto and along spaces of this oblique projector. We manipulate the relation between the representations (2) and (6) to determine the along and onto spaces of the oblique projector (17).

For L and M of the form (6) we have the Minkowski inverse of $\bar{M}L$, according to the equation (9) of [17], is given by

$$(\bar{M}L)^{\oplus} = U \begin{pmatrix} P_{\bar{w}} & -XZ^{\oplus} \\ 0 & 0 \end{pmatrix} U^{\sim}. \tag{18}$$

Furthermore, the equations (14) and (15) of [17] give the onto and along spaces of the oblique projector $(\bar{M}L)^{\oplus}$ i.e., $(\bar{M}L)^{\oplus}$ is an oblique projector onto $R(L) \cap [N(L) + N(M)]$ along $R(M) \oplus^{[\perp]} [N(L) \cap N(M)]$. When the representations (5) and (6) are same, then, by matrix equality, we have $W = J^{\sim}J$, $X = J^{\sim}K$ and $Z = -G_1K^{\sim}K$. Now when H is range disjoint we have from Lemma 3.7, $J^{\sim}J = I_r$ implies $P_{\bar{w}} = I_r$ and $-XZ^{\oplus} = J^{\sim}K(G_1K^{\sim}K)^{\oplus} = (K^{\oplus}J)^{\sim}G_1$. Therefore, it can easily observed that $H(H+H^{\sim})^{\oplus}$ takes the form of $(\bar{M}L)^{\oplus}$. Thus, Theorem 4 of [17] establishes that $H(H+H^{\sim})^{\oplus}$ is an oblique projector onto $R(L) \cap [N(L) + N(M)]$ along $R(M) \oplus^{[\perp]} [N(L) \cap N(M)]$. Furthermore, Theorem 6 from [17] can be used to obtain some more characterizations of $H(H+H^{\sim})^{\oplus}$ when H is range disjoint and of full range.

We can also characterize the onto and along spaces of the oblique projectors having the basic representation originating from (6) in terms of the subspaces originating from H as follows:

Theorem 3.18 Let L and M of the form (6). Then $(M\bar{L})$ is an oblique projector onto $N(H^{\sim}) \cap [R(H) \cap R(H^{\sim})]$ along $N(H) \oplus^{[\perp]} [R(H) \cap R(H^{\sim})]$.

Proof. For L and M having the representation of the form (6), we have

$$(M\bar{L}) = U \begin{pmatrix} W & X \\ -G_1X^{\sim} & Z \end{pmatrix} U^{\sim}U \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} U^{\sim} = U \begin{pmatrix} 0 & X \\ 0 & Z \end{pmatrix} U^{\sim}.$$

The Minkowski inverse of $(M\bar{L})$, as a result of direct conformation, on utilizing the points (iv) of Lemma 2.7, (ii) and (iii) of Theorem 2.9 and (ii) of Theorem 2.10 from [16], is given by

$$(M\bar{L})^{\oplus} = U \begin{pmatrix} 0 & 0 \\ -Z^{\oplus}G_1X^{\sim} & P_z \end{pmatrix} U^{\sim}.$$

On using the statements (iv) of Lemma 2.7 and (iii) of Theorem 2.10 from [16], we have

$$P_{R[(M\bar{L})^{\oplus}]} = U \begin{pmatrix} 0 & 0 \\ 0 & P_z \end{pmatrix} U^{\sim}. \tag{19}$$

Utilizing the fact that the m-projection onto $N(H^{\sim})$ is given by

$$P_{N(H^{\sim})} = I_n - L = U \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} U^{\sim}.$$

and from the statement (i) of Lemma 3.3 in [16], we have

$$P_{R(L)+R(M)} = U \begin{pmatrix} I_r & 0 \\ 0 & P_z \end{pmatrix} U^{\sim}.$$

Applying the statement (ii) of Lemma 3.2 from [16] to the above two projectors, we get

$$P_{N(H^{\sim}) \cap R(L)+R(M)} = U \begin{pmatrix} 0 & 0 \\ 0 & P_z \end{pmatrix} U^{\sim}. \tag{20}$$

Therefore, from equations (19) and (20), we have $R[(M\bar{L})^{\oplus}] = N(H^{\sim}) \cap R(L) + R(M)$. It can easily be observed that

$$P_{N[(M\bar{L})^{\oplus}]} = U \begin{pmatrix} \tilde{P}_{\bar{w}} + \bar{W} & -X \\ G_1X^{\sim} & \bar{Z} \end{pmatrix} U^{\sim}. \tag{21}$$

Considering the projectors $P_{N((H))} = U \begin{pmatrix} \bar{W} & -X \\ G_1X^{\sim} & \bar{Z} \end{pmatrix} U^{\sim}$, $P_{R(H)} = L$ and $P_{R(H^{\sim})} = M$, it can be easily verified, on applying Lemma 3.2 statement (ii) to $P_{R(H)}$ and $P_{R(H^{\sim})}$, that

$$P_{R(H)\cap R(H^{\sim})} = U \begin{pmatrix} \tilde{P}_{\bar{w}} & 0 \\ 0 & 0 \end{pmatrix} U^{\sim}.$$

Finally, applying statement (i) of Lemma 3.2 to $P_{N(H)}$ and $P_{R(H)\cap R(H^{\sim})}$, we get

$$P_{N((H))\oplus^{[\perp]}R(H)\cap R(H^{\sim})} = U \begin{pmatrix} \tilde{P}_{\bar{w}} + \bar{W} & -X \\ G_1X^{\sim} & \bar{Z} \end{pmatrix} U^{\sim}. \tag{22}$$

From equations (21) and (22), we have $N[(M\bar{L})^{\oplus}] = N((H)) \oplus^{[\perp]} R(H) \cap R(H^{\sim})$. This completes the proof. ■

Theorem 3.19 Let $H \in M_n(\mathbb{C})$ is range disjoint if and only if every $\{1^{\sim}\}$-inverse of $H + H^{\sim}$ is a $\{1^{\sim}\}$-inverse of H i.e., $H(H + H^{\sim})^{\{1^{\sim}\}}H = H$ for every $\{1^{\sim}\}$-inverse $(H + H^{\sim})^{\{1^{\sim}\}}$ of $H + H^{\sim}$.

Proof. From Corollary 3.16 and the obtained in the point (ii), i.e., $R(H + H^{\sim}) = R(H) + R(H^{\sim})$, it is clear that $R(H) \subseteq R(H + H^{\sim})$ and $R(H^{\sim}) \subseteq R(H + H^{\sim})$. These two properties of inclusion establish the fact that

$$(H + H^{\sim})(H + H^{\sim})^{\{1^{\sim}\}}H = H \tag{23}$$

for all $\{1^{\sim}\}$-inverse $(H + H^{\sim})^{\{1^{\sim}\}}$ of $H + H^{\sim}$. Also,

$$\begin{aligned} H^{\sim}(H + H^{\sim})^{\oplus}H &= U \begin{pmatrix} J^{\sim}D & 0 \\ -G_1K^{\sim}D & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ K^{\oplus}J & K^{\oplus}K \end{pmatrix} U^{\sim} \\ &= 0. \end{aligned} \tag{24}$$

Thus, $H^{\sim}(H + H^{\sim})^{\oplus}H^{\sim} = 0$ holds for the Minkowski inverse $(H + H^{\sim})^{\oplus}$ of $(H + H^{\sim})$. Therefore, combining (23) and (24), we have

$$\begin{aligned} H &= (H + H^{\sim})(H + H^{\sim})^{\{1^{\sim}\}}H \\ &= H(H + H^{\sim})^{\{1^{\sim}\}}H + H^{\sim}(H + H^{\sim})^{\{1^{\sim}\}}H \\ &= H(H + H^{\sim})^{\{1^{\sim}\}}H. \end{aligned}$$

This establishes the necessary part. To verify the sufficient part of the result, we first claim that $R(H) \subseteq R(H + H^{\sim})$. For if $R(H) \nsubseteq R(H + H^{\sim})$, their exists a nonzero vector $x \in M_{(n,1)}(\mathbb{C})$ such that $x \in R(H)$ and $x \notin R(H + H^{\sim})$. In other words, there is a subspace $\mathbb{S}$ such that $x \in \mathbb{S}$ and $\mathbb{S} \bigoplus R(H + H^{\sim}) = M_{(n,1)}(\mathbb{C})$. Furthermore, from [23, pp. 286] note that their exists a generalized inverse of $(H + H^{\sim})$ say L such $(H + H^{\sim})L(H + H^{\sim}) = (H + H^{\sim})$ and $L(H + H^{\sim})L = L$. From the first equality we observe that $(H + H^{\sim})L$ is an oblique projector onto $R(H + H^{\sim})$ along $\mathbb{S}$. Therefore, $Lx = L(H + H^{\sim})Lx = 0$. Also, $x \in R(H) \Rightarrow x = Hz$ for some $z \in M_{(n,1)}(\mathbb{C})$. This gives $x = Hz = HLHz = HLx = 0$, which is absurd as $x \neq 0$. Therefore, our assumption that $R(H) \nsubseteq R(H + H^{\sim})$ is wrong. Hence $R(H) \subseteq R(H + H^{\sim})$ and as a consequence of this we have $(H + H^{\sim})(H + H^{\sim})^{\{1^{\sim}\}}H = H$ for all $\{1^{\sim}\}$-inverses of $(H + H^{\sim})$. But, $H(H + H^{\sim})^{\{1^{\sim}\}}H = H$, therefore, $H^{\sim}(H + H^{\sim})^{\{1^{\sim}\}}H = 0$, which is equivalent to $R(H) \cap R(H^{\sim}) = 0$ and the result follows. ■

4. Conclusion

In this paper we have utilized the singular value decomposition of matrices in terms of m-projectors to characterize a few classes of matrices in Minkowski space. This study leads to the formulation of Minkowski inverse of sum of two matrices in a particular case and hence the future study will be directed toward generalizing the formula for the Minkowski inverse of the sum of two matrices.

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