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Domination parameters of Cayley graphs of some groups

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Abstract. In this paper, we investigate domination number, γ, as well as signed domination number, γ_S, of all cubic Cayley graphs of cyclic and quaternion groups. In addition, we show that the domination and signed domination numbers of cubic graphs depend on each other.

Keywords: Cayley graph, cubic graph, cyclic group, domination number, signed domination number.

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1. Introduction

Finding domination number and signed domination number of a graph have been investigated by several authors. For instance, the domination number of graphs such as zero divisor graph [15], measure graph [1] and total graph [2] are determined. Cayley graphs are very applicable in many graph applications, but only the domination number of a special case of Cayley graphs are calculated [3, 4, 14]. A similar concept of domination of a graph is signed domination number of graphs which is considered by some authers [5, 6, 8, 16].

This motivated us to calculate the domination number as well as signed domination number of Cayley graphs. One may also think if there is any relation between domination number and signed domination number of a graph. Up to now, no such relation is found. But we will show, in special cases, there is some relation between these two concepts. We recall some notations and definitions of graph theory.

By a graph Γ we mean a simple graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. A graph is said to be connected if each pair of vertices are joined by a walk. The number

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of edges of the shortest walk joining v_i and v_j is called the *distance* between v_i and v_j and denoted by $d(v_i, v_j)$. A graph Γ is said to be regular of degree k or, k-regular if every vertex has degree k. A subset P of vertices of Γ is a $k-$packing if $d(x, y) > k$ for all pairs of distinct vertices x and y of P [12].

A set $D \subseteq V$ of vertices in a graph Γ is a dominating set if every vertex $v \in V$ is an element of D or adjacent to an element of D. The domination number $\gamma(\Gamma)$ of a graph Γ is the minimum cardinality of a dominating set of Γ.

The closed neighborhood $N[v]$ of $v \in V(\Gamma)$ is the set consisting v and all of its neighbors. For a function $g : V(\Gamma) \to \{-1, 1\}$ and a subset W of V we define $g(W) = \sum\limits_{u \in W} g(u)$. A signed dominating function of Γ is a function $f : V(\Gamma) \to \{-1, 1\}$ such that $f(N[v]) > 0$ for all $v \in V(\Gamma)$. The weight of a function f is $\omega(f) = \sum\limits_{v \in V} f(v)$. The signed domination number $\gamma_s(\Gamma)$ is the minimum weight of a signed dominating function of Γ. A signed dominating function of weight $\gamma_s(\Gamma)$ is called a $\gamma_s(\Gamma)-$function. We denote $f(N[v])$ by $f[v]$. Also for $A \subseteq V(\Gamma)$ and signed dominating function f, set $\{v \in A : f(v) = -1\}$ denoted by A_f^-.

Let Γ_1 and Γ_2 be two graphs with vertex sets $V(\Gamma_1)$ and $V(\Gamma_2)$ and edge sets $E(\Gamma_1)$ and $E(\Gamma_2)$, respectively. The Cartesian product of Γ_1 and Γ_2 denoted by $\Gamma = \Gamma_1 \square \Gamma_2$ is a graph with vertex set $V(\Gamma) = V(\Gamma_1) \times V(\Gamma_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of Γ are adjacent if and only if either ($u_1 = v_1$ and $u_2v_2 \in E(\Gamma_2)$) or ($u_2 = v_2$ and $u_1v_1 \in E(\Gamma_1)$).

For a non-trivial group G, and an inverse closed subset S of G which does not contain the identity element of G, i.e. $S = S^{-1} = \{s^{-1} : s \in S\}$. The Cayley graph of G denoted by $Cay(G : S)$, is a graph with vertex set G and two vertices a and b are adjacent if and only if $ab^{-1} \in S$.

2. Preliminary results

Let $\{1, \ldots, n\}$ be the vertex set of a cycle of length n, one can easily check that the set $D = \{k | k \equiv 1 \pmod 3\}$ is a dominating set for C_n and conclude the following result.

Corollary 2.1 For $n \geqslant 3$, $\gamma(C_n) = \lceil \frac{n}{3} \rceil$.

Haas et. al. in [8] found the signed domination number for complete graph K_n.

Corollary 2.2 For any positive integer n,

$$\gamma_s(K_n) = \begin{cases} 1 & n \text{ is odd;} \\ 2 & n \text{ is even.} \end{cases}$$

For a graph $\Gamma = (V, E)$, $\omega(g)$ obtains its largest value when g is the constant function defined by $g(v) = 1$, for every $v \in V$, and in this case $\omega(g) = n$. Hence $\gamma_s(\Gamma) \leqslant n$. In the literature several authors found some lower bounds for special graphs. For example Dunbar et. al. [5] found a lower bound for $k-$regular graph with even k.

Corollary 2.3 For any graph Γ, if Γ is $k-$regular where k is even, then $\gamma_s(\Gamma) \geqslant \frac{n}{k+1}$.

Henning et. al. in [10] found similar result for $k-$ regular graphs of odd k.

Corollary 2.4 For any graph Γ, if Γ is $k-$regular where k is odd, then $\gamma_s(\Gamma) \geqslant \frac{2n}{k+1}$.

But for domination number of a graph we have much more results for both upper and lower bounds. Haynes et. al. in [9] proved the following result.

Corollary 2.5 For any graph Γ, $\left\lceil \frac{n}{1+\Delta(\Gamma)} \right\rceil \leqslant \gamma(\Gamma) \leqslant n - \Delta(\Gamma)$, where $\Delta(\Gamma)$ is the maximum degree of Γ.

Since in regular graphs maximum degree coincide with minimum degree and both equal to the degree of regularity, as a consequence we can rewrite the Corollary 2.5 for $k-$regular graphs as the following.

Corollary 2.6 For a $k-$regular graph Γ, we have $\left\lceil \frac{n}{1+k} \right\rceil \leqslant \gamma(\Gamma) \leqslant n - k$.

In [13], the authors found the domination numbers of the Cartesian product of some kinds of paths and cycles. We mention one of them which we will use later.

Corollary 2.7

$$\gamma(C_m \Box P_2) = \begin{cases} \lceil \frac{m}{2} \rceil + 1 & m \equiv 2 \pmod 4; \\ \lceil \frac{m}{2} \rceil & \text{otherwise.} \end{cases}$$

Haas et. al. [7] found a similar result for signed domination number.

Corollary 2.8 For $m \geqslant 3$,

$$\gamma_s(C_m \Box P_2) = \begin{cases} m & m \equiv 0 \pmod 4; \\ m+2 & m \equiv 2 \pmod 4; \\ m+1 & m \text{ is odd.} \end{cases}$$

Theorem 2.9 [17] Let $K_{a,b}$ be a complete bipartite graph with $b \leqslant a$. Then

$$\gamma_s(K_{a,b}) = \begin{cases} a+1 & \text{if } b = 1; \\ b & \text{if } 2 \leqslant b \leqslant 3 \text{ and } a \text{ is even;} \\ b+1 & \text{if } 2 \leqslant b \leqslant 3 \text{ and } a \text{ is odd ;} \\ 4 & \text{if } b \geqslant 4 \text{ and } a, b \text{ are both even;} \\ 6 & \text{if } b \geqslant 4 \text{ and } a, b \text{ are both odd;} \\ 5 & \text{if } b \geqslant 4 \text{ and } a, b \text{ have different parity.} \end{cases}$$

Zelinka in [18] brought $2-$packing equivalence assertions for the signed dominating function of cubic graphs.

Corollary 2.10 Let Γ be a cubic graph and let $A \subseteq V(\Gamma)$. The following assertions are equivalent:

i. There exists a signed dominating function f of Γ such that $f(x) = -1$ for all $x \in A$, while $f(x) = 1$ for all $x \in V(\Gamma) \setminus A$.
ii. The distance between any two distinct vertices of A in Γ is at least 3 (A is $2-$ packing).

3. Cubic Cayley graphs of cyclic groups

In this section, we classify the cubic connected Cayley graphs of a cyclic group $G \simeq \mathbb{Z}_n$. Since the Cayley graph $Cay(\mathbb{Z}_n : S)$ is connected if and only if S is a generating set of $\mathbb{Z}_n$, in the following Lemma, we determine such S.

Lemma 3.1 If the cubic Cayley graph $Cay(G, S)$ is connected, then n is even and $S = \{a, -a, \frac{n}{2}\}$, where $gcd(n, a) \in \{1, 2\}$.

Proof. Let $S = \{a, b, c\}$. Since $S = S^{-1}$, S has an element of order 2 and so n is even. Let $O(c) = 2$. Thus $c = \frac{n}{2}$. But $\mathbb{Z}_n$ has exactly one element of order 2, and hence, $b = -a$. Since S is a generating set of $\mathbb{Z}_n$, 1 as an element of $\mathbb{Z}_n$ is generated by S. Hence, $gcd(\frac{n}{2}, a) = 1$. Therefore, $ra + s\frac{n}{2} = 1$ for some integers r and s, implying $gcd(n, a) \in \{1, 2\}$. ■

The Cayley graphs $Cay(G : S)$ and $Cay(G : T)$ are isomorphic if for some automorphism σ of G, $\sigma(S) = \sigma(T)$ [11]. Thus in the next Theorem, up to isomorphism, we can classify all connected cubic Cayley graphs.

Lemma 3.2 Let $S_1 = \{1, -1, \frac{n}{2}\}$ and $S_2 = \{a, -a, \frac{n}{2}\}$, where $gcd(a, n) = 1$. Then $Cay(\mathbb{Z}_n : S_1) \simeq Cay(\mathbb{Z}_n : S_2)$.

Proof. Let $\delta : \mathbb{Z}_n \to \mathbb{Z}_n$ be a homomorphism such that $\delta(1) = a$. Since $gcd(a, n) = 1$, we have δ is an automorphism of the group $\mathbb{Z}_n$. Since n is even with $gcd(n, a) = 1$ implies, a is odd and hence $\delta(\frac{n}{2}) = \frac{an}{2} = \frac{n}{2}$, i.e. $\delta(S_1) = S_2$. Now the result follows. ■

Lemma 3.3 Let $n = 2k$, where k is odd and let $S_1 = \{2, -2, k\}$ and $S_2 = \{a, -a, k\}$ such that $gcd(a, n) = 2$. Then $Cay(\mathbb{Z}_n : S_1) \simeq Cay(\mathbb{Z}_n : S_2)$.

Proof. Consider the homomorphism $\delta : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ such that $\delta(a) = 2$ and $\delta(k) = k$. Since $\{2, k\}$ is a minimal generator of $\mathbb{Z}_n$, δ is an automorphism of the group $\mathbb{Z}_n$ and one can easily verify that $\delta(S_2) = S_1$ and the results reach. ■

First, we introduce the following type of cubic graph of order m which is called Type A: A graph of order m which has a Hamiltonian cycle $v_1, v_2, \cdots, v_m, v_1$ such that v_i and $v_{i+\frac{m}{2}}$ are adjacent (the addition is taken modulo m).

***Example* 3.4** If $S = \{1, -1, \frac{n}{2}\}$, then $Cay(\mathbb{Z}_n : S)$ has a Hamiltonian cycle such that i and $i + \frac{n}{2}$ are also adjacent for every $i \in \mathbb{Z}_n$. Therefore, the Cayley graph $Cay(\mathbb{Z}_n : S)$ is isomorphic to graph of Type A. By Lemma 3.2, if $gcd(n, a) = 1$ and $S' = \{a, -a, \frac{n}{2}\}$, then $Cay(\mathbb{Z}_n : S')$ is isomorphic to $Cay(\mathbb{Z}_n : S)$ when $S = \{1, -1, \frac{n}{2}\}$.

The example of $Cay(\mathbb{Z}_n : S)$ for $S = \{a, -a, \frac{n}{2}\}$ can be imagined as the following Lemma.

Lemma 3.5 If $S = \{a, -a, \frac{n}{2}\}$ where $gcd(n, a) = 2$, then $Cay(\mathbb{Z}_n : S) \simeq P_2 \Box C_{\frac{n}{2}}$.

Proof. By Lemma 3.3, we may suppose that $a = 2$. Let $V = \{0, 1, \ldots, n-1\}$ and let $V = V_1 \cup V_2$ such that $V_1 = \{0, 2, \ldots, n-2\}$ and $V_2 = \{\frac{n}{2}, \frac{n}{2}+2, \ldots, n-1\}$. The induced subgraphs on V_1 and V_2 are two cycles. Let C_i be the induced subgraph of $Cay(\mathbb{Z}_n : S)$ on V_i for $i \in \{1, 2\}$. Define $\alpha : V_1 \to V_2$, $\alpha(i) = i + \frac{n}{2}$. It is not hard to see that α is an isomorphism. So $C_1 \simeq C_2$. Also $i \in V_1$ is adjacent to $j \in V_2$ in graph $Cay(\mathbb{Z}_n : S)$ if and only if $j = \alpha(i)$. Thus $Cay(\mathbb{Z}_n : S) \simeq P_2 \Box C_{\frac{n}{2}}$. ■

Now we can find the domination and signed domination numbers of all cubic Cayley graphs of cyclic groups. But for completeness of our talk we also tell about the domination number of such Cayley graphs of valency 2 in the following Remark.

Remark 1 *Let $\mathbb{Z}_n = \langle S \rangle$, where S is an inverse closed subset of $\mathbb{Z}_n \setminus \{0\}$. If $|S| = 2$, then $Cay(\mathbb{Z}_n : S) \simeq C_n$. By Corollary 2.1, $\gamma(Cay(\mathbb{Z}_n : S)) = \lceil \frac{n}{3} \rceil$.*

Theorem 3.6 If n is even and $S = \{a, -a, \frac{n}{2}\}$ where $gcd(n, a) = 2$, then

$$\gamma(Cay(\mathbb{Z}_n : S)) = \begin{cases} \lceil \frac{n}{4} \rceil + 1 & n \equiv 4 \pmod 8; \\ \lceil \frac{n}{4} \rceil & \text{otherwise.} \end{cases}$$

Proof. The proof is straightforward by Corollary 2.7 and Lemma 3.5. ■

Theorem 3.7 If $\mathbb{Z}_n = \langle S \rangle$ where $S = \{a, -a, \frac{n}{2}\}$ and $gcd(n, a) = 2$, then we have $\gamma_s(Cay(\mathbb{Z}_n : S)) = \frac{n}{2} + 1$.

Proof. Since $\{a, -a, \frac{n}{2}\}$ generates group $\mathbb{Z}_n$, $\frac{n}{2}$ is odd. By Lemma 3.5 and Corollary 2.8, $\gamma_s(Cay(\mathbb{Z}_n : S)) = \frac{n}{2} + 1$. ■

Theorem 3.8 If n is even and $S = \{a, -a, \frac{n}{2}\}$ where $gcd(n, a) = 1$, then

$$\gamma(Cay(\mathbb{Z}_n : S)) = \begin{cases} k & n = 4k \text{ and } k \text{ is odd;} \\ k + 1 & \text{otherwise.} \end{cases}$$

Proof. Since n is even, $n = 4k$ or $4k + 2$. We consider three cases:

Case 1. Let $n = 4k$ and k be odd. Set $D = \{4t : 0 \leqslant t \leqslant k - 1\}$. Then $|D| = k$ and $N(D) = \{4t \pm 1, 4t + 2k : 0 \leqslant t \leqslant k - 1\}$. It is easy to check that $|N(D)| = 3k$ and $D \cap N(D) = \emptyset$. So D is a dominating set and so $\gamma(Cay(\mathbb{Z}_n : S)) \leqslant k$. Since $Cay(\mathbb{Z}_n : S)$ is cubic, by Corollary 2.6, $\gamma(Cay(\mathbb{Z}_n : S)) = k$.

Case 2. Let $n = 4k$ and k be even. Set $A = \{4t : 0 \leqslant t \leqslant \frac{k}{2} - 1\}$, $B = \{2k + 4s + 2 : 0 \leqslant s \leqslant \frac{k}{2} - 1\}$ and $C = \{2k - 1\}$. Let $D = A \cup B \cup C$. We show that D is a dominating set. Let $i \in \mathbb{Z}_n$ and $i \leqslant 2k$. If $i \equiv 2 \pmod 4$, then $i \in N(B)$. Otherwise $i \in A \cup N(A) \cup C$. Suppose that $i > 2k$. If $i \equiv 0 \pmod 4$, then $i \in N(A)$. If not $i \in B \cup N(B)$. Hence, $\mathbb{Z}_n \subseteq D \cup N(D)$. Since $|D| = k + 1$, $\gamma(Cay(\mathbb{Z}_n : S)) \leqslant k + 1$. Consider a subset of $V(Cay(\mathbb{Z}_n : S))$ of cardinality k which is a dominating set. On the contrary, let D' be a dominating set and $|D'| = k$. Since $Cay(\mathbb{Z}_n : S)$ is cubic, $|N(D')| \leqslant 3k$. On the other hand, $|D' \cup N(D')| = n$. So $D' \cap N(D') = \emptyset$ and thus $|N(D')| = 3k$. Hence, $N(r) \cap N(r') = \emptyset$ for $r, r' \in D'$. Let $r \in D'$ and be fixed. Then $N(r) = \{r \pm 1, r + 2k\} \subseteq N(D')$. So $r + 1 \in N(r)$ and hence $r + 2 \in D' \cup N(D')$. If $r + 2 \in D'$, then $r + 1 \in N(r) \cap N(r + 2)$. This is impossible. Hence, $r + 2 \in N(D')$. Also $N(r + 2) = \{r + 1, r + 2, r + 2 + 2k\}$. If $r + 3 \in D'$, then $N(r + 3) \subset N(D')$. Thus $r + 3 + 2k \in N(D')$. Since $r + 2k + 1 \notin D'$, $r + 2k + 1 \in N(D')$. Since $N(r + 2k + 1) = \{r + 2k + 2, r + 2k, r + 1\}$ and $r + 1, r + 2 \in N(r)$, $r + 2k + 2 \in D'$. Hence, $r + 2k + 3 \in N(r + 2k + 2) \cap N(r + 3)$. This is not true. So $r + 3 \notin D'$. Hence, for every $r \in D'$, $r + 2, r + 3 \in N(D')$. Since $r + 1, r + 3 \in N(r + 2) \setminus D'$, $r + 2 + 2k \in D'$. Thus $r + 3 + 2k \in N(D')$. Also $r + 3 + 2k, r + 2 \in N(r + 3) \setminus D'$. So $r + 4 \in D'$. By induction $r + 4(\frac{n}{8}) = r + 2k \in D'$. This is contradiction by $r + 2k \in N(D')$. Therefore, $\gamma(Cay(\mathbb{Z}_n : S)) = k + 1$.

Case 3. Let $n = 4k + 2$. Let $A' = \{4t : 0 \leqslant t \leqslant \lfloor \frac{k}{2} \rfloor\}$, $B' = \{2k + 4s + 3 : 0 \leqslant s \leqslant \lfloor \frac{k}{2} \rfloor - 1\}$ and $C' = \{2k - 1\}$. If k is odd, then let $D_1 = A' \cup B' \cup C'$. Otherwise let $D_2 = A' \cup B'$. For $i = 1, 2$, $|D_i| = k + 1$. Likewise Case 2, D_i is a dominating set. So by Corollary 2.6, $\gamma(Cay(\mathbb{Z}_n : S)) = k + 1$.

■

Corollary 3.9 For any connected cubic graph Γ of Type A and order n we have:

$$\gamma(\Gamma) = \begin{cases} k & n = 4k \text{ and } k \text{ is odd;} \\ k + 1 & \text{otherwise.} \end{cases}$$

Depending on the usage of graphs in industry, some kinds of domination number defined, but among all of them, domination and signed domination number is connected more frequency. One can be doubted if there is any relation between such dominations. In the next Theorem we will show in some special case some relation appears.

Theorem 3.10 Let Γ be a connected cubic graph on n vertices. Then $\gamma(\Gamma) = \frac{n}{4}$ if and only if $\gamma_S(\Gamma) = \frac{n}{2}$.

Proof. Let $\gamma(\Gamma) = \frac{n}{4}$. Then $n = 4k$ for positive integer k and there is a dominating set D with cardinality k. Since graph Γ is cubic and D is the minimum dominating set, every $x \in D$ dominates exactly 3 members of $V(\Gamma)$. Hence, $V(\Gamma) = D \cup N(D)$, where $D \cap N(D) = \emptyset$. Also $N[x] \cap N[y] = \emptyset$ for every $x \neq y \in D$. So $d(x, y) \geqslant 3$. Now define $f : V(\Gamma) \longrightarrow \{-1, 1\}$ such that $f(v) = -1$ if and only if $v \in D$. By Corollary 2.10 and since D is a $2-$packing subset of $V(\Gamma)$, f is a signed dominating function by Corollary 2.10. Thus $\gamma_{_S}(\Gamma) \leqslant \omega(f) = \frac{n}{2}$. On the other hand, by Corollary 2.4, $\gamma_{_S}(\Gamma) \geqslant \frac{n}{2}$.

Conversely, let $\gamma_{_S}(\Gamma) = \frac{n}{2}$. If g is a $\gamma_{_S}-$function, then $|V_g^-| = \frac{n}{4}$. So $n = 4k$. Since Γ is cubic, $N(v) \cap V_g^- = \emptyset$ and $N[v] \cap N[u] = \emptyset$ for every $v, u \in V_g^-$. Thus $|N(V_g^-)| = 3k$. So V_g^- is a dominating set of Γ and $\gamma(\Gamma) \leqslant k$. Corollary 2.6 completes the proof. ■

Theorem 3.11 If $\mathbb{Z}_n = \langle S \rangle$ where $S = \{a, -a, \frac{n}{2}\}$ and $gcd(n, a) = 1$, then

$$\gamma_{_S}(Cay(\mathbb{Z}_n : S)) = \begin{cases} \frac{n}{2} & n = 4k \text{ and } k \text{ is odd;} \\ \frac{n}{2} + 1 & n = 4k + 2; \\ \frac{n}{2} + 2 & n = 4k \text{ and } k \text{ is even.} \end{cases}$$

Proof. Since n is even, we consider two cases:

Case I. Let $n = 4k + 2$. By Corollary 2.4, $\gamma_{_S}(Cay(\mathbb{Z}_n : S)) \geqslant \frac{n}{2}$. Since n is even, $\gamma_{_S}(Cay(\mathbb{Z}_n : S))$ is also even. Thus $\gamma_{_S}(Cay(\mathbb{Z}_n : S)) \geqslant \frac{n}{2} + 1$. Let k be odd. Define $g : V(Cay(\mathbb{Z}_n : S)) \to \{-1, 1\}$ such that $f(x) = -1$ if and only if $x \in A \cup B$, where $A = \{4t : 0 \leqslant t \leqslant \lfloor \frac{k}{2} \rfloor\}$ and $B = \{\frac{n}{2} + 4s + 2 : 0 \leqslant s \leqslant \lfloor \frac{k}{2} \rfloor - 1\}$. It is easily seen that $N(A) \cap B$ and $A \cap N(B)$ are empty. Also since $0 \leqslant s, t \leqslant \lfloor \frac{k}{2} \rfloor$, $N(A) \cap N(B) = \emptyset$. So V_g^- is a $2-$packing and so g is a signed dominating function. Hence, $\gamma_{_S}(Cay(\mathbb{Z}_n : S)) \leqslant \omega(g) = \frac{n}{2} + 1$. Now let k be even. Define g' such that $V_{g'}^- = A' \cup B'$ where $A' = \{4t : 0 \leqslant t \leqslant \frac{k}{2} - 1\}$ and $B' = \{\frac{n}{2} + 4s + 2 : 0 \leqslant s \leqslant \frac{k}{2} - 1\}$. Again g' is a dominating function and the result reaches.

Case 2. Let $n = 4k$. If k is an odd integer, then we reach the result by Theorems 3.8 and 3.10. Let k be even. We claim that if f is a function of $Cay(\mathbb{Z}_n : S)$ and $|V_f^-| = k$, then f is not a signed domination function. On the contrary, suppose that $|V_f^-| = k$ and f is a signed dominating function. By Corollary 2.10, $V_f^- \cap N(V_f^-) = \emptyset$. Let $r \in V_f^-$. Then $r+1, r+2 \notin V_f^-$. If $r+3 \in V_f^-$, then $r+3+\frac{n}{2}, r+2+\frac{n}{2} \in N(V_f^-)$. Thus $N(r+1+\frac{n}{2}) = \{r+1, r+2+\frac{n}{2}, r+\frac{n}{2}\} \cap V_f^- = \emptyset$. This is impossible. So $r + 3 \in N(V_f^-)$. But we show that $r+4 \in V_f^-$. Since $r \in V_f^-$, $N[r+2] \cap V_f^- = r+2+\frac{n}{2}$. So $r+1+\frac{n}{2}, r+3+\frac{n}{2} \notin V_f^-$. Hence, $N[r + 3] \cap V_f^- = \{r + 4\}$. Therefore, $r \in V_f^-$ if and only if $r + 4 \in V_f^-$. Since k is even, if $r \in V_f^-$, then $r + \frac{n}{2} \in V_f^-$ i.e. $r + \frac{n}{2} \in V_f^- \cap N(V_f^-)$. This is not true. Thus $|V_f^-| < k$ and $\gamma_{_S}(Cay(\mathbb{Z}_n : S)) \geqslant \frac{n}{2} + 2$. Now we define $f : V(Cay(\mathbb{Z}_n : S)) \to \{-1, 1\}$ such that $V_f^- = \{4t : 0 \leqslant t \leqslant \frac{k}{2} - 1\} \cup \{2 + \frac{n}{2} + 4s : 0 \leqslant s \leqslant \frac{k}{2} - 2\}$. By Lemma 3.2, $N(x) = \{x \pm 1, x + \frac{n}{2}\}$ for every $x \in \mathbb{Z}_n$. If x, y are two distinct elements of V_f^-, then $x - y \notin S$. Also $N(x) \cap N(y) = \emptyset$. Thus V_f^- is a 2-packing and by Corollary 2.10, f is a signed dominating function. Also $|V_f^-| = k-1$. Hence, $\gamma_{_S}(Cay(\mathbb{Z}_n : S)) \leqslant \omega(f) = \frac{n}{2}+2$. This completes the proof.

■

Corollary 3.12 For any connected cubic graph Γ of Type A and order n we have:

$$\gamma_s(\Gamma) = \begin{cases} \frac{n}{2} & n = 4k \text{ and } k \text{ is odd;} \\ \frac{n}{2}+1 & n = 4k+2; \\ \frac{n}{2}+2 & n = 4k \text{ and } k \text{ is even.} \end{cases}$$

4. Cubic Cayley graphs of quaternion groups

In this section we consider the cubic Cayley graphs of quaternion groups which is defined by $Q_{4n} = \langle a, b; a^n = b^2, a^{2n} = 1, b^{-1}ab = a^{-1}\rangle$. Since $S = S^{-1}$ and $|S| = 3$, S is not a generating set of Q_{4n} and so $Cay(Q_{4n} : S)$ has some isomorphic components.

Theorem 4.1 If $S = S^{-1}$, then cubic Cayley graph $Cay(Q_{4n} : S) \simeq r\Gamma$, where Γ is complete graph K_4, $C_{\frac{m}{2}} \Box P_2$ or a graph of Type A.

Proof. Since $S = S^{-1}$ and a^n is the only element of order 2, $a^n \in S$. Hence, there are two cases, $S = \{a^n, a^i, a^{-i}\}$ or $\{a^n, a^ib, a^ib^{-1}\}$, where $1 \leq i < n$. In the first case, $\langle S\rangle$ is a cyclic group of order $2n$. Likewise the argument for connected cubic Cayley graph, $Cay(Q_{4n} : S) \simeq 2\Gamma$, where Γ is isomorphism to a graph of Type A or $C_n \Box P_2$. Let $S = \{a^n, a^ib, a^ib^{-1}\}$. Then $\langle S\rangle$ is isomorphic to $\mathbb{Z}_4$. So $Cay(Q_{4n} : S) \simeq nK_4$. ■

Theorem 4.2 If $S = S^{-1}$, then for cubic Cayley graph $Cay(Q_{4n} : S)$,

$$n \leq \gamma(Cay(Q_{4n} : S)) \leq n + 1.$$

Proof. By Theorem 4.1, $Cay(Q_{4n} : S) \simeq nK_4$, $2C_n \Box P_2$ or 2Γ, where Γ is a graph of Type A. In the first case, $\gamma(Cay(Q_{4n} : S)) = n\gamma(K_4) = n$. In the second case and by Theorem 2.8, $\gamma(Cay(Q_{4n} : S)) = 2\gamma(C_n \Box P_2) \in \{n, n+1\}$. If $Cay(Q_{4n} : S) \simeq 2\Gamma$ and Γ is a graph of Type A, then Corollary 3.9 completes the proof. ■

Theorem 4.3 If $S = S^{-1}$, then for cubic Cayley graph $Cay(Q_{4n} : S)$,

$$\gamma_s(Cay(Q_{4n} : S)) \in \{2n, 2n+2, 2n+4\}.$$

Proof. The proof is straightforward by 2.2, 2.8, 3.12 and 4.1. ■

References

[1] A. Assari, M. Rahimi, Graphs generated by measures, J. of Mathematics. (2016), 2016:1706812.
[2] A. Badawi, On the Total Graph of a Ring and its Related Graphs: a survey, Commutative algebra, Springer, New York, 2014.
[3] T. T. Chelvam, G. Kalaimurugan, Bounds for Domination Parameters in Cayley Graphs on Dihedral Group, Open J. of Discrete Math. 2 (2012), 5-10.
[4] T. T. Chelvam, I. Rani, Dominating sets in Cayley graphs on Z_n, Tamkang J. Math. 38 (4) (2007), 341-345.
[5] J. Dunbar, S. Hedetniemi, M. A. Henning, P. J. Slater, Signed domination in graphs, Graph theory, Combinatorics, and Algorithms, Wiley, New York, 1995.
[6] O. Favaron, Signed domination in regular graphs, Discrete Mathematics. 158 (1996), 287-293.
[7] R. Haas, T. B. Wexler, Bounds on the Signed Domination Number of a Graph, Electronic Notes in Discrete Mathematics, Elsevier, 2002.
[8] R. Haas, T. B. Wexler, Signed domination numbers of a graph and its complement, Discrete Mathematics. 283 (2004), 87-92.
[9] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Fundamentals of Domination in Graphs, Monographs and Textbooks in Pure and Applied Mathematics 208, 1998.

[10] M. A. Henning, P. J. Slater, Inequalities relating domination parameters in cubic graphs, Discrete Mathematics. 158 (1996), 87-98.
[11] C. H. Li, On isomorphisms of finite Cayley graphsa survey, Discrete Mathematics. 256 (2002), 301-334.
[12] A. Meir, J. W. Moon, Relations between packing and covering numbers of a tree, Pacific J. Math. 61 (1) (1975), 225-233.
[13] P. Pavlič, J. A. Žerovnik, Note on the domination number of the Cartesian products of paths and cycles, Krag. J. Math. 37 (2) (2013), 275-285.
[14] E. Vatandoos , F. Ramezani, Domination and signed domination number of Cayley graphs, IJMSI. 14 (1) (2019), 35-42.
[15] E. Vatandoost, F. Ramezani, On the domination and signed domination numbers of zero-divisor graph, EJGTA. 4 (2) (2016), 148-156.
[16] L. Volkmann, B. Zelinka, Signed domatic number of a graph, Discrete. Appl. mathematics. 150 (2005), 261-267.
[17] B. Zelinka, Signed and minus domination in bipartite graphs, Czechoslovak Math. 56 (131) (2006), 587-590.
[18] B. Zelinka, Some remarks on domination in cubic graphs, Discrete Mathematics. 158 (1996), 249-255.